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REMOVABLE SINGULARITIES FOR BOUNDED p-HARMONIC AND QUASI(SUPER)HARMONIC FUNCTIONS ON METRIC SPACES

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Abstract. We study removable singularities for bounded p-harmonic functions in complete doubling metric spaces supporting a Poincaré inequality. We show that a relatively closed set E of an arbitrary open set Ω is removable if it has zero capacity. Moreover, the extensions are unique.

We give several general results showing nonremovability and giving us characterizations of removable sets as sets of capacity zero in various situations.

We also provide examples of removable sets with positive capacity. Such examples can have some unexpected behaviour: e.g., E may disconnect Ω; the extensions may not be unique; and there exists a nonremovable union of two compact disjoint removable sets.

Similar results for superharmonic, quasiharmonic and quasisuperharmonic functions are also given.

1. Introduction

(Relatively closed) sets of zero capacity are removable for bounded p-harmonic and bounded superharmonic functions defined in an open subset of weighted $\mathbf{R}^n$, see Theorems 7.35 and 7.36 in Heinonen–Kilpeläinen–Martio [12]. They also show, p. 143 in [12], that compact sets are removable if and only if they have capacity zero. (See Serrin [28], [29] and Maz'ya [27] for earlier proofs of this fact for unweighted $\mathbf{R}^n$.)

In Björn–Björn–Shanmugalingam [10, Section 8], the study of removable sets for bounded p-harmonic and superharmonic functions was extended to bounded domains Ω (with complement of positive capacity) in metric spaces. Removability was shown for sets of capacity zero.

The sharpness of the removability results in [10] was illustrated by showing nonremovability for a large class of sets with positive capacity. However, they were not able to show nonremovability for all sets with positive capacity, as pointed out on p. 419 in [10]. The reason for this turns out to be quite natural since we here present a simple counterexample, see Section 9.

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Let Ω be an open set in our metric space and $E \subset \Omega$ be relatively closed with zero capacity. In addition to showing that E is removable for bounded p-harmonic functions and superharmonic functions bounded from below, we also show that E is removable for p-harmonic and superharmonic functions belonging to the Newtonian–Sobolev space $N^{1,p}(\Omega \setminus E)$. We also obtain the corresponding results for quasi(super)harmonic functions. (See Section 3 for the definition of quasi(super)harmonic functions and Section 6 for the precise statements of the removability results.)

It should be mentioned that for unweighted $\mathbf{R}^n$ Tolksdorf [33, Theorem 1.5], proved that compact sets of capacity zero are removable for bounded quasisuperminimizers (which is equivalent to removability for bounded quasisuperharmonic functions, by Proposition 6.6 below).

In Section 7 we extend the nonremovability results of [10] to quasi(super)-harmonic functions, as well as providing some more nonremovability results. This leads to characterizations of removable sets as those of capacity zero in several cases, see Section 8.

In Section 9 we give examples of sets with positive capacity that are removable for bounded p-harmonic and bounded Q-quasiharmonic functions. When a set of capacity zero is removed the extensions are unique. However, when a set of positive capacity is removable it is possible that a given p-harmonic function has several extensions. We provide two examples of such nonunique removability in Section 10. (A necessary requirement for nonunique removability is of course nonunique continuation of p-harmonic functions; see Martio [26] and Björn–Björn–Shanmugalingam [10, pp. 426–427].) We also give examples of removable sets E which disconnect Ω.

In Section 11 we point out that removable singularities with positive capacity have some unexpected behaviour, e.g. it is possible to have a nonremovable union of two compact disjoint removable singularities.

For more on quasiminimizers and their importance see the introductions in Kinnunen–Martio [23] and A. Björn [4]. An application of the removability results for quasiharmonic functions will be given in the forthcoming paper A. Björn [7].

For examples of complete metric spaces equipped with a doubling measure supporting a Poincaré inequality, see, e.g., A. Björn [3], [5].

