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ON ENTIRE FUNCTIONS THAT SHARE A VALUE WITH THEIR DERIVATIVES

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Abstract. Let f be a nonconstant entire function, a a finite complex number, k and m two distinct positive integers, and (k,m) the greatest common divisor of k and m. If f, $f^{(k)}$ and $f^{(m)}$ share a CM, then

$$f(z)=\frac{c-1}{c}a+\sum_{j=1}^{q}C_je^{\lambda_j z},$$

where q is a positive integer with $q\le (k,m)$, c and C_j for $1\le j\le q$ are nonzero constants, and λ_j for $1\le j\le q$, are distinct nonzero constants satisfying $(\lambda_j)^k=(\lambda_j)^m=c$, for $a\ne 0$, and $(\lambda_j)^k=c$, $(\lambda_j)^m=d$, for $a=0$, where d is a nonzero constant. This answers a question of Yang and Yi [14] for entire functions, and extends a result of Csillag [2].

1. Introduction and main results

Let f and g be two nonconstant meromorphic functions in the complex plane, and let a be a finite complex number. If $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities, then we say that f and g share a CM.

In 1986, Jank–Mues–Volkmann [8] proved

Theorem A. *Let f be a nonconstant meromorphic function and a a nonzero finite complex number. If f, f' and f'' share a CM, then $f\equiv f'$.*

By Theorem A, the following question was posed.

Question 1 (see [6], [7], [13], [14]). *Let f be a nonconstant meromorphic function, a a nonzero finite complex number, and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Can we get $f\equiv f^{(k)}$?*

The following example [12] shows that the answer to Question 1 is, in general, negative.

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Example 1. Let k, m be positive integers satisfying $m > k+1$, b a nonzero constant such that $b^k = b^m \neq 1$ and $a = b^k$. Set

$$f(z) = e^{bz} + a - 1.$$

Then f, $f^{(k)}$ and $f^{(m)}$ share a CM. But $f \not\equiv f^{(k)}$.

In Example 1, f is an entire function, and f, $f^{(k)}$ and $f^{(m)}$ share a CM. Although $f \not\equiv f^{(k)}$, we have $f^{(k)} \equiv f^{(m)}$.

Naturally, we pose the following question.

Question 2. *Let f be a nonconstant meromorphic function, a a nonzero finite complex number and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Can we get $f^{(k)} \equiv f^{(m)}$?*

In this paper, we give an affirmative answer to Question 2 for entire functions. In fact, we have proved the following more general result.

Theorem 1. *Let f be a nonconstant entire function, a a finite complex number, k and m two distinct positive integers, and (k, m) the greatest common divisor of k and m. If f, $f^{(k)}$ and $f^{(m)}$ share a CM, then*

$$f(z) = \left(1 - \frac{1}{c}\right)a + \sum_{j=1}^{q} C_j e^{\lambda_j z}, \tag{1.1}$$

where q is a positive integer with $q \leq (k, m)$, c and C_j, $1 \leq j \leq q$, are nonzero constants, and λ_j, $1 \leq j \leq q$, are distinct nonzero constants satisfying

$$(\lambda_j)^k = (\lambda_j)^m = c, \quad \text{for } a \neq 0; \tag{1.2}$$

and

$$(\lambda_j)^k = c, \quad (\lambda_j)^m = d, \quad \text{for } a = 0, \tag{1.3}$$

where d is a nonzero constant.

By Theorem 1, we can easily obtain the following results.

Corollary 2. *Let f be a nonconstant entire function, a a nonzero finite complex number, and k, m two distinct positive integers. Suppose that f, $f^{(k)}$ and $f^{(m)}$ share a CM. Then $f^{(k)} \equiv f^{(m)}$.*

Corollary 2 gives an affirmative answer to Question 2 for entire functions.

Corollary 3 ([10, Theorem 1]). *Let f be a nonconstant entire function, a a nonzero finite complex number and k a positive integer. If f, $f^{(k)}$ and $f^{(k+1)}$ share a CM, then $f \equiv f'$.*

Corollary 4 ([10, Theorem 2]). *Let f be a nonconstant entire function, a a nonzero finite complex number and $k \geq 2$ a positive integer. If f, f' and $f^{(k)}$ share a CM, then*

$$f(z) = \left(1 - \frac{1}{c}\right)a + Ce^{cz}, \tag{1.4}$$

where C and c are nonzero constants with $c^{k-1} = 1$.

Corollary 5 (Csillag [2], cf. [4, p. 67]). *Let f be a nonconstant entire function, and k and m two distinct positive integers. If $ff^{(k)}f^{(m)} \neq 0$, then $f = e^{Az+B}$, where A $(\neq 0)$ and B are constants.*

Let f be a nonconstant meromorphic function in the complex plane. Throughout this paper, we use the basic results and notations of Nevanlinna theory (cf. [3], [4], [11], [14]). In particular, $S(r,f)$ denotes any function satisfying

$$S(r,f) = o\{T(r,f)\},$$

as $r \to +\infty$, possibly outside of a set of finite linear measure, where $T(r,f)$ is Nevanlinna's characteristic function.

As usual, the order $\varrho(f)$ of f is defined as

$$\varrho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

2. Some lemmas

We will use $P_d[f]$ to denote a differential polynomial in f of degree $\leq d$ with constant coefficients which may be different at different occurrence. We denote the set of differential polynomials in f with constant coefficients by $\mathscr{P}[f]$.

Lemma 1 (Clunie [1], cf. [4, p. 68]). *Let f be a nonconstant meromorphic function, n be a positive integer, $P[f]$ and $Q[f]$ two differential polynomials in f with constant coefficients, and $P[f] \not\equiv 0$. If the degree of $P[f]$ is at most n and*

$$f^n Q[f] = P[f],$$

then

$$m(r, Q[f]) = S(r,f).$$

Lemma 2 (cf. [9, p. 29–34]). *Let f be a nonconstant entire function, n be a positive integer and a_j, $0 \leq j \leq n$, meromorphic functions with $a_n \not\equiv 0$. Suppose that*

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \equiv 0. \tag{2.1}$$

Then

$$T(r,f) \leq O\left(1 + \sum_{j=0}^{n} T(r, a_n)\right). \tag{2.2}$$

The following result is an instant corollary of Lemma 2.

Lemma 3. *Let f be a nonconstant entire function, n a positive integer and a_j, $0 \le j \le n$, meromorphic functions satisfying $T(r, a_j) = S(r, f)$. If*

$$a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \equiv 0,$$

then $a_j \equiv 0$ for $j = 0, 1, \ldots, n$.

Lemma 4 ([3, Lemma 3.12]). *Let $f_j(z)$ $(\not\equiv 0)$, $j = 1, 2, \ldots, n$, be n meromorphic functions which are linearly independent such that*

$$f_1(z) + f_2(z) + \cdots + f_n(z) \equiv 1. \tag{2.3}$$

Then for every j, $1 \le j \le n$,

$$T(r, f_j) \le \sum_{k=1}^{n} N\left(r, \frac{1}{f_k}\right) + N(r, f_j) + N(r, D) + S(r), \tag{2.4}$$

where $D = W(f_1, f_2, \ldots, f_n)$ is the Wronskian, and $S(r)$ is a function which satisfies

$$S(r) = o\Big(\max_{1 \le k \le n} T(r, f_k)\Big)$$

as $r \to \infty$, possibly outside a set of finite linear measure.

