Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 31, 2006, 13–30

EXTENSION OF INTERNALLY BILIPSCHITZ MAPS IN JOHN DISKS

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Abstract. A map is said to be internally bilipschitz if it is bilipschitz with respect to internal distances. We prove the following extension theorem for John disks: any internally bilipschitz map of the boundary ∂D of a Jordan John disk D extends as an internally bilipschitz map of $\overline{D}$. This result is a partial analogue to a result of Gehring for quasidisks.

1. Introduction

A domain $D \subset \mathbf{R}^2$ is called a c-*John domain* if any two points z_1, z_2 in D may be joined by a rectifiable arc $\alpha \subset D$ such that

$$\min_{j=1,2} \operatorname{length}(\alpha[z_j, z]) \le c \operatorname{dist}(z, \partial D). \tag{1.1}$$

See [9, Section 2.26]. Here, as in the rest of this paper, $\alpha[w_1, w_2]$ denotes the subarc of α from w_1 to w_2. If we are not interested in a particular constant, we will say simply that D is a John domain. A simply connected John domain with at least two boundary points will be called a *John disk*. It is an important and very useful fact that in a John disk we may always, possibly by changing the constant c, use hyperbolic geodesics γ in the definition (1.1); [5, Theorem 4.1], [9, Theorem 5.2].

John disks enjoy many properties similar to those characteristic for *quasidisks*, i.e. images of disks or half planes under quasiconformal self maps of $\overline{\mathbf{R}^2}$. Indeed, John disks may be regarded as one sided quasidisks. See for example [9, Section 9], [10, Theorem 5.9]. They appear in many contexts in analysis.

In this paper we shall concern ourselves with a certain extension problem for John disks. A result of Gehring says that if $D \subset \mathbf{R}^2$ is a quasidisk, then every bilipschitz map of ∂D extends as a bilipschitz map of $\overline{D}$. See [2, Theorem 7]. Gehring also gives a partial converse to this in [3, Lemma 4.4]: A Jordan curve $C \subset \mathbf{R}^2$ is a quasicircle if every bilipschitz map of C extends as a bilipschitz map of $\overline{D}$, where D is a component of $\overline{\mathbf{R}^2} \setminus C$. A homeomorphism f of some subset $A \subset \mathbf{R}^2$ onto another is L-*bilipschitz* if

$$\frac{1}{L}|z_1 - z_2| \le |f(z_1) - f(z_2)| \le L|z_1 - z_2|$$

2000 Mathematics Subject Classification: Primary 30C20; Secondary 30C62.

for all $z_1, z_2 \in A$. In Section 4 of the present paper we prove a partial analogue of Gehring's result for John disks, namely that if D is a Jordan John disk, and if f is an internally bilipschitz map of ∂D, then f extends as an internally bilipschitz map of $\overline{D}$. Cf. Theorem 4.7 below. By an *internally bilipschitz* map f we mean a map which is bilipschitz with respect to internal distances, i.e.

$$\frac{1}{L}\lambda_D(z_1, z_2) \le \lambda_{D'}\big(f(z_1), f(z_2)\big) \le L\lambda_D(z_1, z_2).$$

See Section 2 for the definition of internal distance, λ_D. At present we are only able to prove that this extension property is necessary for Jordan John disks.

Inside a domain D the classes of internally bilipschitz and euclidean locally bilipschitz homeomorphisms are the same, [13, p. 303]. In Section 4 we give examples to show that this is not the case when we consider homeomorphisms of the boundary of a domain. Thus our main result is not included in the results of papers like [3], [6].

We have to clarify what we mean by "internal distance on the boundary". This is the subject of Section 2. In Section 3 we prove a new version of Ahlfors' three point property for John disks, a result which may be interesting in its own right. In particular it is useful for certain quasisymmetry considerations in Section 4.

Our notation is almost self explanatory. $B(z, r)$ denotes the open disk with center z of radius r and $S(z, r)$ denotes its boundary. We reserve $\mathbf{D}$ for the unit disk with boundary $\mathbf{T}$, while $\mathbf{H}$ is the upper half plane. D will always denote a *proper* subdomain of $\mathbf{R}^2$. Whenever D is simply connected, we will use γ for hyperbolic segments. For the euclidean line segment from z_1 to z_2 we write $[z_1, z_2]$. Diameters and lengths are denoted by diam and length, respectively.

2. Internal distances

Let $D \subset \mathbf{R}^2$ be any domain. We define the *internal distance* between z_1 and z_2 in D as

$$\lambda_D(z_1, z_2) = \inf_{\alpha} \operatorname{length}(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha \subset D$ joining z_1 and z_2; cf. [9, Section 3.1]. The internal distance is well suited for studying John disks, since John domains are allowed to have inward cusps. In this paper we will be concerned with boundary conditions, and therefore we would like to measure internal distances between boundary points. This leads to some problems.

Firstly, there may be no rectifiable arc joining two arbitrary boundary points. A point z in the boundary ∂D of D is said to be *rectifiably accessible* if there is a half open rectifiable arc α in D ending at z, which means that there is a path $\varphi\colon [a, b) \to D$, $\alpha = \varphi\big([a, b)\big)$, with $\lim_{t\to b} \varphi(t) = z$ and $\lim_{\tau\to b} \operatorname{length}\big(\varphi([a, \tau])\big) =$

$l < \infty$, $\tau < b$. We let $\partial_r D$ denote the subset of ∂D which consists of all the rectifiably accessbile points, that is

$$\partial_r D = \{z \in \partial D : \text{is rectifiably accessible}\}.$$

Fortunately, rectifiably accessible points are plentiful in the boundary of any domain.

2.1. Lemma. *If D is any domain, then $\partial_r D$ is a dense subset of ∂D.*

See pp. 162–163 in [8]. In fact, in a simply connected domain a boundary point is rectifiably accessible if and only if it is accessible by a rectifiable, hyperbolic ray; see [10, Exercise 4.5.3]. This is an immediate consequence of the following result, known as the Gehring–Hayman theorem, which will be used repeatedly throughout the paper. Here, and elsewhere, we will write

$$D_r = D \cup \partial_r D.$$

2.2. Lemma. *Let D be a simply connected domain in $\mathbf{R}^2$, and let $z_1, z_2 \in D_r$. If γ is the hyperbolic segment from z_1 to z_2 in D and α is any rectifiable Jordan arc from z_1 to z_2 in D, we have that*

$$\operatorname{length}(\gamma) \leq K \operatorname{length}(\alpha),$$

$$\operatorname{diam}(\gamma) \leq K \operatorname{diam}(\alpha),$$

where K is a universal constant.