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2. Notation and preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a complete metric space endowed with a metric d and a doubling measure μ, i.e. there exists a constant $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X (we make the convention that balls are nonempty and open),

$$0 < \mu(2B) \le C\mu(B) < \infty,$$

where $\lambda B = B(x_0, \lambda r)$. We emphasize that the σ-algebra on which μ is defined is obtained by completion of the Borel σ-algebra. We also assume that $1 < p < \infty$ and that $\Omega \subset X$ is a nonempty open set. (At the end of this section we make some further assumptions assumed in the rest of the paper.)

Note that some authors assume that X is proper (i.e. closed bounded sets are compact) rather than complete, but, since μ is doubling, X is proper if and only if it is complete.

A *curve* is a continuous mapping from an interval. We will in addition, throughout the paper, assume that every curve is nonconstant, compact and rectifiable. A curve can thus be parameterized by its arc length ds.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function f on X if for all curves γ: $[0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2.1}$$

whenever both $f(\gamma(0))$ and $f(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every curve, then g is a *p-weak upper gradient* of f.

By saying that (2.1) holds for p-almost every curve we mean that it fails only for a curve family with zero p-modulus, see Definition 2.1 in Shanmugalingam [30]. It is implicitly assumed that $\int_\gamma g\, ds$ is defined (with a value in $[0, \infty]$) for p-almost every curve.

If $g \in L^p(X)$ is a p-weak upper gradient of f, then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j \to g$ in $L^p(X)$, see Lemma 2.4 in Koskela–MacManus [25].

If f has an upper gradient in $L^p(X)$, then it has a *minimal p-weak upper gradient* $g_f \in L^p(X)$ in the sense that $g_f \le g$ μ-a.e. for every p-weak upper gradient $g \in L^p(X)$ of f, see Corollary 3.7 in Shanmugalingam [31].

If $f, h \in N^{1,p}(X)$, then $g_f = g_h$ μ-a.e. in $\{x \in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{f \ne c}$ for $c \in \mathbf{R}$. For these and other facts on p-weak upper gradients, see, e.g., Björn–Björn [8, Section 3].

Definition 2.2. We say that X supports a *weak $(1, q)$-Poincaré inequality* if there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls $B \subset X$, all measurable functions f on X and all upper gradients g of f,

$$\fint_B |f - f_B|\, d\mu \le C \operatorname{diam}(B) \biggl(\fint_{\lambda B} g^q\, d\mu \biggr)^{1/q}, \tag{2.2}$$

where $f_B := \fint_B f\,d\mu := \mu(B)^{-1} \int_B f\,d\mu$.

In the definition of Poincaré inequality we can equivalently assume that g is a q-weak upper gradient—see the comments above. It is also equivalent to require that (2.2) holds for all $f \in \mathrm{Lip}_c(X)$ and all upper gradients $g \in \mathrm{Lip}_c(X)$ of f, see Keith [18, Theorem 2]. We say that $E \Subset A$ if $\overline{E}$ is a compact subset of A, and let $\mathrm{Lip}_c(A) = \{f \in \mathrm{Lip}(A) : \operatorname{supp} f \Subset A\}$.

Following Shanmugalingam [30], we define a version of Sobolev spaces on the metric space X.

Definition 2.3. Whenever $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \left(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

The space $N^{1,p}(X)$ is a Banach space and a lattice, see Shanmugalingam [30].

Definition 2.4. The *capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

The capacity is countably subadditive. For this and other properties as well as equivalent definitions of the capacity we refer to Kilpeläinen–Kinnunen–Martio [20] and Kinnunen–Martio [21], [22].

We say that a property regarding points in X holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, Corollary 3.3 in Shanmugalingam [30] shows that if $u, v \in N^{1,p}(X)$ and $u = v$ μ-a.e., then $u \sim v$.

If X supports a weak $(1,p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$ and the functions in $N^{1,p}(X)$ are quasicontinuous, see Remark 4.4 in [30]. This means that in the Euclidean setting, $N^{1,p}(\mathbf{R}^n)$ is the refined Sobolev space as defined on p. 96 of Heinonen–Kilpeläinen–Martio [12].

