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BANACH SPACES WHOSE BOUNDED SETS ARE BOUNDING IN THE BIDUAL

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Abstract. We discuss necessary conditions for a Banach space to satisfy the property that its bounded sets are bounding in its bidual space. Apart from the classic case of c_0, we prove that, among others, the direct sum $c_0(l_2^n)$ is another example of spaces having such property.

A subset B of a complex Banach space X is said to be *bounding* if every entire function defined on X is bounded on B. Such a set is also a limited set (see below) and, of course, every relatively compact set is bounding. If X is such that every bounding subset is relatively compact, then X is called a *Gelfand–Phillips* space and there is some literature devoted to the topic. Gelfand–Phillips spaces are characterized also as those whose sequences converging against entire functions are norm convergent [7]. B. Josefson [12] and T. Schlumprecht [15] found simultaneously and independently examples of complex Banach spaces containing limited non-bounding (hence non-relatively compact) sets.

By $H(X)$ we denote the space of entire functions defined on X, that is, functions that are Fréchet differentiable at every point of X. $H_b(X)$ is the subspace of all $f \in H(X)$ such that f is bounded on bounded sets; these are the so-called entire functions of bounded type.

Bounding sets are related as well to the extension of holomorphic functions to a larger space in the following way: *Let $E \subset F$ be complex Banach spaces. Every bounded subset of E is bounding in F if, and only if, every $f \in H(E)$ having a holomorphic continuation to F is of bounded type.* In the particular instance of $F = E^{**}$ it is known that every $f \in H_b(E)$ has a holomorphic extension to E^{**}. Therefore, in this case the only holomorphic functions which may be extended to

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the bidual are those of bounded type if, and only if, every bounded set in E is bounding in E^{**}.

As far as we know there is essentially one known case ($E = c_0$) where the above mentioned (equivalent) conditions hold. This was proved among other results by B. Josefson in his 1978 paper [11]. Clearly, no reflexive infinite-dimensional Banach space satisfies those conditions and neither the conditions hold in general for quasi-reflexive Banach spaces, i.e., spaces of finite codimension in its bidual, the conditions may not hold: Consider, for instance, the Q-reflexive (see [1]) and quasi-reflexive Tsirelson–James space T_J^*. Its bidual space is separable and hence there the bounding sets are relatively compact (see [5, 4.26]). So the bounded sets in T_J^* are not bounding in $(T_J^*)^{**}$.

Our aim is to study this situation and to provide further examples of Banach spaces whose bounded sets are bounding in the bidual. So we consider the class $\mathscr{F}$ of Banach spaces

$$\mathscr{F} = \{F : \text{every bounded set } B \subset F \text{ is a bounding set in } F^{**}\}.$$

We show that such a class $\mathscr{F}$ is stable under products and quotients, but not under closed subspaces; however the closed subspaces of c_0 do belong to the class. In the second part of the article we prove that $c_0(l_2^n)$ the predual of C. Stegall's example of a Schur space whose dual lacks the Dunford–Pettis property [16] is a Banach space whose bounded sets are bounding in the bidual. Our argument does not rely on Josefson's paper and so provides another proof of the fact that c_0 is in the class we are dealing with.

1. Generalities

A subset B of a complex Banach space X is bounding if, and only if, $\lim_m \|P_m\|_B^{1/m} = 0$ for all entire functions $f = \sum_m P_m$, where $\sum_m P_m$ represents the Taylor series of $f \in H(X)$ at the origin ([6, Lemma 4.50]). We will use this characterization in Section 2.

A subset A of a Banach space is called *limited* if weak* null sequences in the dual space converge uniformly on A. The unit ball of an infinite-dimensional Banach space is not a limited set by the Josefson–Nissenzweig theorem. An operator $T\colon X \to Y$ is said to be limited if it maps the unit ball of X onto a limited set in Y. It turns out that T is limited if, and only if, T^* is weak*-to-norm sequentially continuous. As we mentioned, each bounding set is a limited one, therefore if $E \in \mathscr{F}$, then the canonical inclusion $E \to E^{**}$ is a limited operator.

Recall that a Banach space X has the property (P) of Pełczyński if for every $K \subset X^*$ that is not weakly relatively compact, there exists a weakly Cauchy series $\sum x_n$ in X such that $\inf_n \|x_n\|_K > 0$. Every $C(K)$ space has the (P) property (see [17, III.D.33]).