Lemma 5 ([3, Lemma 5.1]). *Let $a_j(z)$, $j = 0, 1, \ldots, n$, be entire and of finite order $\le \varrho$ $(< \infty)$. Let $g_j(z)$, $j = 1, \ldots, n$, be also entire such that each of the functions $g_i - g_j$, $i \ne j$, is a transcendental function or a polynomial of degree greater than ϱ. If*

$$\sum_{j=1}^{n} a_j(z) e^{g_j(z)} \equiv a_0(z), \tag{2.5}$$

then

$$a_j(z) \equiv 0, \quad j = 0, 1, \ldots, n. \tag{2.6}$$

Lemma 6. *Let f and α be nonconstant entire functions, a a finite complex number and k a positive integer. Suppose that*

$$f^{(k)} = a + e^{\alpha} f. \tag{2.7}$$

Then for any positive integer j, $1 \le j \le k-1$, we have

$$f^{(k+j)} = \gamma_{0,j} f + \gamma_{1,j} f' + \cdots + \gamma_{j,j} f^{(j)}, \tag{2.8}$$

and $\gamma_{i,j}$ are entire functions satisfying

$$\begin{pmatrix} \gamma_{0,j} \\ \vdots \\ \gamma_{j,j} \end{pmatrix} = \begin{pmatrix} A_{0,1,j}e^{\alpha} \\ \vdots \\ A_{j,1,j}e^{\alpha} \end{pmatrix} \tag{2.9}$$

where

$$\begin{aligned} A_{i,1,j} &= \frac{j!}{i!(j-i)!}e^{-\alpha}(e^{\alpha})^{(j-i)} \\ &= \frac{j!}{i!(j-i)!}\big((\alpha')^{j-i} + P_{j-i-1}[\alpha']\big), \quad 0 \le i \le j, \end{aligned} \tag{2.10}$$

are differential polynomials in α' with constant coefficients. In particular, $A_{j,1,j} \equiv 1$ for $1 \le j \le k-1$. Here $P_d[\alpha'] \equiv 0$ for $d \le 0$.

Proof. We prove this lemma by mathematical induction on j. By (2.7), we have $f^{(k+1)} = \alpha' e^{\alpha} f + e^{\alpha} f'$, so that (2.8)–(2.10) are true for $j = 1$. Now suppose that (2.8)–(2.10) are true for $j \le k-2$. Thus by (2.8), we get

$$\begin{aligned} f^{(k+j+1)} &= \gamma'_{0,j} f + \gamma'_{1,j} f' + \cdots + \gamma'_{j,j} f^{(j)} \\ &\quad + \gamma_{0,j} f' + \cdots + \gamma_{j-1,j} f^{(j)} + \gamma_{j,j} f^{(j+1)} \\ &= \gamma_{0,j+1} f + \gamma_{1,j+1} f' + \cdots + \gamma_{j,j+1} f^{(j)} + \gamma_{j+1,j+1} f^{(j+1)}, \end{aligned} \tag{2.11}$$

where

$$\gamma_{0,j+1} = \gamma'_{0,j}, \tag{2.12}$$

$$\gamma_{1,j+1} = \gamma'_{1,j} + \gamma_{0,j},$$

$$\vdots$$

$$\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}, \tag{2.13}$$

$$\vdots$$

$$\gamma_{j,j+1} = \gamma'_{j,j} + \gamma_{j-1,j},$$

$$\gamma_{j+1,j+1} = \gamma_{j,j}. \tag{2.14}$$

By (2.11)–(2.14), we know that (2.8)–(2.10) are true for $j+1$. Thus (2.8)–(2.10) are true for $j = 1, 2, \ldots, k-1$. Lemma 6 is proved.

Lemma 7. *Let f and α be nonconstant entire functions, a a finite complex number and k a positive integer. Suppose that*

$$f^{(k)} = a + e^{\alpha} f. \tag{2.15}$$

Then for any positive integer j $(\geq k)$ $j = sk + l$, $s \geq 1$, $0 \leq l \leq k-1$, *we have*

$$f^{(k+j)} = \gamma_{-1,j} + \gamma_{0,j} f + \gamma_{1,j} f' + \cdots + \gamma_{k-1,j} f^{(k-1)}, \tag{2.16}$$

and $\gamma_{i,j}$ *are entire functions satisfying*

$$\begin{pmatrix} \gamma_{-1,j} \\ \gamma_{0,j} \\ \vdots \\ \gamma_{l,j} \\ \gamma_{l+1,j} \\ \vdots \\ \gamma_{k-1,j} \end{pmatrix} = \begin{pmatrix} aA_{-1,1,j}e^{\alpha} + a\sum_{t=2}^{s-1} A_{-1,t,j}(e^{\alpha})^t + aA_{-1,s,j}(e^{\alpha})^s \\ A_{0,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{0,t,j}(e^{\alpha})^t + A_{0,s+1,j}(e^{\alpha})^{s+1} \\ \vdots \\ A_{l,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{l,t,j}(e^{\alpha})^t + A_{l,s+1,j}(e^{\alpha})^{s+1} \\ A_{l+1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{l+1,t,j}(e^{\alpha})^t + A_{l+1,s,j}(e^{\alpha})^s \\ \vdots \\ A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s \end{pmatrix}, \tag{2.17}$$

where $A_{i,t,j}$ $(\in \mathscr{P}[\alpha'])$ *satisfy*

$$\begin{pmatrix} A_{-1,s,j} \\ A_{0,s+1,j} \\ \vdots \\ A_{l-1,s+1,j} \\ A_{l,s+1,j} \\ A_{l+1,s,j} \\ \vdots \\ A_{k-1,s,j} \end{pmatrix} = \begin{pmatrix} C_{-1,s,j}(\alpha')^l + P_{l-1}[\alpha'] \\ C_{0,s+1,j}(\alpha')^l + P_{l-1}[\alpha'] \\ \vdots \\ C_{l-1,s+1,j}\alpha' + P_0[\alpha'] \\ 1 \\ C_{l+1,s,j}(\alpha')^{k-1} + P_{k-2}[\alpha'] \\ \vdots \\ C_{k-1,s,j}(\alpha')^{l+1} + P_l[\alpha'] \end{pmatrix}, \tag{2.18}$$

and $C_{i,s+1,j}$, $-1 \leq i \leq l-1$, *and* $C_{i,s,j}$, $l+1 \leq i \leq k-1$, *are positive integers, and*

$$A_{i,1,j} = \frac{j!}{i!(j-i)!} e^{-\alpha}(e^{\alpha})^{(j-i)} = \frac{j!}{i!(j-i)!}\big((\alpha')^{j-i} + P_{j-i-1}[\alpha']\big). \tag{2.19}$$

Here $P_d[\alpha'] \equiv 0$ *for* $d \leq 0$.