To be able to compare the boundary values of Newtonian functions we need a Newtonian space with zero boundary values. We let for a measurable set $E \subset X$,

$$N^{1,p}_0(E) = \{f|_E : f \in N^{1,p}(X) \text{ and } f = 0 \text{ on } X \setminus E\}.$$

One can replace the assumption "$f = 0$ on $X \setminus E$" with "$f = 0$ q.e. on $X \setminus E$" without changing the obtained space $N_0^{1,p}(E)$. Note that if $C_p(X \setminus E) = 0$, then $N_0^{1,p}(E) = N^{1,p}(E)$.

We say that $f \in N_{\mathrm{loc}}^{1,p}(\Omega)$ if $f \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$.

By a continuous function we always mean a real-valued continuous function, whereas a semicontinuous function is allowed to be extended real-valued, i.e. to take values in the extended real line $\overline{\mathbf{R}} := [-\infty, \infty]$. We let $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$.

In addition to the assumptions made in the beginning of this section, from now on we assume that X supports a weak $(1,p)$-Poincaré inequality. By Keith–Zhong [19] it follows that X supports a weak $(1,q)$-Poincaré inequality for some $q \in [1,p)$, which was earlier a standard assumption. Throughout the paper we also let $Q \geq 1$ be a real number.

3. Quasi(super)harmonic functions

We follow Kinnunen–Martio [23, Section 3], making the following definition.

Definition 3.1. A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *Q-quasiminimizer* in Ω if for all open $\Omega' \Subset \Omega$ and all $\varphi \in N_0^{1,p}(\Omega')$ we have

$$\int_{\Omega'} g_u^p \, d\mu \leq Q \int_{\Omega'} g_{u+\varphi}^p \, d\mu. \tag{3.1}$$

A function $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$ is a *Q-quasisuperminimizer* in Ω if (3.1) holds for all nonnegative functions $\varphi \in N_0^{1,p}(\Omega')$, and a *$Q$-quasisubminimizer* in Ω if (3.1) holds for all nonpositive functions $\varphi \in N_0^{1,p}(\Omega')$.

A function is a Q-quasiminimizer in Ω if and only if it is both a Q-quasisubminimizer and a Q-quasisuperminimizer in Ω.

We will need some characterizations of Q-quasisuperminimizers. The following result was proved in A. Björn [4].

Proposition 3.2. *Let $u \in N_{\mathrm{loc}}^{1,p}(\Omega)$. Then the following are equivalent*:

(a) *The function u is a Q-quasisuperminimizer in Ω.*

(b) *For all nonnegative $\varphi \in \mathrm{Lip}_c(\Omega)$ we have*

$$\int_{\varphi \neq 0} g_u^p \, d\mu \leq Q \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu.$$

(c) *For all nonnegative $\varphi \in N_0^{1,p}(\Omega)$ we have*

$$\int_{\varphi \neq 0} g_u^p \, d\mu \leq Q \int_{\varphi \neq 0} g_{u+\varphi}^p \, d\mu.$$

If we omit "super" from (a) and "nonnegative" from (b) and (c) we have a corresponding characterization for Q-quasiminimizers.

By Proposition 3.8 and Corollary 5.5 in Kinnunen–Shanmugalingam [24], a Q-quasiminimizer can be modified on a set of capacity zero so that it becomes locally Hölder continuous in Ω. A *Q-quasiharmonic* function is a continuous Q-quasiminimizer.

Kinnunen–Martio [23, Theorem 5.3], showed that if u is a Q-quasisuperminimizer in Ω, then its *lower semicontinuous regularization*

$$u^*(x) := \operatorname{ess}\liminf_{y\to x} u(y)$$

is also a Q-quasisuperminimizer in Ω in the same equivalence class as u in $N^{1,p}_{\mathrm{loc}}(\Omega)$. Furthermore, u^* is Q-quasisuperharmonic in Ω.