We refer to [5] for infinite-dimensional holomorphy background.

Proposition 1.1. *If the canonical inclusion $E \to E^{**}$ is a limited operator, then E^* is a Schur space. Conversely, if E^* is a Schur space and* (i) *E^{**} is a Grothendieck space or* (ii) *E has the (P) property, then the canonical embedding of E into E^{**} is limited.*

Proof. If the canonical inclusion $E \to E^{**}$ is a limited operator, then the natural projection $\pi_{E^*}\colon E^{***} \to E^*$ is weak*-to-norm sequentially continuous. Therefore the identity on E^* as a composition of the canonical embedding in E^{***} and the natural projection is weak-to-norm sequentially continuous, i.e., E^* is a Schur space. For the converse statement (i), note that $\pi_{E^*}\colon E^{***} \to E^*$ is weak-weak sequentially continuous, hence weak*-to-norm sequentially continuous by the assumptions on E^* and E^{**}. And for the (ii) case, $\pi_{E^*}\colon E^{***} \to E^*$ is weak*-to-weak sequentially continuous by [10, Proposition 3.6, p. 131]. ▫

Note. *If the canonical inclusion $E \to E^{**}$ is a limited operator, then $H_b(E) = H_{wu}(E)$.* This is so because then E^* is a Schur space, hence E has the Dunford–Pettis property and does not contain copies of l_1.

In the sequel, we aim at studying whether important classes of Banach spaces belong to the class $\mathscr{F}$.

Firstly, we observe that for no (isomorphic) infinite-dimensional dual space X^*, the canonical inclusion $X^* \to X^{***}$ can be limited, since otherwise by composing it with the restriction to X operator, $\varrho\colon X^{***} \to X^*$, it would turn out that the identity on X^* would be a limited operator. Therefore, no dual space belongs to $\mathscr{F}$.

When does a $C(K)$-space belong to $\mathscr{F}$? The above proposition shows that in this setting of Banach spaces of continuous functions the condition of E^* being a Schur space is equivalent to the bounded sets in E being bounding in the bidual: Indeed, if $C(K)^*$ is a Schur space, then $C(K)$ does not contain a copy of l_1, hence by [13] its bidual is isomorphic to $l_\infty(I)$ for some set I. But then $C(K)^{**}$ is a Grothendieck space, so the bounded sets in $C(K)$ are limited in $C(K)^{**}$ and furthermore bounding there because of [11, Theorem 1].

Since $c_0 \in \mathscr{F}$ and it is an M-ideal in l^∞ one wonders whether there are any connections between $\mathscr{F}$ and the class of Banach spaces E which are an M-ideal in E^{**}. We are to see that there is no a clear connection.

On one hand, for the Lorentz sequence space $d(w,1)$ ([10, p. 103]), its predual $E = d_*(w,1)$ is known to be an M-ideal in its bidual ([10, III 1.4]). However, the unit basis of $d(w,1)$ is weakly null, so $d(w,1)$ is not a Schur space. Hence, it turns out that the canonical inclusion $E \to E^{**}$ is not a limited operator. Thus, not every bounded set in E is bounding in E^{**}. Another (simpler) example is the following: According to [10, III Examples 1.4(f) and (g)], for $1 < p < \infty$, the space $K(l_p)$ of compact linear operators of l_p, is an M-ideal in its bidual. However $K(l_p)^*$ is not a Schur space since $K(l_p)$ does not have the Dunford–Pettis property since it contains a complemented copy of the reflexive space l_p.

On the other hand, the condition of E being an M-ideal in E^{**} is not necessary for their bounded sets being bounding in E^{**}. The Bourgain–Delbaen Banach space Y—introduced in [2]—is shown there to satisfy $Y^* \approx l_1$ and not to contain any copy of c_0. Thus $Y^{**} \approx l_\infty$ is a Grothendieck space and Proposition 1.1 applies. Further, again by Josefson's result the limited sets in Y^{**} are bounding sets, so it turns out that the bounded sets in Y are bounding in Y^{**}. However, Y is not an M-ideal in its bidual, since if it were, then it would be isomorphic to c_0 as a consequence of [10, III Theorem 3.11]. We also mention in passing that Y fails the (P) property.