For a proof see [10, Theorem 4.20]. We will use the letter K for the Gehring–Hayman constant throughout. Another way to state this theorem is to say that $\operatorname{length}(\gamma[z_1, z_2]) \leq K\lambda_D(z_1, z_2)$, or that hyperbolic segments are *quasigeodesics* for the metric λ_D. Note also that in John disks (as well as many other kinds of domains) all finite boundary points are rectifiably accessible, [9, Remark 6.6]. We will use this fact without further reference throughout.

Secondly, we feel it is necessary to show that the internal distance λ_D is in fact a metric. This is readily proved when all points lie inside the domain in question. When we allow points to lie on the boundary, however, the proof of the triangle inequality is no longer entirely trivial. In fact, the triangle inequality may not hold for λ_D if D is not a Jordan domain. (Consider a disk with a radius removed. Let z_3 be the point exactly in the join of the radius with the circle, and let z_1 and z_2 be points on the circle close to z_3, but on opposite sides of the slit. Then $\lambda_D(z_1, z_2) > \lambda_D(z_1, z_3) + \lambda_D(z_2, z_3)$.)

We will now give a proof that λ_D is a metric in D_r if D is a Jordan domain. Recall that a *cross cut* in a domain D is an arc with both end points in ∂D.

2.3. Lemma. *For every Jordan domain D, λ_D is a metric in D_r.*

Proof. By definition $\lambda_D(z_1, z_2) < \infty$ for all pairs $z_1, z_2 \in \partial_r D$. The only non-obvious property is the triangle inequality, which we prove next.

Take two arbitrary points $z_1, z_2 \in \partial_r D$, and let z_3 be any other point in $\partial_r D$. For every $\varepsilon > 0$ there exist arcs β_1 and β_2 from z_1 and z_2 to z_3, respectively, with

$$\text{length}(\beta_j) < \lambda_D(z_j, z_3) + \varepsilon, \quad j = 1, 2.$$

We will show that there is an arc β_ε from z_1 to z_2 such that

$$\text{length}(\beta_\varepsilon) \leq \text{length}(\beta_1) + \text{length}(\beta_2) + M\varepsilon,$$

for some fixed constant M. To this end we may assume that β_1 and β_2 never meet in D, for if they *do* meet, we obtain an arc from z_1 to z_2 in D with the desired properties. For the rest of the proof, see Figure 1.

Let α denote that part of $\partial D \backslash \{z_1, z_2\}$ which does not contain z_3. Then $C = \beta_1 \cup \beta_2 \cup \alpha$ is a Jordan curve. Denote by $\widetilde{D}$ the component of $\overline{\mathbf{R}^2} \backslash C$ which lies inside D. If ε is sufficiently small we may assume that $S = \partial B(z_3, \varepsilon)$ meets $\widetilde{D}$ but not α. Here we use the fact that a Jordan curve is locally connected. Furthermore there is a unique component U of $\widetilde{D} \backslash S$ which has z_3 as a boundary point. To see this, assume that $\widetilde{D} = \mathbf{D}$ by applying a preliminary homeomorphism of $\mathbf{R}^2$ first if necessary; now S will not necessarily be a circle anymore. We may do this because D and hence $\widetilde{D}$ is a Jordan domain. Then any sufficiently small circular neighbourhood around $z_3 \in \mathbf{T}$ in $\mathbf{R}^2 \backslash S$ will meet only one component of $\mathbf{D} \backslash S$.

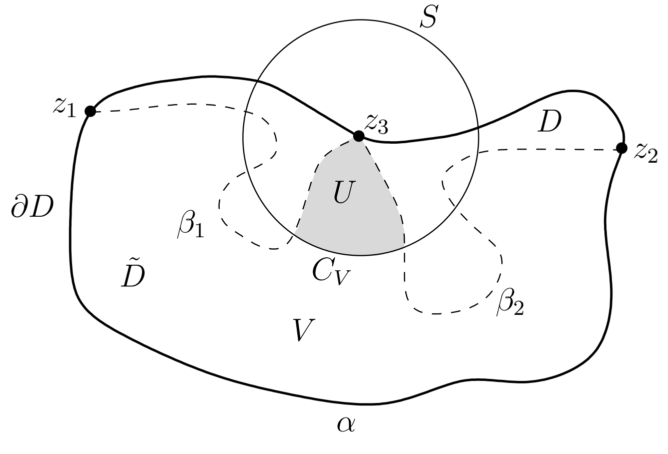

Figure 1.

By Proposition 2.13 in [10] there are countably many cross cuts $C_k \subset S$ of $\widetilde{D}$ such that

$$\widetilde{D} = U \cup \Big(\bigcup_k D_k\Big) \cup \Big(\bigcup_k C_k\Big),$$

where D_k are disjoint domains with

$$C_k = \widetilde{D} \cap \partial D_k.$$

Let V be that component D_k which has α on the boundary. Then $C_V = \widetilde{D} \cap \partial V \cap \partial U \subset S$ is a cross cut in $\widetilde{D}$, and one of the C_k. V must be a Jordan domain because it is a residual component of a cross cut of the Jordan domain $\widetilde{D}$ (e.g. [8, Theorem V.11.8]). Furthermore $(\partial\widetilde{D}\backslash C_V) \subset \beta_1 \cup \beta_2 \cup \alpha$. Now C_V is a cross cut in $\widetilde{D}$ that separates z_3 from z_1 and z_2. Hence $\overline{C}_V$ meets β_1 and β_2.

We obtain the arc β_ε by going along β_1 from z_1 until it meets C_V, then along C_V to β_2 and to z_2 along β_2. Since $C_V \subset S$,

$$\operatorname{length}(\beta_\varepsilon) < \operatorname{length}(\beta_1) + \operatorname{length}(\beta_2) + 2\pi\varepsilon.$$

This implies that

$$\lambda_D(z_1, z_2) < \operatorname{length}(\beta_\varepsilon) < \lambda_D(z_1, z_3) + \lambda_D(z_2, z_3) + (2\pi + 2)\varepsilon,$$

and the result follows upon letting $\varepsilon \to 0$. Similar arguments dispense with the cases when one or two of the points z_1, z_2, z_3 lie in $\partial_r D$. □

3. The three point property

In this section we will prove a version of the three point property for John disks. We start by quoting a result of Gehring, Hag, and Martio [5, Theorem 4.4]; it was proved independently by Näkki and Väisälä [9, Theorem 4.5].