Proof. We prove this lemma by mathematical induction on j. First we prove that (2.16)–(2.19) are true for $j = k$. By Lemma 6, we have

$$f^{(2k-1)} = \gamma_{0,k-1} f + \gamma_{1,k-1} f' + \cdots + \gamma_{k-1,k-1} f^{(k-1)}. \tag{2.20}$$

This together with (2.15) yields

$$\begin{aligned} f^{(2k)} &= (f^{(2k-1)})' \\ &= \gamma'_{0,k-1} f + \gamma'_{1,k-1} f' + \cdots + \gamma'_{k-1,k-1} f^{(k-1)} \\ &\quad + \gamma_{0,k-1} f' + \cdots + \gamma_{k-2,k-1} f^{(k-1)} + a\gamma_{k-1,k-1} + e^{\alpha}\gamma_{k-1,k-1} f \\ &= \gamma_{-1,k} + \gamma_{0,k} f + \gamma_{1,k} f' + \cdots + \gamma_{k-1,k} f^{(k-1)}. \end{aligned} \tag{2.21}$$

By Lemma 6, we get

$$
\begin{aligned}
\gamma_{-1,k} &= a\gamma_{k-1,k-1} = ae^{\alpha}, && (2.22)\\
\gamma_{0,k} &= \gamma'_{0,k-1} + e^{\alpha}\gamma_{k-1,k-1}\\
&= (e^{\alpha})^{(k)} + (e^{\alpha})^2, && (2.23)\\
\gamma_{i,k} &= \gamma'_{i,k-1} + \gamma_{i-1,k-1}\\
&= \frac{(k-1)!}{i!(k-1-i)!}(e^{\alpha})^{(k-i)} + \frac{(k-1)!}{(i-1)!(k-i)!}(e^{\alpha})^{(k-i)}\\
&= \frac{k!}{i!(k-i)!}(e^{\alpha})^{(k-i)}, \quad i = 1, \dots, k-1. && (2.24)
\end{aligned}
$$

Thus (2.16)–(2.19) are true for $j = k$.

Now we assume that this lemma is true for a given $j = sk + l$ with $s \geq 1$ and $0 \leq l \leq k-1$. Next we show that this lemma is true for $j+1$. First by (2.15) and (2.16), we get

$$
\begin{aligned}
f^{(k+j+1)} = {} & \gamma'_{-1,j} + \gamma'_{0,j} f + \gamma'_{1,j} f' + \cdots + \gamma'_{k-1,j} f^{(k-1)}\\
& + \gamma_{0,j} f' + \cdots + \gamma_{k-2,j} f^{(k-1)} + a\gamma_{k-1,j} + e^{\alpha}\gamma_{k-1,j} f.
\end{aligned}
$$

It follows that (2.16) is true for $j+1$ with

$$\gamma_{-1,j+1} = \gamma'_{-1,j} + a\gamma_{k-1,j}, \tag{2.25}$$

$$\gamma_{0,j+1} = \gamma'_{0,j} + e^{\alpha}\gamma_{k-1,j}, \tag{2.26}$$

$$\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}, \quad i = 1, 2, \dots, k-1. \tag{2.27}$$

Thus for $l \leq k-2$, by the assumptions,

$$
\begin{aligned}
\gamma_{-1,j+1} = {} & \left(aA_{-1,1,j}e^{\alpha} + a\sum_{t=2}^{s-1} A_{-1,t,j}(e^{\alpha})^t + aA_{-1,s,j}(e^{\alpha})^s\right)'\\
& + a\left(A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s\right) && (2.28)\\
= {} & a(A'_{-1,1,j} + \alpha' A_{-1,1,j} + A_{k-1,1,j})e^{\alpha}\\
& + a\sum_{t=2}^{s-1}(A'_{-1,t,j} + t\alpha' A_{-1,t,j} + A_{k-1,t,j})(e^{\alpha})^t + aA_{-1,s,j+1}(e^{\alpha})^s,
\end{aligned}
$$

where $A_{-1,s,j+1} = A'_{-1,s,j} + s\alpha' A_{-1,s,j} + A_{k-1,s,j}$,

$$\begin{aligned}\gamma_{0,j+1} &= \left(A_{0,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{0,t,j}(e^{\alpha})^t + A_{0,s+1,j}(e^{\alpha})^{s+1}\right)' \\ &\quad + e^{\alpha}\left(A_{k-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{k-1,t,j}(e^{\alpha})^t + A_{k-1,s,j}(e^{\alpha})^s\right) \\ &= A_{0,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{0,t,j} + t\alpha' A_{0,t,j} + A_{k-1,t-1,j})(e^{\alpha})^t \\ &\quad + A_{0,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.29}$$

where $A_{0,1,j+1} = A'_{0,1,j} + A_{0,1,j}\alpha'$, $A_{0,s+1,j+1} = A'_{0,s+1,j} + (s+1)\alpha' A_{0,s+1,j} + A_{k-1,s,j}$, and for $1 \le i \le l$,

$$\begin{aligned}\gamma_{i,j+1} &= \gamma'_{i,j} + \gamma_{i-1,j} \\ &= \left(A_{i,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i,t,j}(e^{\alpha})^t + A_{i,s+1,j}(e^{\alpha})^{s+1}\right)' \\ &\quad + A_{i-1,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{i-1,t,j}(e^{\alpha})^t + A_{i-1,s+1,j}(e^{\alpha})^{s+1} \\ &= A_{i,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{i,t,j} + t\alpha' A_{i,t,j} + A_{i-1,t,j})(e^{\alpha})^t \\ &\quad + A_{i,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.30}$$

where $A_{i,1,j+1} = A'_{i,1,j} + \alpha' A_{i,1,j} + A_{i-1,1,j}$, $A_{i,s+1,j+1} = A'_{i,s+1,j} + (s+1)\alpha' A_{i,s+1,j} + A_{i-1,s+1,j}$, and for $i = l+1$,

$$\begin{aligned}\gamma_{l+1,j+1} &= \gamma'_{l+1,j} + \gamma_{l,j} \\ &= \left(A_{l+1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{l+1,t,j}(e^{\alpha})^t + A_{l+1,s,j}(e^{\alpha})^s\right)' \\ &\quad + A_{l,1,j}e^{\alpha} + \sum_{t=2}^{s} A_{l,t,j}(e^{\alpha})^t + A_{l,s+1,j}(e^{\alpha})^{s+1} \\ &= A_{l+1,1,j+1}e^{\alpha} + \sum_{t=2}^{s}(A'_{l+1,t,j} + t\alpha' A_{l+1,t,j} + A_{l,t,j})(e^{\alpha})^t \\ &\quad + A_{l+1,s+1,j+1}(e^{\alpha})^{s+1},\end{aligned} \tag{2.31}$$

where $A_{l+1,1,j+1} = A'_{l+1,1,j} + \alpha' A_{l+1,1,j} + A_{l,1,j}$, $A_{l+1,s+1,j+1} = A_{l,s+1,j}$, and for $l+2 \le i \le k-1$,

$$\gamma_{i,j+1} = \gamma'_{i,j} + \gamma_{i-1,j}$$

$$
\begin{aligned}
&= \Big(A_{i,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{i,t,j}(e^{\alpha})^t + A_{i,s,j}(e^{\alpha})^s\Big)' \\
&\quad + A_{i-1,1,j}e^{\alpha} + \sum_{t=2}^{s-1} A_{i-1,t,j}(e^{\alpha})^t + A_{i-1,s,j}(e^{\alpha})^s \\
&= A_{i,1,j+1}e^{\alpha} + \sum_{t=2}^{s-1}(A'_{i,t,j} + t\alpha' A_{i,t,j} + A_{i-1,t,j})(e^{\alpha})^t + A_{i,s,j+1}(e^{\alpha})^s,
\end{aligned} \tag{2.32}
$$

where $A_{i,1,j+1} = A'_{i,1,j} + \alpha' A_{i,1,j} + A_{i-1,1,j}$, $A_{i,s,j+1} = A'_{i,s,j} + s\alpha' A_{i,s,j} + A_{i-1,s,j}$.