Corollary 1.2. *If the Banach space E is an M-ideal in its bidual, then the canonical inclusion $E \to E^{**}$ is a limited operator if, and only if, E^* is a Schur space.*

Proof. Since any Banach space which is an M-ideal in its bidual has the (P) property ([10, III Theorem 3.4]), it suffices to apply Proposition 1.1. ▫

Is the class $\mathscr{F}$ stable by closed subspaces? The Bourgain–Delbaen space Y also shows that $\mathscr{F}$ is not stable by closed subspaces since, as Haydon [9] proved, Y contains a subspace isomorphic to some l_p, which clearly does not belong to $\mathscr{F}$.

Recall ([10, III Theorem 1.6]) that every subspace M of $K(l_p)$ is also an M-ideal in its bidual and those subspaces which embed into c_0 satisfy that M^* are Schur spaces ([14, p. 415]). Moreover, every closed subspace E of c_0 is also an M-ideal in its bidual, has the Dunford–Pettis property and does not contain any copy of l_1, hence E has the (P) property and E^* is a Schur space. Therefore, the canonical inclusion $E \to E^{**}$ is a limited operator. So the arising question is: Does every closed subspace of c_0 have its bounded sets bounding in the bidual? Actually this question has a positive answer which reduces to Josefson's result that $c_0 \in \mathscr{F}$.

Proposition 1.3. *The bounded sets in every closed subspace E of c_0 are bounding in E^{**}.*

Proof. To begin with, note that since E has the DP property and does not contain any copy of l_1, $H_b(E) = H_{\mathrm{wu}}(E)$, the space of entire functions on E which are weakly uniformly continuous on bounded sets. Moreover E^* is a separable space, hence by Corollary 12 in [3], $H_{\mathrm{wu}}(E) = H_{\mathrm{wsc}}(E)$, the space of weakly sequentially continuous entire functions on E.

Let f be entire on E with extension, $\bar{f}$, to E^{**}. If $f \notin H_b(E)$, f is not weakly sequentially continuous, thus there is a sequence $(x_n) \subset E$ weakly convergent to $a \in E$ such that $\big(f(x_n)\big)$ does not converge to $f(a)$. By considering the function $g(x) = f(x + a)$ we may assume that $a = 0$. Clearly, (x_n) cannot be a norm null sequence, hence by the Bessaga–Pełczyński selection principle (see [4, Chapter V, p. 46]) there is a subsequence of (x_n) which is a basic sequence equivalent to a block basic sequence taken with respect to the unit vector basis of c_0, and further such subsequence spans a Banach subspace X of E which is complemented in c_0 along a projection p (see [8, Proposition 249]). There is no loss of generality assuming that the sequence (x_n) is such a subsequence. Now $\bar{f}|_{X^{**}}$ is an extension to X^{**} of the function $f|_X$, and therefore, $\bar{f}|_{X^{**}} \circ p^{**}$ is an extension of $f|_X \circ p$ to l_∞. Further, since $c_0 \in \mathscr{F}$, it follows that $f|_X \circ p$ belongs to $H_b(c_0)$, so it has to be a weakly sequentially continuous function and this is prevented by the choice of $(x_n) \subset X$. □

We end this section with another positive result.

Proposition 1.4. *$\mathscr{F}$ is stable by products and quotients.*

Proof. (a) Let $E, F \in \mathscr{F}$. Let $f \in H(E^{**} \times F^{**})$ and consider a bounded set in $E \times F$ which we may suppose to be $A \times B$ with $A \subset E$ and $B \subset F$ both bounded sets. We check that the collection $\{f(x, \cdot)\}_{x \in A} \subset H(F^{**})$ is τ_0 bounded: Indeed, for any compact subset K of F^{**}, the collection $\{f(\cdot, y)\}_{y \in K} \subset \big(H(E^{**}), \tau_0\big)$ is bounded, hence is τ_δ bounded ([5, 2.44, 2.46]). In addition, since A is bounding in E^{**}, $\|f\|_A = \sup\{|f(x)| : x \in A\}$ defines a τ_δ continuous seminorm in $H(E^{**})$ by [5, 4.18], so we have

$$\sup_{x \in A} \sup_{y \in K} |f(x, y)| = \sup_{y \in K} \sup_{x \in A} |f(x, y)| = \sup_{y \in K} \|f(\cdot, y)\|_A < \infty,$$

as we wanted. Now, since B is bounding in F^{**}, we have that $\{f(x, \cdot)\}_{x \in A}$ is bounded for the $\|\cdot\|_B$ seminorm, hence $\{|f(x, y)|\}_{x \in A, y \in B}$ is bounded and so $A \times B$ is shown to be bounding in $E^{**} \times F^{**}$.