Definition 3.3. A function $u\colon \Omega \to (-\infty,\infty]$ is *Q-quasisuperharmonic* in Ω if u is not identically ∞ in any component of Ω, and there is a sequence of open sets Ω_j and Q-quasisuperminimizers $v_j\colon \Omega_j \to (-\infty,\infty]$ such that

(i) $\Omega_j \Subset \Omega_{j+1}$;
(ii) $\Omega = \bigcup_{j=1}^\infty \Omega_j$;
(iii) $v_j \le v_{j+1}$ in Ω_j;
(iv) $u = \lim_{j\to\infty} v_j^*$ in Ω,

where v_j^* is the lower semicontinuous regularization of v_j.

This definition is due to Kinnunen–Martio [23, Definition 7.1]. The following characterization, Theorem 7.10 in [23], is often useful. (Note that there are misprints in Definition 7.1 and Theorem 7.10 in [23]—which have been corrected here.)

Theorem 3.4. *A function $u\colon \Omega \to (-\infty,\infty]$ is Q-quasisuperharmonic in Ω if u is not identically ∞ in any component of Ω, u is lower semicontinuously regularized, and $\min\{u,k\}$ is a Q-quasisuperminimizer in Ω for every $k\in\mathbf{R}$.*

If u_j is a Q_j-quasisuperminimizer in Ω, $j=1,2$, then, by Corollary 3.8 in [23], $\min\{u_1,u_2\}$ is a $\min\{Q_1+Q_2, Q_1Q_2\}$-quasisuperminimizer in Ω; there is also a corresponding result for quasisuperharmonic functions, see Theorem 7.6 in [23]. We will use these facts mainly with u_2 constant.

By Lemma 5.2 in [23], a quasisuperharmonic function u in Ω obeys the minimum principle: If $u(x) = \inf_\Omega u$ for some $x\in\Omega$, then u is constant in the component of Ω containing x.

Corollary 3.5. *Assume that X is bounded and that u is a quasisuperharmonic function on X. Then u is constant.*

It may be worth to observe that even if our removability results later in the paper will show that if $C_p(X \setminus \Omega) = 0$, then any quasisuperharmonic function on Ω which is either bounded from below or in $N^{1,p}(\Omega)$ is constant, there may still exist unbounded quasisuperharmonic functions on Ω. Let, e.g., $X = \overline{B(0,1)}$ in (unweighted) $\mathbf{R}^3$, $p = 2$, $\Omega = X \setminus \{0\}$ and $u(x) = -|x|^{-1}$. Then it is not difficult to show that u is 2-superharmonic in Ω. (Note however that by, e.g, the maximum principle u is not 2-harmonic in Ω.)

Proof. Since X is compact and u lower semicontinuous, there is x such that $u(x) = \inf_X u$. As X is connected (which follows from the Poincaré inequality), the minimum principle gives that u is constant. $\square$

The following boundary minimum principle was proved in A. Björn [4].

Lemma 3.6. *Assume that Ω is bounded, $C_p(X \setminus \Omega) > 0$ and $f \in C(\partial\Omega) \cap N_0^{1,p}(\overline{\Omega})$. Let u be a quasisuperharmonic function in Ω satisfying $u-f \in N_0^{1,p}(\Omega)$. Then $\inf_\Omega u \geq \inf_{\partial\Omega} f$.*

We will need the following convergence result.

Proposition 3.7. *Let $\{u_j\}_{j=1}^\infty$ be a nondecreasing sequence of Q-quasisuperharmonic functions in Ω and let $u = \lim_{j\to\infty} u_j$. Then either u is identically ∞ in some component of Ω or u is Q-quasisuperharmonic in Ω.*

Proof. Let $k \in \mathbf{R}$, $w_k = \min\{u, k\}$ and $w_{k,j} = \min\{u_j, k\}$. By Theorem 3.4, $w_{k,j}$ is a lower semicontinuously regularized Q-quasisuperminimizer in Ω.