3.1. Lemma ([5], [9]). *Suppose that D is a simply connected domain in $\mathbf{R}^2$. Then D is a c-John disk if and only if there exists a constant a such that for each cross cut α of D,*

$$\min_{j=1,2} \operatorname{diam} D_j \le a \operatorname{diam} \alpha,$$

where D_1 and D_2 are the two components of $D\backslash\alpha$. The constants a and c depend only on each other.

3.2. Lemma. *Suppose D is a Jordan domain in $\mathbf{R}^2$. Then D is a c-John disk if and only if there exists a constant b such that for every pair of points $z_1, z_2 \in \partial_r D$*

$$\min_{j=1,2} \operatorname{diam} \beta_j \le b\lambda_D(z_1, z_2), \tag{3.3}$$

where β_j are the components of $\partial D\backslash\{z_1, z_2\}$. The constants b and c depend only on each other.

Proof. Suppose that D is a c-John disk. Then $\partial_r D = \partial D$. Choose a cross cut α from z_1 to z_2. We know from Lemma 3.1 that

$$\min_{j=1,2} \operatorname{diam} D_j \leq a \operatorname{diam} \alpha,$$

$a = a(c)$, and trivially

$$\operatorname{diam} \beta_j \leq \operatorname{diam} D_j$$

for $j = 1, 2$. Let $\varepsilon > 0$. We may choose α such that $\operatorname{length}(\alpha) < \lambda_D(z_1, z_2) + \varepsilon$. Trivially we have that $\operatorname{diam} \alpha \leq \operatorname{length}(\alpha)$. It follows that

$$\min_{j=1,2} \operatorname{diam} \beta_j \leq \min_{j=1,2} \operatorname{diam} D_j \leq a \operatorname{length}(\alpha) \leq a\bigl(\lambda_D(z_1, z_2) + \varepsilon\bigr).$$

Since ε is arbitrary we conclude:

$$\min_{j=1,2} \operatorname{diam} \beta_j \leq a\lambda_D(z_1, z_2).$$

So we may let $b = a$ in (3.3).

Now assume that (3.3) holds for every pair of points z_1, z_2 in $\partial_r D$. Take a *straight* (*rectilinear* in Pommerenke's, [10], terminology) cross cut $[z_1, z_2]$ of D. Then $z_1, z_2 \in \partial_r D$. Let β_1 be the component of $\partial D \backslash \{z_1, z_2\}$ of least diameter. We have $\operatorname{diam} D_1 \leq \operatorname{diam} \beta_1$. According to (3.3)

$$\min_{j=1,2} \operatorname{diam} D_j \leq \operatorname{diam} D_1 \leq b\lambda_D(z_1, z_2) = b|z_1 - z_2|.$$

Thus the condition in Lemma 3.1 holds for every straight cross cut, and by [9, Theorem 4.5] D is a c-John disk with $c = c(b)$. □

3.4. Lemma. *Let D be a c-John disk. Then there is a constant $M = M(c)$ such that for any hyperbolic geodesic segment γ joining two points in $\overline{D}$ we have*

$$\operatorname{length}(\gamma) \leq M \operatorname{diam} \gamma.$$

Thus by Lemma 2.2 the internal length distance $\lambda_D(z_1, z_2)$ is comparable to the *internal diameter distance* defined by

$$\delta_D(z_1, z_2) = \inf\{\operatorname{diam} \alpha : \alpha \text{ arc from } z_1 \text{ to } z_2\}.$$

This has been proved earlier in [6, Theorem 5.14], although it is stated only for D, and not $\overline{D}$. Väisälä later showed that $\lambda_D(z_1, z_2) \leq L\, \delta_D(z_1, z_2)$ holds in all *airy* domains in $\mathbf{R}^n$. See [15, Lemma 3.3, Theorem 3.4]. For completeness we include a straightforward proof of Lemma 3.4 for John disks.

Proof of Lemma 3.4. By 2.25 in [9] we could have used diameters instead of lengths in (1.1), by changing the constants. Thus if D is a c-John disk, there are $c_1 = c_1(c)$ and $c_2 = c_2(c)$ such that

$$\min_{j=1,2} \operatorname{diam} \gamma[z_j, z] \le c_1 \operatorname{dist}(z, \partial D) \quad \text{and} \quad \min_{j=1,2} \operatorname{length}(\gamma[z_j, z]) \le c_2 \operatorname{dist}(z, \partial D),$$

for the hyperbolic geodesic γ of finite euclidean length jointing any two points $z_1, z_2 \in \overline{D}$. Let z_0 be the mid point of γ with respect to arc length. Then

$$\operatorname{length}(\gamma[z_1, z_0]) = \operatorname{length}(\gamma[z_2, z_0]) \le c_2 \operatorname{dist}(z_0, \partial D).$$

Suppose first that we have

$$|z_1 - z_0| \ge \operatorname{dist}(z_0, \partial D), \quad \text{or} \quad |z_2 - z_0| \ge \operatorname{dist}(z_0, \partial D)$$

or both. Without loss of generality assume that $|z_1 - z_0| \ge \operatorname{dist}(z_0, \partial D)$. Then $\operatorname{diam} \gamma[z_1, z_0] \ge \operatorname{dist}(z_0, \partial D)$, and by our assumption

$$\operatorname{diam} \gamma[z_1, z_0] \ge \operatorname{length}(\gamma[z_1, z_0])/c_2.$$

Therefore we obtain:

$$\operatorname{length}(\gamma) = 2 \operatorname{length}(\gamma[z_1, z_0]) \le 2c_2 \operatorname{diam} \gamma[z_1, z_0] \le 2c_2 \operatorname{diam} \gamma.$$

Now assume that

$$|z_0 - z_1| < \operatorname{dist}(z_0, \partial D) \quad \text{and} \quad |z_0 - z_2| < \operatorname{dist}(z_0, \partial D).$$

We have: $\operatorname{length}([z_1, z_0]) = \operatorname{diam}[z_0, z_1] = |z_0 - z_1|$ and

$$\operatorname{length}([z_2, z_0]) = \operatorname{diam}[z_0, z_2] = |z_0 - z_2|.$$

Lemma 2.2 gives

$$\begin{aligned} \operatorname{length}(\gamma[z_1, z_0]) &\le K \operatorname{length}([z_1, z_0]) = K|z_1 - z_0|, \\ \operatorname{length}(\gamma[z_2, z_0]) &\le K \operatorname{length}([z_2, z_0]) = K|z_2 - z_0| \end{aligned}$$

while

$$\begin{aligned} \operatorname{diam} \gamma[z_1, z_0] &\ge |z_1 - z_0|, \\ \operatorname{diam} \gamma[z_2, z_0] &\ge |z_2 - z_0|. \end{aligned}$$

Then we have

$$\operatorname{diam} \gamma \ge \operatorname{diam} \gamma[z_1, z_0] \ge \frac{1}{K} \operatorname{length}(\gamma[z_1, z_0]) = \frac{1}{2K} \operatorname{length}(\gamma[z_1, z_2]).$$

So choose $M = \max\{2c_2, 2K\}$. $\square$

In general, even if D is a bounded (in the euclidean metric) Jordan domain, (D_r, λ_D) does not have to be a bounded metric space (to see this, think of a Jordan curve with a spiral on the boundary, like [8, Figure 30]). If however, D is John, we have the following.