(b) If F is a quotient of a Banach space $E \in \mathscr{F}$, then F belongs to $\mathscr{F}$ as well: Let $q \colon E \to F$ be the quotient mapping and $f \in H(F)$ with a holomorphic extension g to F^{**}. Then $g \circ q^{tt}$ is a holomorphic function in E^{**} which extends $f \circ q$. Therefore, $f \circ q \in H_b(E)$ and since any bounded subset $A \subset F$ is contained in $q(B)$ for some bounded set B, it follows that f is bounded in A. □

2. The example $c_0(l_2^n)$

In this section we consider $X = c_0(l_2^n)$. X^* enjoys the Schur property, yet its dual fails the Dunford–Pettis property, and it was the first example of a space with the Dunford–Pettis property whose dual lacks it, a fact discovered by C. Stegall [16]. Thus in view of Proposition 1.1 we found it suitable to explore whether X belongs to $\mathscr{F}$.

We now set some notation. Consider a partition of $\mathbf{N}$ by defining for each $n \in \mathbf{N}$ the interval

$$I_n = \left[\frac{n(n-1)}{2} + 1, \frac{n(n+1)}{2}\right] \cap \mathbf{N}.$$

Stegall's example is

$$X^* := \left\{(x_i) \in l^\infty : \sum_{n=1}^{\infty} \left(\sum_{i \in I_n} |x_i|^2\right)^{1/2} < \infty\right\}.$$

The canonical predual of X^* is the space

$$X := \left\{ (x_i) \in l^\infty : \lim_{k\to\infty} \sup_{n\geq k} \left(\sum_{i\in I_n} |x_i|^2 \right)^{1/2} = 0 \right\}.$$

Observe that X coincides with the closed linear span in X^{**} of the unit vectors $\{e_j = (\delta_{nj})_{n\in\mathbf{N}},\ j \in \mathbf{N}\}$.

The dual of X^* can be represented by

$$X^{**} = \left\{ (x_i) \in l^\infty : \sup_{n\in\mathbf{N}} \left(\sum_{i\in I_n} |x_i|^2 \right)^{1/2} < \infty \right\}.$$

Let S be a subset of $\mathbf{N}$. A sequence of complex numbers $x = (x_i)$ is said to have *support* in S if $x_i = 0$ for each $i \notin S$. If, moreover, x belongs to a sequence space E, then we write $x \in E_S$.

$B(E)$ denotes the closed unit ball of E whatever the normed space E be.

For $f\colon E \to \mathbf{C}$ we denote $\|f\|_S = \sup_{x\in B(E_S)} |f(x)|$.

Let $(p_i)_{i\in\mathbf{N}}$ be a strictly increasing sequence of natural numbers. For each $i \in \mathbf{N}$, let $\alpha_{p_i+1}, \ldots, \alpha_{p_{i+1}}$ be scalars at least one of which is nonzero, and let $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ be a block sequence taken from $(e_i)_{i\in\mathbf{N}}$. The sequence $(u_i)_{i\in\mathbf{N}}$ is called *totally disjoint* with respect to $(I_n)_{n\in\mathbf{N}}$ whenever $p_i \in I_{n_i}$, $p_{i+1} \in I_{n'_i}$ and $n_i \neq n'_i$.

We remark that a block sequence $(u_i)_{i\in\mathbf{N}}$ may be written as $u_i = \sum_{j\in U_i} \alpha_j e_j$, where $U_i = \bigcup_{j\in F_i} I_j$ and F_i is a finite subset of $\mathbf{N}$. Further, if $p_i \in I_s$, $p_{i+1} \in I_r$ such that $I_r \cap I_s = \emptyset$ we can choose the subsets F_i such that $\max F_i < \min F_{i+1}$ and in this case $(u_i)_{i\in\mathbf{N}}$ is a totally disjoint sequence.

In the sequel we consider a sequence $(U_n)_{n\in\mathbf{N}}$ of subsets of $\mathbf{N}$ such that for each n

(i) $U_n = \bigcup_{j\in F_n} I_j$, where F_n is a finite subset of $\mathbf{N}$ and

(ii) $\max F_n < \min F_{n+1}$.