Let further $\Omega_1 \Subset \Omega_2 \Subset \cdots$, be open and such that $\Omega = \bigcup_{j=1}^\infty \Omega_j$. Using w_k and $w_{k,j}$ in the place of u and v_j, respectively, in Definition 3.3 we see that w_k is Q-quasisuperharmonic in Ω.

Hence w_k is lower semicontinuously regularized and it follows easily that also u is lower semicontinuously regularized. Thus using Theorem 3.4 we see that u is Q-quasisuperharmonic in Ω, or identically ∞ in some component. $\square$

The corresponding result for locally uniformly convergent sequences can be obtained as a corollary.

Corollary 3.8. *Let $\{u_j\}_{j=1}^\infty$ be a sequence of Q-quasisuperharmonic functions in Ω which converges locally uniformly and let $u = \lim_{j\to\infty} u_j$. Then u is Q-quasisuperharmonic in Ω.*

Proof. Let $\Omega' \Subset \Omega$ be open. Then $u_j \to u$ uniformly in Ω', and by choosing a subsequence if necessary we may assume that $\sup_{\Omega'} |u_j - u| < 2^{-j}$. Let now $v_j = u_j - 3 \cdot 2^{-j}$. It is easy to check that $v_j \leq v_{j+1}$ in Ω'. Moreover $v_j \to u$ in Ω', and thus by Proposition 3.7, u is Q-quasisuperharmonic in Ω'. Since Ω' was an arbitrary compactly contained open subset of Ω we conclude that u is Q-quasisuperharmonic in Ω. $\square$

4. A Caccioppoli type estimate for quasisubminimizers

We will need a Caccioppoli type estimate for quasisubminimizers. This estimate was proved for unweighted $\mathbf{R}^n$ by Tolksdorf [33, Theorem 1.4]. The proof given here is an easy adaptation of Tolksdorf's proof to metric spaces.

Theorem 4.1. *Let $u \geq 0$ be a Q-quasisubminimizer in Ω. Then for all nonnegative $\eta \in \mathrm{Lip}_c(\Omega)$,*

$$\int_\Omega g_u^p \eta^p \, d\mu \leq c \int_\Omega u^p g_\eta^p \, d\mu,$$

where c only depends on p and Q.

Proof. After multiplying η with a constant we may assume that $0 \leq \eta \leq 1$. Let $\varphi = \eta u \in N_0^{1,p}(\Omega)$. By the quasisubminimizing property of u we have

$$\int_{\eta=1} g_u^p \, d\mu \leq \int_{\varphi>0} g_u^p \, d\mu \leq Q \int_{\varphi>0} g_{u-\varphi}^p \, d\mu \leq Q \int_{\eta>0} g_{u(1-\eta)}^p \, d\mu.$$

Using that $g_{u(1-\eta)} \leq u g_{1-\eta} + (1-\eta) g_u = u g_\eta + (1-\eta) g_u$ we get with $Q_2 = 2^{p-1} Q$,

$$\begin{aligned}\int_{\eta=1} g_u^p \, d\mu &\leq Q_2 \int_{\eta>0} u^p g_\eta^p \, d\mu + Q_2 \int_{\eta>0} (1-\eta)^p g_u^p \, d\mu \\ &\leq Q_2 \int_\Omega u^p g_\eta^p \, d\mu + Q_2 \int_{0<\eta<1} g_u^p \, d\mu.\end{aligned}$$

By adding Q_2 times the left-hand side to both sides, and then dividing by $Q_2 + 1$, we obtain

$$\int_{\eta=1} g_u^p \, d\mu \leq \theta \int_\Omega u^p g_\eta^p \, d\mu + \theta \int_{\eta>0} g_u^p \, d\mu, \tag{4.1}$$

where $\theta = Q_2/(Q_2+1) < 1$.