3.5. Corollary. *If D is a bounded Jordan John disk, then $(\overline{D}, \lambda_D)$ is a bounded metric space.*

Proof. Clearly $\operatorname{diam}\gamma \le \operatorname{diam} D$ for any hyperbolic geodesic γ in D with end points in $\overline{D}$, while Lemma 3.4 assures us that $\lambda_D(z_1, z_2) \le \operatorname{length}(\gamma[z_1, z_2]) \le M \operatorname{diam}\gamma[z_1, z_2]$ for any $z_1, z_2 \in \overline{D}$. □

We next give another version of Lemma 2.2.

3.6. Lemma. *Let $D \subset \mathbf{R}^2$ be a Jordan domain and let $z_1, z_2 \in \partial D$. If γ is the hyperbolic geodesic from z_1 to z_2 and β is a bounded component of $\partial D \backslash \{z_1, z_2\}$, then*

$$\operatorname{diam}\gamma \le K \operatorname{diam}\beta,$$

where K is the Gehring–Hayman constant.

Proof. Let β be a bounded component of $\partial D \backslash \{z_1, z_2\}$ and $f\colon \mathbf{D} \to D$ a conformal map. Because D is Jordan, f extends as a homeomorphism $f\colon \overline{\mathbf{D}} \to \overline{D}$. Since β is bounded, $\bar{\beta}$ is contained in the interior of a compact disk W. Let $V = \overline{D} \cap W$, $V' = f^{-1}(V) \subset \overline{\mathbf{D}}$. Then $f\colon V' \to V$ is uniformly continuous. Thus, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f\bigl(B(\zeta, \delta) \cap V'\bigr) \subset B\bigl(f(\zeta), \varepsilon\bigr) \cap V \quad \text{for all } \zeta \in \overline{\mathbf{D}} \cap V'.$$

Let β' be the preimage of β in $\overline{\mathbf{D}}$ and let $\zeta_1, \zeta_2 \in \mathbf{T}$ with $z_j = f(\zeta_j)$, $j = 1, 2$. Consider the union

$$U' = \bigcup_{\zeta \in \beta'} \bigl(B(\zeta, \delta) \cap V'\bigr).$$

Then we have

$$U = f(U') \subset \bigcup_{\zeta \in \beta'} \bigl(B\bigl(f(\zeta), \varepsilon\bigr) \cap V\bigr).$$

Join ζ_1, ζ_2 in U' by α' such that $\alpha' \backslash \{\zeta_1, \zeta_2\} \subset \mathbf{D} \cap U'$, and let $\alpha = f(\alpha')$. Now

$$\operatorname{diam}\alpha \le \operatorname{diam} U \le \operatorname{diam}\beta + 2\varepsilon. \tag{3.7}$$

Let γ be the hyperbolic geodesic from z_1 to z_2 in D. By Lemma 2.2 there exists an absolute constant K such that $\operatorname{diam}\gamma \le K \operatorname{diam}\alpha$. Inequality (3.7) then implies that $\operatorname{diam}\gamma \le K(\operatorname{diam}\beta + 2\varepsilon)$. Letting $\varepsilon \to 0$ we conclude that

$$\operatorname{diam}\gamma \le K \operatorname{diam}\beta. \ \square$$

Finally we can prove the main result in this section, a reformulation of the three point property for John disks. It looks just like the corresponding formulation for quasidisks, but with euclidean distances replaced by internal distances; see e.g. [4, Section 2.2]. The theorem is interesting in its own right, as it may be useful for various calculations in John disks.

3.8. Theorem. *Let D be a Jordan domain. If D is a c-John disk, then there is a constant a such that*

$$\lambda_D(z_1, z_2) \le a\lambda_D(z_1, z_3) \tag{3.9}$$

for any $z_1, z_3 \in \partial D$, with z_2 in the component of $\partial D\backslash\{z_1, z_3\}$ with the least euclidean diameter.

Conversely, suppose that there is a constant a such that whenever $z_1, z_3 \in \partial_r D$, (3.9) holds for every z_2 in a bounded component of $\partial_r D\backslash\{z_1, z_3\}$. Then D is a c-John disk.

The constants a and c depend only on each other.

Proof. If D is a c-John disk, then by Lemma 3.2 there is a constant $b = b(c)$ so that

$$\min_{j=1,2} \operatorname{diam} \beta_j \le b\lambda_D(z_1, z_3) \tag{3.10}$$

for $z_1, z_3 \in \partial D$. Here β_1 and β_2 denote the two components of $\partial D\backslash\{z_1, z_3\}$. Assume that β_1 is the component with the least euclidean diameter. Pick any point z_2 in $\beta_1\backslash\{z_1, z_3\}$ and let γ be the unique hyperbolic geodesic that joins z_1 and z_2 in D. Then by Lemma 3.6 we have

$$\operatorname{diam} \gamma \le K \operatorname{diam} \beta_1[z_1, z_2] \tag{3.11}$$

for some universal $K \ge 1$. Since we are in a John disk there exists, by Lemma 3.4, a constant $M = M(c)$ so that

$$\operatorname{length}(\gamma) \le M \operatorname{diam} \gamma. \tag{3.12}$$

Combining (3.12), (3.11), (3.10), and using Lemma 3.2, we get

$$\begin{aligned}
\lambda_D(z_1, z_2) &\le \operatorname{length}(\gamma) \\
&\le M \operatorname{diam} \gamma \\
&\le KM \operatorname{diam} \beta_1[z_1, z_2] \\
&\le KM \operatorname{diam} \beta_1 \\
&\le bKM\lambda_D(z_1, z_3).
\end{aligned}$$

Now let $a = bKM$.