Condition (ii) implies $U_n \cap U_m = \emptyset$ for each $m \neq n$.

It is not difficult to show that if the sequence $(u_i)_{i\in\mathbf{N}}$, $u_i = \sum_{j\in U_i} \alpha_j e_j$, is totally disjoint and $\sup_{i\in\mathbf{N}} \|u_i\| \leq C$ for some positive constant C, then the partial sums of the series $\sum \theta_i u_i$ are bounded by $C \cdot \sup_i |\theta_i|$ and so the formal series $\sum \theta_i u_i$ defines an element in X^{**}.

To prove the next theorem we will make use of the following result which extends a similar one from Dineen ([6, p. 299]) valid for l_∞.

Lemma 2.1. *Let Q be a continuous polynomial in X^{**} such that $Q(0) = 0$. Let $N' \subset \mathbf{N}$ be an infinite set. Then for each $\varepsilon > 0$ there is an infinite subset $S \subset N'$ such that*

$$\|Q\|_{\cup_{n\in S} U_n} < \varepsilon.$$

Proof. First we consider a homogeneous polynomial P. Suppose that there is an $\varepsilon > 0$ such that for each infinite subset $S \subset N'$ we have

$$\|P\|_{\cup_{n\in S} U_n} > \varepsilon.$$

Let $(S_i)_i$ be a disjoint partition of N' into an infinite number of infinite sets. Then

$$\|P\|_{\cup_{n\in S_i} U_n} > \varepsilon \quad \text{for all } i \in \mathbf{N}.$$

Thus, given $i \in \mathbf{N}$ there is an x_i in $X^{**}_{\cup_{n\in S_i} U_n}$ with $\|x_i\| \le 1$ such that $|P(x_i)| > \varepsilon$.

Now, by Lemma 1.9 of [6] we have that

$$\sup_{|\theta_i|=1} \left| P\left(\sum_{i=1}^{l} \theta_i x_i\right)\right|^2 \ge \sum_{i=1}^{l} |P(x_i)|^2 > l\varepsilon^2.$$

Since the sequence (x_i) has pairwise disjoint support, it follows that for $|\theta_i| = 1$, $i = 1, \dots, l$ and any $l \in \mathbf{N}$, the combination $\sum_{n=1}^{l} \theta_i x_i \in B(X^{**})$. So, for all $l \in \mathbf{N}$ we have

$$\|P\|^2 \ge \sup_{|\theta_i|=1} \left| P\left(\sum_{i=1}^{l} \theta_i x_i\right)\right|^2 \ge \sum_{i=1}^{l} |P(x_i)|^2 > l\varepsilon^2.$$

This leads to a contradiction when we let $l \to \infty$.

Finally, let Q be an arbitrary continuous polynomial with $Q(0) = 0$. Then there are homogeneous continuous polynomials P_j such that $Q = \sum_{j=0}^{m} P_j$ and, by the above, we may find inductively infinite subsets $S_j \subset S_{j-1} \subset N'$, $j = 1, \dots, m$ such that $\|P_j\|_{\cup_{n\in S_j} U_n} < \varepsilon/m$. Consequently, $\|Q\|_{\cup_{n\in S_m} U_n} < \varepsilon$. ▫

Theorem 2.2. *Let $(u_i)_{i\in\mathbf{N}}$ be a totally disjoint sequence in $B(X^{**})$ given by $u_i = \sum_{j\in U_i} \alpha_j e_j$ for each i. Then $A = \{u_i : i \in \mathbf{N}\}$ is a bounding set in X^{**}.*

Proof. Suppose the set A is not bounding in X^{**}. Then there exists an entire function $f = \sum_m P_m \in H(X^{**})$ such that f is not bounded in A. By Lemma 4.50 in [6] this implies the existence of $\delta > 0$ and strictly increasing subsequences $(m_n) \subset \mathbf{N}$, $(\gamma_n) \subset \mathbf{N}$ such that

$$|P_{\gamma_n}(u_{m_n})|^{1/\gamma_n} > \delta \quad \text{for all } n \in \mathbf{N}.$$

We will show that this inequality cannot hold by using an inductive argument.