By (2.28)–(2.32), we know that (2.16)–(2.19) are true for $j+1$ when $j = sk+l$ with $0 \le l \le k-2$. Similarly, we can prove (2.16)–(2.19) are true for $j+1$ when $j = sk+k-1$. We omit the details here. Thus Lemma 7 is proved.

Lemma 8. *Let*

$$
\Delta_j = \begin{pmatrix} \gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1} \\ \gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1} \end{pmatrix}, \tag{2.33}
$$

where $\gamma_{i,j}$ are entire functions defined in Lemmas 6–7 (for $1 \le j \le k-1$ and $i > j$ set $\gamma_{i,j} = 0$). Denote the determinant of Δ_j by $\det(\Delta_j)$. Then for $j = sk+l$ (≥ 1) with $s \ge 0$, $0 \le l \le k-1$, we have

$$
\begin{aligned}
\det(\Delta_j) &= \big((\alpha')^{kj} + P_{kj-1}[\alpha']\big)(e^{\alpha})^k \\
&\quad + \sum_{t=k+1}^{(s+1)k+l-1} A_{t,j}(e^{\alpha})^t + (-1)^{l(k-l)}(e^{\alpha})^{(s+1)k+l},
\end{aligned} \tag{2.34}
$$

where $A_{t,j} \in \mathscr{P}[\alpha']$.

Proof. Obviously, by Lemmas 6–7, we have

$$
\det(\Delta_j) = \sum_{t=k}^{\nu} A_{t,j}(e^{\alpha})^t \tag{2.35}
$$

with $\nu \ge k$ and $A_{t,j} \in \mathscr{P}[\alpha']$. Thus we need only to show that

$$
\nu = (s+1)k + l, \quad A_{\nu,j} = (-1)^{l(k-l)}, \tag{2.36}
$$

and

$$
A_{k,j} = (\alpha')^{kj} + P_{kj-1}[\alpha']. \tag{2.37}
$$

First we prove (2.36). By Lemmas 6–7, we have

$$M_1=\begin{pmatrix}\gamma_{0,j} & \gamma_{0,j+1} & \cdots & \gamma_{0,j+k-1-l}\\ \gamma_{1,j} & \gamma_{1,j+1} & \cdots & \gamma_{1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{l-1,j} & \gamma_{l-1,j+1} & \cdots & \gamma_{l-1,j+k-1-l}\end{pmatrix}_{l\times(k-l)}$$

$$=(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times(k-l)},$$

$$M_2=\begin{pmatrix}\gamma_{0,j+k-l} & \gamma_{0,j+k-l+1} & \cdots & \gamma_{0,j+k-1}\\ \gamma_{1,j+k-l} & \gamma_{1,j+k-l+1} & \cdots & \gamma_{1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{l-1,j+k-l} & \gamma_{l-1,j+k-l+1} & \cdots & \gamma_{l-1,j+k-1}\end{pmatrix}$$

$$=\begin{pmatrix}1 & A_{0,s+2,j+k-l+1} & \cdots & A_{0,s+2,j+k-1}\\ 0 & 1 & \cdots & A_{1,s+2,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}_{l\times l}(e^{\alpha})^{s+2}$$

$$+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times l}$$

$$=A_{l\times l}(e^{\alpha})^{s+2}+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s+1)_{l\times l},$$

$$M_3=\begin{pmatrix}\gamma_{l,j} & \gamma_{l,j+1} & \cdots & \gamma_{l,j+k-1-l}\\ \gamma_{l+1,j} & \gamma_{l+1,j+1} & \cdots & \gamma_{l+1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{k-1,j} & \gamma_{k-1,j+1} & \cdots & \gamma_{k-1,j+k-1-l}\end{pmatrix}_{(k-l)\times(k-l)}$$

$$=\begin{pmatrix}1 & A_{l,s+1,j+1} & \cdots & A_{l,s+1,j+k-1-l}\\ 0 & 1 & \cdots & A_{l+1,s+1,j+k-1-l}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}_{(k-l)\times(k-l)}(e^{\alpha})^{s+1}$$

$$+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s)_{(k-l)\times(k-l)}$$

$$=B_{(k-l)\times(k-l)}(e^{\alpha})^{s+1}+(\text{polynomials in } e^{\alpha} \text{ of degrees } \le s)_{(k-l)\times(k-l)},$$

$$M_4=\begin{pmatrix}\gamma_{l,j+k-l} & \gamma_{l,j+k-l+1} & \cdots & \gamma_{l,j+k-1}\\ \gamma_{l+1,j+k-l} & \gamma_{l+1,j+k-l+1} & \cdots & \gamma_{l+1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{k-1,j+k-l} & \gamma_{k-1,j+k-l+1} & \cdots & \gamma_{k-1,j+k-1}\end{pmatrix}_{(k-l)\times l}$$

$$=\begin{pmatrix}A_{l,s+1,j+k-l} & A_{l,s+1,j+k-l+1} & \cdots & A_{l,s+1,j+k-1}\\ A_{l+1,s+1,j+k-l} & A_{l+1,s+1,j+k-l+1} & \cdots & A_{l+1,s+1,j+k-1}\\ \vdots & \vdots & \ddots & \vdots\\ A_{k-1,s+1,j+k-l} & A_{k-1,s+1,j+k-l+1} & \cdots & A_{k-1,s+1,j+k-1}\end{pmatrix}_{(k-l)\times l}(e^{\alpha})^{s+1}$$

$$
\begin{aligned}
&\quad + (\text{polynomials in } e^{\alpha} \text{ of degrees } \leq s)_{(k-l)\times l}\\
&= C_{(k-l)\times l}(e^{\alpha})^{s+1} + (\text{polynomials in } e^{\alpha} \text{ of degrees } \leq s)_{(k-l)\times l},
\end{aligned}
$$

where $A_{l\times l}, B_{(k-l)\times(k-l)}, C_{(k-l)\times l}$ are matrices whose elements are differential polynomials in α'. In particular, $A_{l\times l}$ and $B_{(k-l)\times(k-l)}$ are upper triangular matrices whose principal diagonal elements equal 1. Thus by (2.33) we get

$$
\begin{aligned}
\det(\Delta_j) &= \begin{vmatrix} M_1 & M_2 \\ M_3 & M_4 \end{vmatrix}\\
&= \begin{vmatrix} 0 & A \\ B & C \end{vmatrix} (e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1)\\
&= (-1)^{l(k-l)} \det(A)\det(B)(e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1)\\
&= (-1)^{l(k-l)}(e^{\alpha})^{(s+1)k+l} + (\text{terms of degree} \leq (s+1)k+l-1),
\end{aligned}
$$

where $0 = 0_{l\times(k-l)}$ is the zero matrix, $A = A_{l\times l}$, $B = B_{(k-l)\times(k-l)}$, $C = C_{(k-l)\times l}$. This proves (2.36).