Conversely, assume that $\lambda_D(z_1, z_2) \le a\lambda_D(z_1, z_3)$ for any point $z_2 \in \beta_1 \cap \partial_r D$, where β_1 is a boundary arc from z_1 to z_3. For a fixed $0 < \varepsilon < \operatorname{diam} \beta_1$ there exist $z_{21}, z_{22} \in \partial_r D \cap (\beta_1\backslash\{z_1, z_3\})$ such that $|z_{21} - z_{22}| > \operatorname{diam} \beta_1 - \varepsilon$. (This is because, by Lemma 2.1, $\partial_r D$ is dense in ∂D.) Then:

$$\operatorname{diam} \beta_1 \le \lambda_D(z_{21}, z_{22}) + \varepsilon \le \lambda_D(z_1, z_{21}) + \lambda_D(z_1, z_{22}) + \varepsilon \le 2a\lambda_D(z_1, z_3) + \varepsilon.$$

If we let $\varepsilon \to 0$ we may use Lemma 3.2 and conclude that D is c-John with $c = c(a)$. □

To close this section we prove an "obvious" result, a fact which will be needed below. It is a corollary of Lemmas 3.4 and 3.6.

3.13. Lemma. *Suppose that D is a Jordan John domain. If $z \in \partial D$ and (z_n) is a sequence in D, then $\lambda_D(z_n, z) \to 0$ if and only if $|z_n - z| \to 0$.*

Proof. If $\lambda_D(z_n, z) \to 0$, then obviously $|z_n - z| \to 0$, since $|z_n - z| \le \lambda_D(z_n, z)$.

Now assume that $|z_n - z| \to 0$. Let ζ_n be the first point in $[z_n, z] \cap \partial D$ when we go from z_n to z. (If $\zeta_n = z$ there is nothing to prove.) Since ∂D is locally connected (in the sense of the definition on p. 19 in [10]), for every $\varepsilon > 0$ there exists a δ such that any two points $a, b \in \partial D$ with $|a - b| < \delta$ can be joined by a continuum β in ∂D of diameter less than ε. Choose any $\varepsilon > 0$. If n is so large that $|z_n - z| < \delta$, then $|\zeta_n - z| < \delta$. Thus there is an arc β of ∂D with $\zeta_n, z \in \beta$ and $\operatorname{diam} \beta < \varepsilon$. Lemma 3.6 then gives $\operatorname{diam} \gamma[\zeta_n, z] \le K \operatorname{diam} \beta < K\varepsilon$. Lemma 3.4, on the other hand, provides an M such that $\operatorname{length}(\gamma[\zeta_n, z]) \le M \operatorname{diam} \gamma[\zeta_n, z]$. Combining what we have gives:

$$\lambda_D(z_n, z) \le \lambda_D(z_n, \zeta_n) + \lambda_D(\zeta_n, z) \le |z_n - \zeta_n| + \operatorname{length}(\gamma[\zeta_n, z]) \le (1 + KM)\varepsilon,$$

where we also used that $|z_n - \zeta_n| < |z_n - z| < \delta \le \varepsilon$. This is what we needed to prove. □

3.14. Remark. It follows from Lemma 3.13 that if D is a Jordan John domain, then $(\overline{D}, \lambda_D)$ is complete. In particular, if D is bounded, then $(\overline{D}, \lambda_D)$ is compact, by Corollary 3.5.

4. Extension properties

In this section we prepare for and prove the main result of this paper.

Given two metric spaces (X, d_X) and (Y, d_Y) and a homeomorphism $f\colon X \to Y$. The homeomorphism f is called *η-quasisymmetric* if there exists a strictly increasing homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ with $\eta(0) = 0$ such that

$$\frac{d_Y\big(h(x_1), h(x_2)\big)}{d_Y\big(h(x_1), h(x_3)\big)} \le \eta\left(\frac{d_X(x_1, x_2)}{d_X(x_1, x_3)}\right)$$

for all $x_1, x_2, x_3 \in X$.

We say that f is *L-bilipschitz* if there exists a constant L such that

$$\frac{1}{L} d_X(x_1, x_2) \le d_Y\big(f(x_1), f(x_2)\big) \le L d_X(x_1, x_2)$$

for all x_1, x_2 in X, i.e. if both f and f^{-1} are L-lipschitz. If f is L-bilipschitz for some L we sometimes say simply that f is bilipschitz, and if we say that f

is quasisymmetric, we mean that f is η-quasisymmetric for some η. Note that every L-bilipschitz map is η-quasisymmetric with $\eta(t) = L^2 t$; see e.g. [7, p. 78]. A quasisymmetric map is not lipschitz in general.

Tukia and Väisälä gave the definition of quasisymmetric maps between general metric spaces in [11]. (See also Näkki and Väisälä [9, 3.4] and Heinonen [7, Chapter 10]). If $X = \mathbf{T}$ and $Y \subset \mathbf{R}^2$, both equipped with the euclidean standard metrics, we get close to the original concept of quasisymmetry studied by Beurling and Ahlfors in [1]. In this situation $h(\mathbf{T})$ is a quasicircle if and only if the homeomorphism h is a quasisymmetric map. See [10, Proposition 5.10 and Theorem 5.11].

For later reference we recall the following properties of quasisymmetric maps. See [17, Propositions 10.6, 10.8, and 10.11], and of course [11].

4.1. Lemma. *Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are η and ϑ-quasisymmetric, respectively. Then*

(1) *$f(A)$ is bounded for every bounded $A \subset X$,*
(2) *f extends to an η-quasisymmetric homeomorphism of the completions of X and Y,*
(3) *$g \circ f\colon X \to Z$ is $\vartheta(\eta)$-quasisymmetric,*
(4) *$f^{-1}\colon Y \to X$ is $1/\eta^{-1}(1/t)$-quasisymmetric.*

We will now consider the metric spaces (D, λ_D), $(\partial_r D, \lambda_D)$, and (D_r, λ_D), where D is some simply connected Jordan domain, and maps between these spaces. If we say something like "$f\colon \mathbf{D} \to (D, \lambda_D)$ is quasisymmetric", we use euclidean distance in $\mathbf{D}$ and internal distance in D.

Our point of departure is the following result of Näkki and Väisälä. (Section 7 of [9]. The proof is based on several results in Section 3 of [9] and on Theorem 2.20 in [14].)