Given the polynomial $y \in X^{**} \mapsto P_{\gamma_1}(u_{m_1} + y) - P_{\gamma_1}(u_{m_1})$, Lemma 2.1 shows that there is an infinite subset $S_1 \subset S_0 := \{m_n : n \in \mathbf{N}\}$ such that

$$|P_{\gamma_1}(u_{m_1} + y) - P_{\gamma_1}(u_{m_1})| \le \left(\frac{\delta}{\gamma_1 !}\right)^{\gamma_1} \quad \text{for all } y \in B(X^{**}_{\cup_{n\in S_1} U_n}).$$

Since S_1 is an infinite subset of S_0 there is $m_j \in S_1$ with $j > 1$. Without loss of generality we may suppose that $j = 2$. Thus $u_{m_2} \in B(X^{**}_{\cup_{n\in S_1} U_n})$. Put $\theta_1 = 1$. Now, by Lemma 1.9(b) in [6], there is $\theta_2 \in \mathbf{C}$ with $|\theta_2| = 1$ such that

$$|P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})|^2 \geq |P_{\gamma_2}(\theta_1 u_{m_1})|^2 + |P_{\gamma_2}(u_{m_2})|^2 \geq \delta^{2\gamma_2}.$$

The next step is to consider the polynomial $y \in X^{**} \mapsto P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y) - P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})$. Again Lemma 2.1 shows that there is an infinite subset $S_2 \subset S_1$, such that

$$|P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y) - P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})| \leq \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2}$$

for all $y \in B(X^{**}_{\cup_{n\in S_2} U_n})$. Hence for all $y \in B(X^{**}_{\cup_{n\in S_2} U_n})$,

$$\begin{aligned}|P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2} + y)| &\geq |P_{\gamma_2}(\theta_1 u_{m_1} + \theta_2 u_{m_2})| - \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2} \\ &\geq \delta^{\gamma_2} - \left(\frac{\delta}{\gamma_2!}\right)^{\gamma_2} \geq \left(\frac{\delta}{2}\right)^{\gamma_2}.\end{aligned}$$

Arguing by induction we obtain a decreasing sequence (S_j) of infinite subsets of $\mathbf{N}$, and a sequence $(\theta)_i$, $\theta_1 = 1$, in the unit sphere of $\mathbf{C}$ such that

$$\left|P_{\gamma_j}\left(\sum_{k=1}^{j} \theta_k u_{m_k} + y\right)\right| \geq \left(\frac{\delta}{2}\right)^{\gamma_j} \quad \text{for all } y \in B(X^{**}_{\cup_{n\in S_j} U_n}).$$

We set $a = (a_l) \in X^{**}$ defined by $a_l = \theta_k \alpha_l e_l$ if $l \in U_{m_k}$ for some $k \in \mathbf{N}$, and $a_l = 0$ otherwise. That is, a is the formal series $\sum_{k=1}^{\infty} \theta_k u_{m_k}$.

Since the sets U_k are disjoint, the vectors u_k have disjoint supports, so $a \in B(X^{**})$ and also $a - \sum_{k=1}^{j} \theta_k u_{m_k} \in B(X^{**}_{\cup_{n\in S_j} U_n})$. Then from the latter inequality we conclude

$$|P_{\gamma_j}(a)|^{1/\gamma_j} = \left|P_{\gamma_j}\left(\sum_{k=1}^{j} \theta_k u_{m_k} + \left(a - \sum_{k=1}^{j} \theta_k u_{m_k}\right)\right)\right|^{1/\gamma_j} \geq \frac{\delta}{2}.$$

Since $\{a\}$ is a compact set in X^{**}, it follows that $\lim_{j\to\infty} |P_{\gamma_j}(a)|^{1/\gamma_j} = 0$. A contradiction. □

To prove the next theorem we need the following lemma.

Lemma 2.3. *Let $(u_i)_{i\in\mathbf{N}}$ be a totally disjoint bounded sequence given by $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ for each $i \in \mathbf{N}$. Then there exists a projection $Q\colon X \to \overline{\operatorname{span}[u_i]}$.*

Proof. Let $(u_i')_{i\in\mathbf{N}}$ be a sequence in X^* such that $\|u_i'\| = 1, u_i'u_i = 1$ for each $i \in \mathbf{N}$ and $u_i'(e_j) = 0$ for $j \notin \{p_i + 1, \ldots, p_{i+1}\}$.