Next let τ be such that $0 < \tau^{-p}\theta < \tau^{-2p}\theta < 1$ (note that it follows that $0 < \tau < 1$), $\eta_j = (\min\{\eta, \tau^j\} - \tau^{j+1})_+$ and $\hat{\eta}_j = \eta_j/(\tau^j - \tau^{j+1})$. Applying (4.1) to $\hat{\eta}_j$ gives

$$\begin{aligned}\int_\Omega g_u^p \hat{\eta}_{j-1}^p \, d\mu \leq \int_{\hat{\eta}_j=1} g_u^p \, d\mu &\leq \theta \int_\Omega u^p g_{\hat{\eta}_j}^p \, d\mu + \theta \int_{\hat{\eta}_j>0} g_u^p \, d\mu \\ &\leq \theta \int_\Omega u^p g_{\hat{\eta}_j}^p \, d\mu + \theta \int_\Omega g_u^p \hat{\eta}_{j+1}^p \, d\mu\end{aligned}$$

from which it follows that

$$\int_\Omega g_u^p \eta_{j-1}^p \, d\mu \leq \tau^{-p}\theta \int_\Omega u^p g_{\eta_j}^p \, d\mu + \tau^{-2p}\theta \int_\Omega g_u^p \eta_{j+1}^p \, d\mu.$$

Using that $g_{\eta_j} \le g_\eta$ and iterating gives us

$$\int_{\eta=1} g_u^p \, d\mu \le (1-\tau)^{-p} \int_\Omega g_u^p \eta_0^p \, d\mu \le Q_3 \int_\Omega u^p g_\eta^p \, d\mu, \tag{4.2}$$

where $Q_3 = (1-\tau)^{-p}\tau^{-p}\theta(1-\tau^{-2p}\theta)^{-1}$. Using (4.2) with η replaced by $\psi = \min\{(4\eta-1)_+, 1\}$ gives

$$\int_{\eta \ge 1/2} g_u^p \, d\mu = \int_{\psi=1} g_u^p \, d\mu \le Q_3 \int_\Omega u^p g_\psi^p \, d\mu \le 4^p Q_3 \int_{1/4<\eta<1/2} u^p g_\eta^p \, d\mu.$$

Finally, let $\psi_j = \min\{2^j \eta, 1\}$. Then we get

$$\begin{aligned} \int_\Omega g_u^p \eta^p \, d\mu &\le \sum_{j=0}^\infty 2^{-jp} \int_{\psi_j \ge 1/2} g_u^p \, d\mu \\ &\le 4^p Q_3 \sum_{j=0}^\infty 2^{-jp} \int_{1/4<\psi_j<1/2} u^p g_{\psi_j}^p \, d\mu \le 4^p Q_3 \int_\Omega u^p g_\eta^p \, d\mu. \qquad \square \end{aligned}$$

5. p-harmonic functions

If $Q = 1$, "quasi" is omitted from the notation and a function is, e.g., *p-harmonic* if it is 1-quasiharmonic, and *superharmonic* if it is 1-quasisuperharmonic. For equivalent definitions and characterizations of superharmonic functions, see A. Björn [5].

We need some results for p-harmonic functions, see, e.g., Shanmugalingam [31] or Björn–Björn–Shanmugalingam [9]. Assume throughout this section that Ω is bounded and $C_p(X \setminus \Omega) > 0$.

If $f \in N^{1,p}(X)$, then there is a unique solution to the Dirichlet problem with boundary values f, i.e. there is a unique function $Hf = H_\Omega f$ such that $Hf = f$ in $X \setminus \Omega$ and Hf is p-harmonic in Ω.

Lemma 5.1. *If $f_1, f_2 \in N^{1,p}(X)$ and $f_1 \le f_2$ q.e. on $\partial\Omega$, then $Hf_1 \le Hf_2$ in Ω.*

It follows that for $f \in N^{1,p}(X)$, $(Hf)|_\Omega$ only depends on $f|_{\partial\Omega}$. A Lipschitz function f on $\partial\Omega$ can be extended to a function $\tilde f \in \mathrm{Lip}_c(X)$ such that $f = \tilde f$ on $\partial\Omega$. As $(H\tilde f)|_\Omega$ does not depend on the choice of extension, we define $Hf := (H\tilde f)|_\Omega$.