4.2. Theorem. *A bounded (respectively unbounded) conformal disk $D \subset \mathbf{R}^2$ is a c-John disk if and only if there is a homeomorphism $h\colon \mathbf{D} \to (D, \lambda_D)$ (respectively $h\colon \mathbf{H} \to (D, \lambda_D)$) which is η-quasisymmetric, c and η depending on each other.*

Recall that a conformal disk is a domain $D \subset \mathbf{R}^2$ such that the complement $\overline{\mathbf{R}^2} \setminus D$ is a continuum with at least two points.

4.3. Remark. The *proof* of Theorem 4.2 shows that if $D \subset \mathbf{R}^2$ is a bounded (unbounded) John disk, then any conformal map $h\colon \mathbf{D} \to D$ ($h\colon \mathbf{H} \to D$) is quasisymmetric with respect to internal distances. This will be of importance later on, most notably in the proof of Theorem 4.7.

We can now prove a one-sided analogue of Proposition 5.10 and Theorem 5.11 in [10].

4.4. Lemma. *Let D be a bounded Jordan domain.*

If there exists a homeomorphism $h\colon \mathbf{T} \to \partial D$ such that $h\colon h^{-1}(\partial_r D) \to (\partial_r D, \lambda_D)$ is η-quasisymmetric, then D is a c-John disk. Conversely, if D is a c-John disk then any conformal map $h\colon \mathbf{D} \to D$ has a homeomorphic extension to the boundary such that $h\colon \mathbf{T} \to (\partial D, \lambda_D)$ is η-quasisymmetric. η and c depend only on each other.

The same result holds for unbounded domains if we replace $\mathbf{T}$ by $\mathbf{R}$.

Proof. First assume that we have an η-quasisymmetric map h of $h^{-1}(\partial_r D) \subset \mathbf{T}$ to $\partial_r D$. The proof is essentially that of Proposition 5.10 in [10], although we have a slightly more flexible three point property.

Pick $w_1 = h(z_1)$, $w_3 = h(z_3)$ on $\partial_r D$, $z_j \in \mathbf{T}$. Let z_2 lie on the shorter arc between z_1 and z_3 with $w_2 = h(z_2) \in \partial_r D$. Then $|z_2 - z_3| < |z_1 - z_3|$. By quasisymmetry we have

$$\lambda_D(w_2, w_3) \leq \eta(1)\lambda_D(w_1, w_3).$$

Then D is a c-John disk, with $c = c(\eta(1))$ by Theorem 3.8.

If D is a Jordan John disk the conformal map f from $\mathbf{D}$ onto D is internally quasisymmetric (Theorem 4.2 and Remark 4.3), and this map extends to the boundary since by Lemma 4.1(2) any quasisymmetric map of a metric space extends to a quasisymmetric map of the completion of that space. (By Remark 3.14 the completion of (D, λ_D) is $(\overline{D}, \lambda_D)$.)

The extension can be done without reference to Lemma 4.1 as follows. Since D is Jordan, f extends as a homeomorphism $f\colon \overline{\mathbf{D}} \to \overline{D}$. Pick any three points $\zeta_1, \zeta_2, \zeta_3 \in \mathbf{T}$ and let $z_j = f(\zeta_j)$. Now, since f is quasisymmetric with respect to internal distance in D, we have, for every $0 < r < 1$

$$\frac{\lambda_D\big(f(r\zeta_1), f(r\zeta_2)\big)}{\lambda_D\big(f(r\zeta_2), f(r\zeta_3)\big)} \leq \eta\biggl(\frac{|r\zeta_1 - r\zeta_2|}{|r\zeta_2 - r\zeta_3|}\biggr).$$

As $r \to 1$, $f(r\zeta_j) \to z_j$, and by Lemma 3.13 we have

$$\frac{\lambda_D(z_1, z_2)}{\lambda_D(z_2, z_3)} \leq \eta\biggl(\frac{|\zeta_1 - \zeta_2|}{|\zeta_2 - \zeta_3|}\biggr). \square$$

4.5. Corollary. *If D and D' are c and c' Jordan John disks, respectively, and if $f\colon (D, \lambda_D) \to (D', \lambda_{D'})$ is an η-quasisymmetric map, then f extends as an η'-quasisymmetric map $\tilde{f}\colon (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'})$, where η' depends only on η, c, and c'.*

Proof. We only give the proof for the case when both D and D' are bounded. The unbounded case is similar. (D' is bounded if and only if D is bounded, by Lemma 4.1(1) and (4).) Let $\varphi\colon \mathbf{D} \to D$ be a conformal map. It is ϱ-quasisymmetric, according to Theorem 4.2 and Remark 4.3. Thus $g = f \circ \varphi\colon \mathbf{D} \to D'$ is $\eta(\varrho)$-quasisymmetric (Lemma 4.1(3)). Furthermore, g extends as a quasisymmetric map $\tilde{g}\colon \overline{\mathbf{D}} \to \overline{D'}$, just like in the proof of Lemma 4.4. Then $\tilde{g}^{-1} \circ \varphi^{-1}$ provides the extension. $\square$

Using results of Ghamsari–Näkki–Väisälä, [6], we can say a bit more:

4.6. Lemma. *Suppose D is a Jordan c-John disk. Let $f\colon \partial D \to \partial D' = f(\partial D)$ be a homeomorphism such that $f\colon \big(f^{-1}(\partial_r D'), \lambda_D\big) \to (\partial_r D', \lambda_{D'})$ is quasisymmetric. Then D' is a Jordan c'-John disk, $c' = c'(c)$, and there exists an extension $\tilde f\colon (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'})$ of f such that $\tilde f|_D$ is*

(1) *quasisymmetric with respect to internal distances,*
(2) *quasiconformal,*
(3) *bilipschitz with respect to hyperbolic distances.*

Note that à priori $f(\partial D)$ is just a Jordan curve in $\mathbf{R}^2$ with two complementary domains, and that we have to specify the domain in which we want to use the internal distance.

Proof of Lemma 4.6. Take w_1, w_3 in $\partial_r D'$, let $w_j = f(z_j)$, $z_j \in \partial D$, and then let $w_2 \in \partial_r D' \setminus \{w_1, w_3\}$ lie in that component of $\partial D'$ which is the image of the component of $\partial D \setminus \{z_1, z_3\}$ of least diameter. Then we have

$$\frac{\lambda_D(z_1, z_2)}{\lambda_D(z_1, z_3)} \le a = a(c),$$

which leads to

$$\frac{\lambda_{D'}(w_1, w_2)}{\lambda_{D'}(w_1, w_3)} \le \tilde c = \eta(a).$$

Thus $\lambda_{D'}(w_1, w_2) \le \tilde c \lambda_{D'}(w_1, w_3)$, and D' is a c'-John disk by Theorem 3.8, with $c' = c'(\tilde c)$. Then $\partial_r D' = \partial D'$, and f is a quasisymmetric map $(\partial D, \lambda_D) \to (\partial'_D, \lambda_{D'})$.