If we define $Q(x) = \sum_{i=1}^{\infty} u_i'(x)u_i$ for each $x = (x_j) \in X$, since $u_i'(x) = u_i'\big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\big)$, we have that

$$|u_i'(x)| \leq \Big\|\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\Big\| \leq 2\Lambda\Big\|\sum_{i=1}^{\infty}\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\Big\| \leq 2\Lambda\|x\|,$$

where Λ is the basis constant for $(e_n)_{n\in\mathbf{N}}$. Then

$$\Big\|\sum_{i=1}^{\infty} u_i'(x)u_i\Big\| = \Big\|\sum_{i=1}^{\infty} u_i'\Big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\Big)u_i\Big\| \leq \sup_i\Big|u_i'\Big(\sum_{j=p_i+1}^{p_{i+1}} x_j e_j\Big)\Big| \leq 2\Lambda\|x\|.$$

So Q is a continuous projection. ▫

Theorem 2.4. *Every bounded subset of X is bounding in X^{**}.*

Proof. Notice that X^* is separable. So, by a result of [3], it is sufficient to prove that every weakly compact subset of X is bounding in X^{**}. So by Eberlein's theorem, it suffices to show that every weakly convergent sequence $(x_i) \subset X$ is a bounding set in X^{**}. We may assume that (x_i) is weakly null since bounding sets are also bounding after translation.

Suppose that (x_i) is not a bounding subset in X^{**}. Then, there is an entire function $f = \sum_{n\in\mathbf{N}} P_n \in H(X^{**})$ which is unbounded on the set $\{x_i : i \in \mathbf{N}\}$. Thus, by Lemma 4.50 in [6] there exist a subsequence of (x_i) (which we are going to denote in the same way), a subsequence (P_{γ_i}), and $\delta > 0$ such that

$$|P_{\gamma_i}(x_i)|^{1/\gamma_i} > \delta.$$

Since (x_i) is a weakly null non null sequence, we find by the Bessaga–Pełczynski selection principle, a subsequence (x_{k_i}) equivalent to a basic block sequence taken from (e_i). That is, there is a strictly increasing sequence $(p_i) \subset \mathbf{N}$ such that $(x_{k_i}) \approx (u_i)$, where $u_i = \sum_{j=p_i+1}^{p_{i+1}} \alpha_j e_j$ for $i \in \mathbf{N}$. Since (p_i) is strictly increasing we may assume, passing to subsequences if necessary, that (u_i) is totally disjoint, and henceforth a bounding set in X^{**}.

Since (x_{k_i}) is equivalent to (u_i), there exists an isomorphism T from $[u_i,\ i \in \mathbf{N}]$ onto $[x_{k_i},\ i \in \mathbf{N}]$ such that $T(u_i) = x_{k_i}$ for each $i \in \mathbf{N}$. Let $T^{**}\colon [u_i,\ i \in \mathbf{N}]^{**} \longrightarrow [x_{k_i},\ i \in \mathbf{N}]^{**}$ be the double transpose of T and let $Q^{**}\colon X^{**} \longrightarrow [u_i,\ i \in \mathbf{N}]^{**}$ be the double transpose of the projection defined in Lemma 2.3. Since $f \in H(X^{**})$ we have that $f \circ T^{**} \circ Q^{**} \in H(X^{**})$ and $f \circ T^{**} \circ Q^{**}|_{[u_i]}$ coincides with $f \circ T \circ Q$.

Finally, since $\{u_i : i \in \mathbf{N}\}$ is a bounding subset in X^{**} by Theorem 2.2, we get that $\lim_i |P_{\gamma_{k_i}} \circ T^{**} \circ Q^{**}(u_i)|^{1/\gamma_{k_i}} = 0$ for each $i \in \mathbf{N}$, but $\lim_i |P_{\gamma_{k_i}} \circ T^{**} \circ Q^{**}(u_i)|^{1/\gamma_{k_i}} = \lim_i |P_{\gamma_{k_i}} \circ T \circ Q(u_i)|^{1/\gamma_{k_i}} = \lim_i |P_{\gamma_{k_i}}(x_{k_i})|^{1/\gamma_{k_i}}$. So, $\lim_i |P_{\gamma_{k_i}}(x_{k_i})|^{1/\gamma_{k_i}} = 0$. This contradicts our assumption and completes the proof. ▫

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