Next we prove (2.37). By Lemmas 6–7, we have

$$
\begin{aligned}
A_{k,j} &= \begin{vmatrix}
A_{0,1,j} & A_{0,1,j+1} & \cdots & A_{0,1,j+k-1}\\
A_{1,1,j} & A_{1,1,j+1} & \cdots & A_{1,1,j+k-1}\\
\vdots & \vdots & \ddots & \vdots\\
A_{k-1,1,j} & A_{k-1,1,j+1} & \cdots & A_{k-1,1,j+k-1}
\end{vmatrix} \qquad (2.38)\\
&= \begin{vmatrix}
1 & 1 & \cdots & 1\\
\binom{j}{1} & \binom{j+1}{1} & \cdots & \binom{j+k-1}{1}\\
\vdots & \vdots & \ddots & \vdots\\
\binom{j}{i} & \binom{j+1}{i} & \cdots & \binom{j+k-1}{i}\\
\vdots & \vdots & \ddots & \vdots\\
\binom{j}{k-1} & \binom{j+1}{k-1} & \cdots & \binom{j+k-1}{k-1}
\end{vmatrix} (\alpha')^{kj} + P_{kj-1}[\alpha'],
\end{aligned}
$$

where

$$\binom{j}{i} = \frac{j!}{i!(j-i)!}$$

are the binomial coefficients. Since

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}$$

is a polynomial in x of degree i, by the calculating properties of determinant and the well-known Vandermonde's determinant, we see that $A_{k,j} = C(\alpha')^{kj} + P_{kj-1}[\alpha']$, where C is a nonzero constant which is equal to

$$\prod_{s=1}^{k-1}\frac{1}{s!}\prod_{1\leq i<t\leq k}(t-i) = 1.$$

This proves (2.37). Thus Lemma 8 is proved.

3. Proof of Theorem 1

By the assumptions, there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that

$$\frac{f^{(k)}(z)-a}{f(z)-a}=e^{\alpha(z)}, \tag{3.1}$$

$$\frac{f^{(m)}(z)-a}{f(z)-a}=e^{\beta(z)}. \tag{3.2}$$

Next we consider two cases.

Case 1. Either α or β is a constant. Without loss of generality, we assume that α is a constant. Set $e^{\alpha}=c$. Then by (3.1), we get

$$f^{(k)}-cf=(1-c)a. \tag{3.3}$$

Solving (3.3), we get

$$f(z)=\left(1-\frac{1}{c}\right)a+\sum_{j=1}^{q}C_j e^{\lambda_j z}, \tag{3.4}$$

where q $(\le k)$ is a positive integer, and C_j, λ_j are nonzero constants satisfying $(\lambda_j)^k=c$ and $\lambda_i\ne\lambda_j$, $i\ne j$. By (3.4) and (3.2), it follows that $\varrho(e^{\beta})\le 1$, where ϱ (e^{β}) is the order of e^{β}, so that $e^{\beta}=de^{\mu z}$, where d $(\ne 0)$ and μ are constants. Thus by (3.2) and (3.4), we get

$$-a+\sum_{j=1}^{q}(\lambda_j)^m C_j e^{\lambda_j z}=-\frac{da}{c}e^{\mu z}+\sum_{j=1}^{q}C_j de^{(\lambda_j+\mu)z}. \tag{3.5}$$

Applying Lemma 5 to (3.5), we deduce that $\mu=0$ and $(\lambda_j)^m=d$. Further, if $a\ne 0$, then $c=d$.

By $(\lambda_j)^k=c$, $(\lambda_j)^m=d$ and the fact that λ_j, $1\le j\le q$, are distinct, we know that $q\le (k,m)$, where (k,m) is the greatest common divisor of k and m. In fact, by Euclidean division algorithm, there exist integers k_0 and m_0 such that $(k,m)=k_0k+m_0m$. Thus $(\lambda_j)^{(k,m)}=\big[(\lambda_j)^k]^{k_0}[(\lambda_j)^m\big]^{m_0}=c^{k_0}d^{m_0}$. Hence by the fact that λ_j, $1\le j\le q$, are distinct, it follows that $q\le (k,m)$.

Case 2. Both α and β are not constants.

We will prove that this case cannot occur. Without loss of generality, we assume $k<m$. Let

$$F(z)=f(z)-a. \tag{3.6}$$

Then by (3.1) and (3.2), we have

$$F^{(k)} = a + e^{\alpha} F, \tag{3.7}$$

$$F^{(m)} = a + e^{\beta} F. \tag{3.8}$$

Set

$$\phi = \frac{F^{(m)} - F^{(k)}}{F}. \tag{3.9}$$

Then by (3.7) and (3.8), we get

$$\phi = e^{\beta} - e^{\alpha}. \tag{3.10}$$

Next we consider two subcases.

Case 2.1: $\phi \equiv 0$. Then by (3.10), we get

$$e^{\beta} = e^{\alpha}. \tag{3.11}$$

Thus by (3.1), (3.2) and (3.11), we get

$$f^{(m)} - f^{(k)} = 0. \tag{3.12}$$

Solving (3.12), we get

$$f(z) = b(z) + \sum_{j=1}^{s} C_j e^{\lambda_j z}, \tag{3.13}$$

where b is a polynomial with $\deg b \le k-1$, $s \le m-k$ is a positive integer, and C_j, λ_j are nonzero constants with $(\lambda_j)^{m-k} = 1$ and $\lambda_i \ne \lambda_j$, $i \ne j$. By (3.1) and (3.13), we know that $\varrho(e^{\alpha}) \le 1$. This together with that α is nonconstant yields that $e^{\alpha} = Ce^{cz}$, where C and c are nonzero constants. Thus by (3.1) and (3.13), we get

$$-a + \sum_{j=1}^{s} C_j (\lambda_j)^k e^{\lambda_j z} = C[b(z) - a]e^{cz} + \sum_{j=1}^{s} CC_j e^{(\lambda_j + c)z}. \tag{3.14}$$

Applying Lemma 5 to (3.14), we get that $c = 0$, a contradiction.

Case 2.2: $\phi \not\equiv 0$. Then by the logarithmic derivative lemma, it follows from (3.9) that

$$m(r, \phi) = S(r, F). \tag{3.15}$$

By (3.10), ϕ is an entire function. Thus by (3.15), we get

$$T(r,\phi)=S(r,F). \tag{3.16}$$

Since $\phi\not\equiv 0$, by (3.10), we get

$$\frac{e^{\beta}}{\phi}=1+\frac{e^{\alpha}}{\phi}. \tag{3.17}$$

Thus by (3.16), (3.17) and the second fundamental theorem we deduce that

$$\begin{aligned} T\left(r,\frac{e^{\beta}}{\phi}\right) &\le \overline{N}\left(r,\frac{e^{\beta}}{\phi}\right)+\overline{N}\left(r,\frac{\phi}{e^{\beta}}\right)+\overline{N}\left(r,\frac{1}{\frac{e^{\beta}}{\phi}-1}\right)+S\left(r,\frac{e^{\beta}}{\phi}\right)\\ &\le \overline{N}\left(r,\frac{e^{\beta}}{\phi}\right)+\overline{N}\left(r,\frac{\phi}{e^{\beta}}\right)+\overline{N}\left(r,\frac{1}{\frac{e^{\alpha}}{\phi}}\right)+S\left(r,\frac{e^{\beta}}{\phi}\right)\\ &\le S(r,F)+S\left(r,\frac{e^{\beta}}{\phi}\right). \end{aligned} \tag{3.18}$$