Definition 5.2. A point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \to x_0} Hf(y) = f(x_0) \quad \text{for all } f \in \mathrm{Lip}(\partial\Omega).$$

If $x_0 \in \partial\Omega$ is not regular, then it is *irregular.*

For equivalent characterizations of regular boundary points see Björn–Björn [8, Theorem 6.2]. Recall the following result from Björn–Björn–Shanmugalingam [9, Theorem 3.9].

Theorem 5.3 (The Kellogg property). *The set of all irregular points on $\partial\Omega$ has capacity zero.*

6. Removability

From now on let $E \subsetneq \Omega$ be relatively closed and such that no component of Ω is completely contained in E.

Definition 6.1. The set E is (*uniquely*) *removable* for bounded Q-quasi(super)harmonic functions in $\Omega \setminus E$ if every bounded Q-quasi(super)harmonic function u in $\Omega \setminus E$ has a (unique) bounded Q-quasi(super)harmonic extension to Ω.

Similarly, E is (*uniquely*) *removable* for Q-quasi(super)harmonic functions in $N^{1,p}(\Omega \setminus E)$ if every Q-quasi(super)harmonic function $u \in N^{1,p}(\Omega \setminus E)$ has a (unique) Q-quasi(super)harmonic extension in $N^{1,p}(\Omega)$.

Here, when we, e.g., talk about a Q-quasiharmonic function $u \in N^{1,p}(\Omega)$, we mean that it is Q-quasiharmonic in Ω.

In view of Theorems 6.2 and 6.3 below one can get the feeling that it is not essential to stress the set $\Omega \setminus E$ in the definition, and that the notation could therefore be simplified. However, in Section 11, we see in (a$'$) that for removable singularities of positive capacity it is essential to always stress on which set we discuss removability.

In this section we show that sets of capacity zero are removable for Q-quasi(super)harmonic functions that are either bounded or in $N^{1,p}$. Moreover the extensions are always unique in these cases.

In Björn–Björn–Shanmugalingam [10, Propositions 8.2 and 8.3], the following two results were shown under the additional assumption that $Q = 1$, u is bounded and Ω is a bounded domain with $C_p(X \setminus \Omega) > 0$. (The uniqueness was given as a comment in the text.) In the proof of Proposition 8.3 (which was also used to derive Proposition 8.2) in [10] it was not explicitly shown that $u \in N^{1,p}_{\rm loc}(\Omega)$. How this can be observed is pointed out in the proof of Theorem 6.3 below.

Theorem 6.2. *Assume that $C_p(E) = 0$. Let u be a Q-quasiharmonic function in $\Omega \setminus E$. Assume that one of the following conditions hold*:

(a) $u \in N^{1,p}(\Omega \setminus E)$;
(b) *u is the restriction of a function in $N^{1,p}_{\rm loc}(\Omega)$*;
(c) *$u \in N^{1,p}(B \setminus E)$ for every ball $B \Subset \Omega$*;
(d) *u is bounded.*

Then u has a unique quasisuperharmonic extension U to Ω. Moreover, U is Q-quasiharmonic in Ω, U is bounded if u is bounded, and $U \in N^{1,p}(\Omega)$ if $u \in N^{1,p}(\Omega \setminus E)$.

By saying that u is quasisuperharmonic we, of course, mean that there is some Q' such that u is Q'-quasisuperharmonic.

Theorem 6.3. *Assume that $C_p(E)=0$. Let u be a Q-quasisuperharmonic function in $\Omega\setminus E$. Assume that one of the following conditions hold:*

(a) $u\in N^{1,p}(\Omega\setminus E)$;
(b) *u is the restriction of a function in $N^{1,p}_{\mathrm{loc}}(\Omega)$;*
(c) *$u\in N^{1,p}(B\setminus E)$ for every ball $B\Subset\Omega$;*
(d) *u is bounded from below.*

Then u has a unique quasisuperharmonic extension U to Ω. Moreover, U is Q-quasisuperharmonic in Ω, U is bounded if u is bounded, and $U\in N^{1,p}(\Omega)$ if $u\in N^{1,p}(\Omega\setminus E)$.