Suppose that D is bounded. Then D' is bounded too, by Lemma 4.1(1). (The proof of the unbounded case is similar, replacing $\mathbf{D}$ by $\mathbf{H}$.) Let $\varphi\colon \mathbf{D} \to D$ and $\psi\colon \mathbf{D} \to D'$ be conformal maps; since D and D' are both Jordan and John, these maps extend as homeomorphisms of the boundaries. Then the map $g\colon \mathbf{T} \to \mathbf{T}$ defined by

$$g = \psi^{-1} \circ f \circ \varphi$$

is quasisymmetric with respect to the euclidean metric. This is because φ and ψ are quasisymmetric by Theorem 4.2 and Remark 4.3. The boundary extensions are still quasisymmetric by Lemma 4.1 or Lemma 4.4, and the composition of quasisymmetric maps is again quasisymmetric. By Lemma 2.10 in [6] g extends as a homeomorphism $\tilde g\colon \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ with the properties (1)–(3) above. Then

$$\tilde f = \psi \circ \tilde g \circ \varphi^{-1}\colon \overline{D} \to \overline{D'}$$

has the same properties because φ and ψ are conformal maps onto John disks (remember that conformal maps are hyperbolic isometries); $\tilde f$ coincides with f on ∂D. $\square$

Next we state and prove the main result of this paper. It is an internal, or one sided, version of Theorem 6 in [2]. It is also reminiscent of Theorem 3.1 in [6].

4.7. Theorem. *Suppose D is a Jordan c-John disk, and that $f\colon \partial D \to \partial D'$ is a homeomorphism. If $f\colon \big(f^{-1}(\partial_r D'), \lambda_D\big) \to (\partial_r D', \lambda_{D'})$ is L-bilipschitz, then D' is a c'-John domain, and f extends as an M-bilipschitz map $\tilde f\colon (\overline{D}, \lambda_D) \to (\overline{D'}, \lambda_{D'})$. M and c' depend only on L and c.*

Proof. Assume that $f\colon \big(f^{-1}(\partial_r D'), \lambda_D\big) \to (\partial_r D', \lambda_{D'})$ is L-bilipschitz. Then f is quasisymmetric, and D' is a c'-John disk, so that $f\colon (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is L-bilipschitz. Furthermore f has an extension $\tilde f$ which is η-quasisymmetric with respect to the internal distances and L'-bilipschitz with respect to the hyperbolic distances in D and D'. All this follows from Lemma 4.6. Here $L' = L'(L)$, and η depends only on c and L. Let $\varrho_D(z)$ and $\varrho_{D'}(w)$ denote the hyperbolic densities in D and D', respectively, and let

$$l_{\tilde f}(z_0) = \liminf_{z\to z_0} \frac{|\tilde f(z) - \tilde f(z_0)|}{|z - z_0|}, \quad L_{\tilde f}(z_0) = \limsup_{z\to z_0} \frac{|\tilde f(z) - \tilde f(z_0)|}{|z - z_0|}.$$

Then because $\tilde f$ is hyperbolic L'-bilipschitz we have for any $z_0 \in D$

$$\frac{1}{L'}\frac{\varrho_D(z_0)}{\varrho_{D'}\big(\tilde f(z_0)\big)} \le l_{\tilde f}(z_0) \le L_{\tilde f}(z_0) \le L'\frac{\varrho_D(z_0)}{\varrho_{D'}\big(\tilde f(z_0)\big)}; \tag{4.8}$$

see e.g. [3, p. 250], [4, Lemma 7.7.2]. From Koebe's theorem and Schwarz' lemma it follows that

$$\frac{1}{2\,\mathrm{dist}(z_0, \partial D)} \le \varrho_D(z_0) \le \frac{2}{\mathrm{dist}(z_0, \partial D)}$$

and

$$\frac{1}{2\,\mathrm{dist}\big(\tilde f(z_0), \partial D'\big)} \le \varrho_{D'}\big(\tilde f(z_0)\big) \le \frac{2}{\mathrm{dist}\big(\tilde f(z_0), \partial D'\big)}.$$

Combining (4.8) with the above yields

$$\frac{1}{4L'}\frac{\mathrm{dist}\big(\tilde f(z_0), \partial D'\big)}{\mathrm{dist}(z_0, \partial D)} \le l_{\tilde f}(z_0) \le L_{\tilde f}(z_0) \le 4L'\frac{\mathrm{dist}\big(\tilde f(z_0), \partial D'\big)}{\mathrm{dist}(z_0, \partial D)}. \tag{4.9}$$

In addition, there is an N such that:

$$\frac{1}{N} \le \frac{\mathrm{dist}\big(\tilde f(z_0), \partial D'\big)}{\mathrm{dist}(z_0, \partial D)} \le N \tag{4.10}$$

for every $z_0 \in D$. To see this, let $z_0 \in D$ and choose an $a \in \partial D$ with $|z_0 - a| = \mathrm{dist}(z_0, \partial D)$. Now z_0 lies in $S(a, |z_0 - a|)$. Traverse $S(a, |z_0 - a|)$ from z_0 (in any direction) and denote by b the first point where $S(a,|z_0-a|)$ meets ∂D. Then clearly

$$|z_0-a| \le |z_0-b| \le \lambda_D(z_0,b) \le \pi|z_0-a|,$$

and

$$|z_0-a| = |a-b| \le \lambda_D(a,b) \le |z_0-a| + \pi|z_0-a| = (\pi+1)|z_0-a|. \tag{4.11}$$

Thus

$$\frac{\lambda_D(z_0,b)}{\lambda_D(a,b)} \le \frac{\pi|z_0-a|}{|z_0-a|} = \pi,$$

which implies

$$\frac{\lambda_{D'}\big(\tilde f(z_0),\tilde f(b)\big)}{\lambda_{D'}\big(\tilde f(a),\tilde f(b)\big)} \le \eta(\pi)$$

because $\tilde f$ is η-quasisymmetric with respect to internal distances. Then

$$\operatorname{dist}\big(\tilde f(z_0),\partial D'\big) \le \lambda_{D'}\big(\tilde f(z_0),\tilde f(b)\big) \le \eta(\pi)\lambda_{D'}\big(\tilde f(a),\tilde f(b)\big). \tag{4.12}$$