This together with (3.16) yields that $T(r,e^{\beta})=S(r,F)$. It follows from (3.10) and (3.16) that $T(r,e^{\alpha})=T(r,e^{\beta}-\phi)=S(r,F)$. Thus we get

$$T(r,e^{\alpha})+T(r,e^{\beta})=S(r,F). \tag{3.19}$$

Now, for $0\le j\le k-1$, set

$$p_{i,j}=\gamma_{i,m-k+j},\quad i=-1,0,1,\dots,k-1, \tag{3.20}$$

where $\gamma_{i,j}$ are defined as in Lemmas 6–8. Then by Lemmas 6–7, we have

$$\begin{aligned} F^{(m+j)} &= F^{(k+m-k+j)}\\ &= p_{-1,j}+p_{0,j}F+p_{1,j}F'+\cdots+p_{k-1,j}F^{(k-1)},\quad j=0,1,\dots,k-1. \end{aligned} \tag{3.21}$$

On the other hand, by (3.8) and Lemma 6, for $1\le j\le k-1$, we have

$$F^{(m+j)}=q_{0,j}e^{\beta}F+q_{1,j}e^{\beta}F'+\cdots+q_{j,j}e^{\beta}F^{(j)}, \tag{3.22}$$

where $q_{i,j}$, $i\le j$, are differential polynomials in β' with constant coefficients. In particular, $q_{j,j}\equiv 1$ for $j=1,2,\dots,k-1$. Thus by (3.8), (3.21) and (3.22), we get

$$(\,F,F',\dots,F^{(k-1)}\,)\,(e^{\beta}Q-P)=\Gamma, \tag{3.23}$$

where

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} & \cdots & p_{0,k-1} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ p_{k-1,0} & p_{k-1,1} & \cdots & p_{k-1,k-1} \end{pmatrix}, \tag{3.24}$$

$$Q = \begin{pmatrix} 1 & q_{0,1} & \cdots & q_{0,k-1} \\ 0 & 1 & \cdots & q_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3.25}$$

$$\Gamma = (\, p_{-1,0} - a, p_{-1,1}, \ldots, p_{-1,k-1}\,)\,. \tag{3.26}$$

By (3.23) and the theory of linear equations, we get

$$\det(e^{\beta}Q - P)F = \det(T), \tag{3.27}$$

where T is a matrix whose first line is Γ and the other lines are the same as those of $e^{\beta}Q - P$.

Thus by (3.19) and (3.27), we know that

$$\det(e^{\beta}Q - P) = 0. \tag{3.28}$$

This yields that

$$\det(e^{\beta}I - R) = 0, \tag{3.29}$$

where $I = I_{k\times k}$ is the kth unit matrix, $R = Q^{-1}P$ and Q^{-1} is the inverse matrix of Q. Obviously, the matrix Q^{-1} is also an upper triangular matrix whose elements are differential polynomial in β'. By (3.29), we get

$$(e^{\beta})^k - a_1(e^{\beta})^{k-1} + \cdots + (-1)^t a_t(e^{\beta})^{k-t} + \cdots + (-1)^k a_k = 0, \tag{3.30}$$

where a_t is the sum of all the principle minors of order t of R. In particular, $a_k = \det(R) = \det(P)$. Here, for a matrix

$$A = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

and t integers $1 \leq i_1 < i_2 < \cdots < i_t \leq n$, we call

$$\begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_t} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_t} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_t,i_1} & a_{i_t,i_1} & \cdots & a_{i_t,i_t} \end{vmatrix}$$

a principle minor of order t of A.

Obviously, by (3.24), (3.25) and the definition of a_t, a_t, $1 \leq t \leq k$, are polynomials in e^α whose coefficients are differential polynomials in α' and β' with constant coefficients.

Next we consider the degrees of these polynomials a_t. Since $m > k$, there exist integers $s \geq 1$ and $0 \leq l \leq k-1$ such that

$$m = sk + l. \tag{3.31}$$

It is obvious that if $l = 0$ then $s > 1$. We claim that for $l \geq 1$,

$$\deg(a_t) \leq ts + l - 1, \quad t = 1, 2, \ldots, k-1, \tag{3.32}$$

and for $l = 0$,

$$\deg(a_t) \leq ts, \quad t = 1, 2, \ldots, k-1. \tag{3.33}$$

and

$$\deg(a_k) = m = ks + l. \tag{3.34}$$

In order to prove (3.32)–(3.34), we first consider the degree of the elements of $R = (r_{i,j})$ which are polynomials in e^α. By (3.20), we see that for $0 \leq j \leq k-1-l$, $p_{i,j} = \gamma_{i,(s-1)k+j+l}$, while for $k-l \leq j \leq k-1$, $p_{i,j} = \gamma_{i,sk+j+l-k}$. Thus by Lemmas 6–7, for $0 \leq i, j \leq k-1$,

$$\deg(p_{i,j}) \leq \begin{cases} s & \text{if } 0 \leq j \leq k-1-l,\ 0 \leq i \leq j+l, \\ s-1 & \text{if } 0 \leq j \leq k-1-l,\ j+l+1 \leq i \leq k-1, \\ s+1 & \text{if } k-l \leq j \leq k-1,\ 0 \leq i \leq j+l-k, \\ s & \text{if } k-l \leq j \leq k-1,\ j+l-k+1 \leq i \leq k-1. \end{cases} \tag{3.35}$$

By (3.25), we may assume that

$$Q^{-1} = \begin{pmatrix} 1 & q^*_{0,1} & \cdots & q^*_{0,k-1} \\ 0 & 1 & \cdots & q^*_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{3.36}$$

where $q^*_{i,j}$, $0 \le i < j \le k-1$, are differential polynomials in β' with constant coefficients. Thus by $(r_{i,j}) = R = Q^{-1}P$, we get

$$r_{i,j} = p_{i,j} + q^*_{i,i+1}p_{i+1,j} + q^*_{i,i+2}p_{i+2,j} + \cdots + q^*_{i,k-1}p_{k-1,j}. \tag{3.37}$$

Thus by (3.35) and (3.37), we see that for $0 \le i, j \le k-1$,

$$\deg(r_{i,j}) \le \begin{cases} s & \text{if } 0 \le j \le k-1-l,\ 0 \le i \le j+l, \\ s-1 & \text{if } 0 \le j \le k-1-l,\ j+l+1 \le i \le k-1, \\ s+1 & \text{if } k-l \le j \le k-1,\ 0 \le i \le j+l-k, \\ s & \text{if } k-l \le j \le k-1,\ j+l-k+1 \le i \le k-1. \end{cases} \tag{3.38}$$

Now let

$$L_{i_1,i_2,\ldots,i_t} = \begin{vmatrix} r_{i_1,i_1} & r_{i_1,i_2} & \cdots & r_{i_1,i_t} \\ r_{i_2,i_1} & r_{i_2,i_2} & \cdots & r_{i_2,i_t} \\ \vdots & \vdots & \ddots & \vdots \\ r_{i_t,i_1} & r_{i_t,i_2} & \cdots & r_{i_t,i_t} \end{vmatrix}$$

be a principle minor of order $t \le k-1$ of R, where $0 \le i_1 < i_2 < \cdots < i_t \le k-1$.