By assumption, $f=\tilde f|_{\partial D}$ is L-bilipschitz with respect to internal distances, so that

$$\lambda_{D'}\big(\tilde f(a),\tilde f(b)\big) \le L\lambda_D(a,b). \tag{4.13}$$

Combining (4.12), (4.13) and (4.11) gives

$$\operatorname{dist}\big(\tilde f(z_0),\partial D'\big) \le \eta(\pi)L(\pi+1)|z_0-a| = N\operatorname{dist}(z_0,\partial D).$$

The lower inequality in (4.10) follows by considering the inverse of $\tilde f$. Combining (4.9) and (4.10) we have

$$\frac{1}{M} \le l_{\tilde f}(z_0) \le L_{\tilde f}(z_0) \le M,$$

where $M = 4L'N$. This is enough to ensure that

$$\frac{1}{M}\operatorname{length}(\alpha) \le \operatorname{length}\big(\tilde f(\alpha)\big) \le M\operatorname{length}(\alpha) \tag{4.14}$$

for any arc $\alpha \subset D$. See [12, Theorem 5.3]. But then

$$\lambda_{D'}\big(\tilde f(z_1),\tilde f(z_2)\big) \le M\lambda_D(z_1,z_2).$$

The lower inequality follows by looking at the inverse of $\tilde f$. This completes the proof. □

4.15. Remark. Maps that satisfy (4.14) are also known as *maps of bounded length distortion*; see [13]. Homeomorphisms of bounded length distortion are locally bilipschitz (in the euclidean metric), and conversely. These maps are important, for instance, if one wants to map a ball quasiconformally onto a cylindrical domain in $\mathbf{R}^3$. See [14].

In Remark 4.15 we just mentioned that internally bilipschitz maps are locally euclidean bilipschitz in the interior of domains (and conversely). On the boundary of a domain the situation is more interesting. Examples 4.16 to 4.18 below show that a boundary map that is bilipschitz with respect to the euclidean metric does not have to be bilipschitz with respect to internal distances, nor does an internally bilipschitz map have to be locally euclidean bilipschitz there. Thus Theorem 4.7 does not follow immediately from previously known results on bilipschitz maps like, for instance, Theorem 6 in [2] or Theorem 3.1 in [6].

There is another difference between [6, Theorem 3.1] and our Theorem 4.7. A euclidean bilipschitz map does not "notice" the direction of cusps, and so a John disk is not necessarily mapped onto a John disk via such a map. Therefore, in order to have an extension, the authors of [6] have to assume that the image is a John disk. Internally bilipschitz maps preserve the John condition (1.1), since the direction of a cusp is already implicit in the choice of internal metric.

4.16. Example. Let D^* denote the half strip, $D^* = \{x+iy : x > 0,\ |y| < 1\}$, and let $D = \mathbf{R}^2 \setminus \overline{D^*}$. Consider $f\colon \partial D \to \partial D^*$ defined by $f(z) = z$. This is a euclidean isometry, thus a bilipschitz map, while the "induced" map $f\colon (\partial D, \lambda_D) \to (\partial D^*, \lambda_{D^*})$ is not bilipschitz.

4.17. Example. Let again D be the outside of the half strip, as in Example 4.16. Consider $g\colon \mathbf{R} \to \partial D$ defined by "wrapping" the real line without stretching around the half strip:

$$g(x) = \begin{cases} ix, & \text{if } |x| < 1, \\ x - 1 + i, & \text{if } x \geq 1, \\ -(x+1) - i, & \text{if } x \leq -1. \end{cases}$$

Now $g\colon \mathbf{R} \to (\partial D, \lambda_D)$ is an isometry, while g is not bilipschitz with respect to the euclidean distance.

4.18. Example. Consider the domain D exterior to the triangle with vertices i, $-i$, and 1. Let $D' = \mathbf{R}^2 \setminus \{z = x+iy : 0 \leq x \leq 1,\ |y| \leq (1-x)^2\}$. See Figure 2. Consider the map $h\colon \partial D \to \partial D'$ defined by

$$h(x+iy) = \begin{cases} iy, & \text{if } x = 0, \\ x + iy^2, & \text{if } x > 0,\ y \geq 0, \\ x - iy^2, & \text{if } x > 0,\ y < 0. \end{cases}$$

Then $h\colon \partial D \to \partial D'$ is not locally bilipschitz at $z = 1$, while $h\colon (\partial D, \lambda_D) \to (\partial D', \lambda_{D'})$ is 2-bilipschitz.

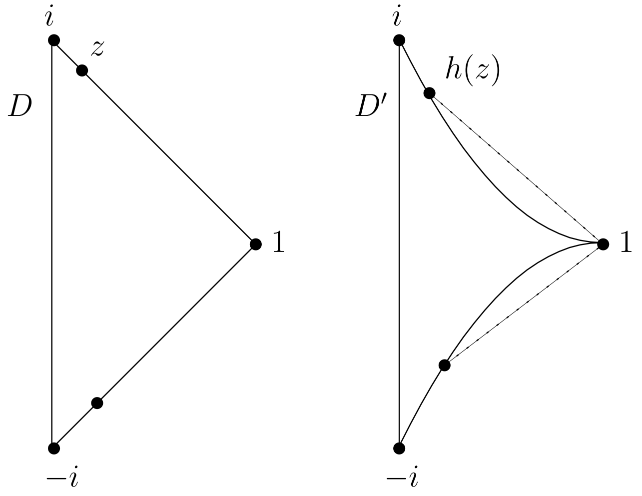

Figure 2.

We mentioned in the introduction Gehring's result which says that any bilipschitz map of the boundary of a quasidisk extends as a bilipchitz map of the whole closed quasidisk, [2, Theorem 7]. The converse holds for bounded Jordan domains, [3, Lemma 4.4, Theorem 4.9]. To conclude this paper we record the following question.

4.19. Question. Is there some sort of converse of Theorem 4.7 for *bounded domains*?

We have to assume the domain to be bounded: the half strip, in which internal and euclidean distances are the same, has the property that any euclidean bilipschitz map of its boundary extends as a bilipschitz map of the closed half strip. See Remark 4.10 in [3]. Because the half strip is not a John disk, we see that the converse of Theorem 4.7 cannot hold for unbounded domains.

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Received 23 April 2004