By (3.38), for the case of $l = 0$, the degrees of all $r_{i,j}$ are at most s, so that the degree of $L_{i_1,i_2,\ldots,i_t}$ is at most ts. It follows that the degree of a_t is at most ts. This proves (3.33).

Next we consider the case of $1 \le l \le k-1$. By the definition of determinant, we have

$$L_{i_1,i_2,\ldots,i_t} = \sum \delta_{j_1,j_2,\ldots,j_t} r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t},$$

where the sum takes over all the permutations of $(i_1, i_2, \ldots, i_t)$, and $\delta_{j_1,j_2,\ldots,j_t} = \pm 1$ according to the permutation $(j_1, j_2, \ldots, j_t)$ of $(i_1, i_2, \ldots, i_t)$ is even or odd. Let

$$L_t = r_{i_1,j_1} r_{i_2,j_2} \cdots r_{i_t,j_t}.$$

For $t \le l-1$, by (3.38), the degree of L_t is at most $t(s+1) \le ts + l - 1$.

For $t \ge l$, if there exist $x \le l-1$ polynomials in $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ with degree $s+1$, then by (3.38), the degree of L_t is at most $x(s+1)+(t-x)s = ts+x \le ts+l-1$. If there exist l polynomials in $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ with degree $s+1$, then by (3.38), $\{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\}$. It follows that there exists at least one of $r_{i_1,j_1}, r_{i_2,j_2}, \ldots, r_{i_t,j_t}$ whose degree is $s-1$ (for otherwise, we must have $\{l, \ldots, k-1\} \subset \{i_1, i_2, \ldots, i_t\}$. This together with $\{0, 1, \ldots, l-1\} \subset \{i_1, i_2, \ldots, i_t\}$ yields that $t \ge k$, which contradicts $t \le k-1$). Hence the degree of L_t is at most $l(s+1)+(s-1)+(t-l-1)s = ts+l-1$. It follows that $\deg(L_{i_1,i_2,\ldots,i_t}) \le ts+l-1$. Thus (3.32) is proved.

Next we prove (3.34). In fact, it can be seen from (3.20), (3.24) and Lemma 8 that

$$\begin{aligned} a_k &= \det(P) = \det(\Delta_{m-k}) \\ &= \big((\alpha')^{k(m-k)} + P_{k(m-k)-1}[\alpha']\big)(e^\alpha)^k \\ &\quad + \sum_{t=k+1}^{m-1} A_{t,m-k}(e^\alpha)^t + (-1)^{l(k-l)}(e^\alpha)^m. \end{aligned} \tag{3.39}$$

Thus we get (3.34).

By (3.32)–(3.34), we see that for $1 \le t \le k-1$, $\deg(a_t) < \deg(a_k)$. Thus by (3.30) and Lemma 2, it follows that

$$T(r, e^{\beta}) = O\big(T(r, e^{\alpha})\big) + S(r, e^{\beta}),$$
$$T(r, e^{\alpha}) = O\big(T(r, e^{\beta})\big) + S(r, e^{\alpha}).$$

Hence we get

$$S(r, e^{\alpha}) = S(r, e^{\beta}) = S(r) \quad \text{(say)}. \tag{3.40}$$

Next we prove the following claims.

Claim I. For any rational number $\theta = \nu/\mu$ with $\nu \in \mathbf{Z}$ and $\mu \in \mathbf{N}$,

$$T(r, e^{\beta - \theta\alpha}) \ne S(r). \tag{3.41}$$

Suppose on the contrary that there exists a rational number $\theta = \nu/\mu$ such that

$$T(r, e^{\beta - \theta\alpha}) = S(r). \tag{3.42}$$

Let

$$b(z) = e^{\beta - \theta\alpha}. \tag{3.43}$$

Then $b(z) \ne 0$ is entire and $T(r, b) = S(r)$. By (3.43),

$$e^{\beta} = b(z)e^{\theta\alpha} = b(z)(e^{\alpha/\mu})^{\nu}. \tag{3.44}$$

On the other hand, by (3.20), (3.24) and Lemmas 6–7, we have

$$P = (e^{\alpha})^{s+1}P_0 + (e^{\alpha})^{s}P_1 + \cdots + (e^{\alpha})P_s, \tag{3.45}$$

where P_j are $k \times k$ matrices whose elements are differential polynomials in α'. In particular, $\det(P_s) \equiv A_{k,m-k}$, where $A_{k,m-k}$ is defined by (2.38). By (3.28), (3.44) and (3.45), we get

$$\det\big(b(e^{\alpha/\mu})^{\nu}Q - (e^{\alpha})^{s+1}P_0 - (e^{\alpha})^{s}P_1 - \cdots - (e^{\alpha})P_s\big) = 0. \tag{3.46}$$

If $\nu > \mu$, then by (3.46), we get

$$\det\big(b(e^{\alpha/\mu})^{\nu-\mu}Q - (e^{\alpha/\mu})^{s\mu}P_0 - \cdots - (e^{\alpha/\mu})^{\mu}P_{s-1} - P_s\big) = 0. \tag{3.47}$$

Since the left side of (3.47) is a polynomial in $e^{\alpha/\mu}$ whose "constant" term is $\det(-P_s) = (-1)^k A_{k,m-k}$, by Lemma 3, we get $A_{k,m-k} = 0$. Thus by (2.10), (2.19) and the fact that α is nonconstant, α' is nonconstant. For otherwise, let $\alpha' = c$. Then $c \neq 0$, and by (2.10), (2.19), we have

$$A_{i,1,j} = \binom{j}{i}(c)^{j-i},$$

so that by (2.38), it follows that $A_{k,m-k} = (c)^{k(m-k)} \neq 0$, which contradicts $A_{k,m-k} = 0$. Hence α' is nonconstant. Thus by (2.37) and Lemma 1, we deduce that $T(r,\alpha') = m(r,\alpha') = S(r,\alpha')$, a contradiction.

If $\nu < \mu$, then by (3.46), we get

$$\det\big(bQ - (e^{\alpha/\mu})^{(s+1)\mu-\nu}P_0 - (e^{\alpha/\mu})^{s\mu-\nu}P_1 - \cdots - (e^{\alpha/\mu})^{\mu-\nu}P_s\big) = 0.$$

Using the same argument as that in case $\nu > \mu$, we deduce that $\det(bQ) = 0$. Thus by $\det(Q) = 1$, we get that $b = 0$, a contradiction.

If $\nu = \mu$, then $e^\beta = b(z)e^\alpha$. Thus by (3.32)–(3.34), we see that the left side of (3.30) is a polynomial in e^α whose leading term is $\varepsilon(e^\alpha)^m$, where $\varepsilon = \pm 1$ is a constant. Thus applying Lemma 2 to (3.30), we get a contradiction: $T(r,e^\alpha) = S(r)$.

Hence Claim I is proved.

Claim II. We have

$$H = \sum_{t=1}^{k-1}(-1)^t a_t (e^\beta)^{k-t} \equiv 0. \tag{3.48}$$

Suppose that $H \not\equiv 0$. Then by the fact that a_t are polynomials in e^α, we can rewrite H as

$$H = \sum_{(t,i)\in T\times I} a_{t,i}e^{(k-t)\beta + i\alpha}, \tag{3.49}$$

where $T \subset \{1,\ldots,k-1\}$ and I are finite index sets, $a_{t,i} \not\equiv 0$ are differential polynomials in α' and β' such that all the functions $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i) \in T\times I$ are linearly independent.

By (3.39), we rewrite a_k as

$$(-1)^k a_k = \sum_{i\in J} a_{k,i}e^{i\alpha}, \tag{3.50}$$

where $J \supset \{m\}$ is a finite index set, and $a_{k,i}$ ($\not\equiv 0$), $i \in J$, are differential polynomials in α'.

Hence by (3.30), (3.48)–(3.50), we get

$$e^{k\beta} + \sum_{(t,i)\in T\times I} a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J} a_{k,i}e^{i\alpha} = 0. \tag{3.51}$$

By (3.51), we get

$$\sum_{(t,i)\in T\times I}(-a_{t,i})e^{-t\beta+i\alpha}+\sum_{i\in J}(-a_{k,i})e^{-k\beta+i\alpha}=1. \tag{3.52}$$

If the functions $(-a_{t,i})e^{-t\beta+i\alpha}$, $(t,i)\in T\times I$ and $(-a_{k,i})e^{-k\beta+i\alpha}$, $i\in J$ are linearly independent, then by Lemma 4 and the fact that $m\in J$, we get

$$T\big(r,(-a_{k,m})e^{-k\beta+m\alpha}\big)=S(r),$$

so that

$$T(r,e^{-k\beta+m\alpha})=S(r),$$

which contradicts Claim I.

Hence the functions $(-a_{t,i})e^{-t\beta+i\alpha}$, $(t,i)\in T\times I$ and $(-a_{k,i})e^{-k\beta+i\alpha}$, $i\in J$ are linearly dependent. That is, there exist constants $C_{t,i},(t,i)\in T\times I$ and $C_{k,i}$, $i\in J$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T\times I}C_{t,i}a_{t,i}e^{-t\beta+i\alpha}+\sum_{i\in J}C_{k,i}a_{k,i}e^{-k\beta+i\alpha}=0,$$

so that

$$\sum_{(t,i)\in T\times I}C_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha}+\sum_{i\in J}C_{k,i}a_{k,i}e^{i\alpha}=0. \tag{3.53}$$

By Lemma 3, at least one of $C_{t,i},(t,i)\in T\times I$ is not equal to 0. Set $T_1\times I_1=\big\{(t,i)\in T\times I : C_{t,i}\neq 0\big\}$. Then $T_1\times I_1\neq\emptyset$ (empty set). By the assumption that $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i)\in T\times I$ are linearly independent, at least one of $C_{k,i}, i\in J$ is not equal to 0. Set $J_1=\big\{i\in J : C_{k,i}\neq 0\big\}$. Then $J_1\neq\emptyset$. Let $i_1\in J_1$. Then by (3.53), we get

$$\sum_{(t,i)\in T_1\times I_1}\frac{-C_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}}e^{(k-t)\beta+(i-i_1)\alpha}+\sum_{i\in J_1\setminus\{i_1\}}\frac{-C_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}}e^{(i-i_1)\alpha}=1.$$

If the functions

$$\begin{aligned}&\frac{-C_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}}e^{(k-t)\beta+(i-i_1)\alpha},(t,i)\in T_1\times I_1\quad\text{and}\\&\frac{-C_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}}e^{(i-i_1)\alpha},i\in J_1\setminus\{i_1\}\end{aligned} \tag{3.54}$$

are linearly independent, then by Lemma 4, we get for $(t_0,i_0)\in T_1\times I_1$,

$$T\left(r,\frac{C_{t_0,i_0}a_{t_0,i_0}}{C_{k,i_1}a_{k,i_1}}e^{(k-t_0)\beta+(i_0-i_1)\alpha}\right)=S(r),$$

so that

$$T\big(r, e^{(k-t_0)\beta+(i_0-i_1)\alpha}\big) = S(r),$$

which again contradicts Claim I. Thus the functions showed in (3.54) are linearly dependent. Thus there exist constants $D_{t,i}, (t,i) \in T_1 \times I_1$ and $D_{k,i}, i \in J_1 \setminus \{i_1\}$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T_1\times I_1} \frac{D_{t,i}a_{t,i}}{C_{k,i_1}a_{k,i_1}} e^{(k-t)\beta+(i-i_1)\alpha} + \sum_{i\in J_1\setminus\{i_1\}} \frac{D_{k,i}a_{k,i}}{C_{k,i_1}a_{k,i_1}} e^{(i-i_1)\alpha} = 0,$$

so that

$$\sum_{(t,i)\in T_1\times I_1} D_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J_1\setminus\{i_1\}} D_{k,i}a_{k,i}e^{i\alpha} = 0. \tag{3.55}$$

By Lemma 3, we see that at least one of $D_{t,i}, (t,i) \in T\times I$ is not equal to 0, so that $T_2 \times I_2 = \big\{(t,i) \in T_1 \times I_1 : D_{t,i} \neq 0\big\} \neq \emptyset$. By the assumption that $a_{t,i}e^{(k-t)\beta+i\alpha}$, $(t,i) \in T\times I$ are linearly independent, at least one of $D_{k,i}, i \in J\setminus\{i_1\}$ is not equal to 0, so that $J_2 = \big\{i \in J_1 \setminus \{i_1\} : D_{k,i} \neq 0\big\} \neq \emptyset$. Let $i_2 \in J_2$. Then using an argument similar to that in the above step, there exist constants $E_{t,i}, (t,i) \in T_2\times I_2$ and $E_{k,i}, i \in J_2 \setminus \{i_2\}$, at least one of them is not equal to 0, such that

$$\sum_{(t,i)\in T_2\times I_2} E_{t,i}a_{t,i}e^{(k-t)\beta+i\alpha} + \sum_{i\in J_2\setminus\{i_2\}} E_{k,i}a_{k,i}e^{i\alpha} = 0.$$

Step by step, it follows that J is an infinite set. It is impossible. Hence we have proved Claim II.

Next we continue to prove Theorem 1. By (3.30), (3.50) and Claim II, we get

$$e^{k\beta} + \sum_{i\in J} a_{k,i}e^{i\alpha} = 0,$$

so that

$$\sum_{i\in J} (-a_{k,i})e^{i\alpha-k\beta} = 1. \tag{3.56}$$

If the functions $(-a_{k,i})e^{i\alpha-k\beta}$, $i \in J$, are linearly independent, then by Lemma 3, we get

$$T(r, a_{k,m}e^{m\alpha-k\beta}) = S(r),$$

so that

$$T(r, e^{m\alpha-k\beta}) = S(r).$$

This contradicts Claim I.

Hence the functions $(-a_{k,i})e^{i\alpha-k\beta}$, $i\in J$, are linearly dependent. Thus there exist constants C_i, $i\in J$, at least one of them is not equal to 0, such that

$$\sum_{i\in J} C_i a_{k,i} e^{i\alpha-k\beta}=0,$$

so that

$$\sum_{i\in J} C_i a_{k,i} e^{i\alpha}=0.$$

This contradicts Lemma 3.

The proof of Theorem 1 is complete.

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