Note that to find the extension U we only need to find the unique semicontinuously regularized extension of u to Ω; $U(x)=\operatorname{ess\,lim\,inf}_{\Omega\setminus E\ni y\to x}u(y)$. In Theorem 6.2 part of the conclusion is that this extension is continuous in Ω.

Note also that some condition of the type (a)–(d) is needed: Consider a bounded open set $\Omega\subset\mathbf{R}^n$ (unweighted), $n\geq 3$, $p=2$, and let v be the Green function of Ω with respect to some $y\in\Omega$ (or $v(x)=|x-y|^{2-n}$). Then $-v$ is a harmonic function in $\Omega\setminus\{y\}$ with no quasisuperharmonic extension to Ω.

Let us next observe that unique removability for bounded Q-quasisuperharmonic functions is the same as for Q-quasisuperharmonic functions bounded from below. Note that in Section 10 we give examples of sets which are nonuniquely removable for p-harmonic functions, but we do not know if there are any nonuniquely removable sets for superharmonic functions (or for Q-quasisuperharmonic functions).

Proposition 6.4. *The set E is uniquely removable for Q-quasisuperharmonic functions bounded from below on $\Omega\setminus E$ if and only if it is uniquely removable for bounded Q-quasisuperharmonic functions on $\Omega\setminus E$.*

Proof. The necessity is obvious. As for the sufficiency assume that E is uniquely removable for bounded Q-quasisuperharmonic functions on $\Omega\setminus E$. Let u be a Q-quasisuperharmonic function bounded from below on $\Omega\setminus E$. Then $u_j:=\min\{u,j\}$ has a Q-quasisuperharmonic extension U_j to Ω. As $\min\{U_{j+1},U_j\}=u_j$ in $\Omega\setminus E$, both U_j and $\min\{U_{j+1},U_j\}$ are Q-quasisuperharmonic extensions of u_j to Ω. By uniqueness $U_j=\min\{U_{j+1},U_j\}$ and thus $U_{j+1}\geq U_j$ in Ω. Let $U=\lim_{j\to\infty}U_j$. Then $U=u$ in Ω and hence U is not identically ∞ in any component of Ω. By Proposition 3.7, U is a Q-quasisuperharmonic extension of u to Ω.

As for the uniqueness, let U be a Q-quasisuperharmonic extension of u to Ω. Then $\min\{U,k\}$ is the unique Q-quasisuperharmonic extension of $\min\{u,k\}$ to Ω. It follows that U is unique. □

Proof of Theorem 6.3. The uniqueness follows from the observation above, we let

$$U(x) = \operatorname*{ess\,lim\,inf}_{\Omega\setminus E \ni y \to x} u(y), \quad x \in \Omega.$$

Assume first that (c) holds. It follows that $u \in N^{1,p}_{\mathrm{loc}}(\Omega \setminus E)$, moreover g_u is a p-weak upper gradient of U in $\Omega \setminus E$. Let Γ_E be the set of curves passing through E. By Lemma 3.6 in Shanmugalingam [30], $\operatorname{Mod}_p(\Gamma_E) = 0$, and hence g_u is a p-weak upper gradient of U in Ω. Let $B \Subset \Omega$. Since $\mu(E) = 0$, we see that $\|U\|_{N^{1,p}(B)} = \|u\|_{N^{1,p}(B\setminus E)} < \infty$, and thus $U \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Moreover, if $u \in N^{1,p}(\Omega \setminus E)$, then $U \in N^{1,p}(\Omega)$.

We shall now show that U is a Q-quasisuperminimizer in Ω. Let $\varphi \in N^{1,p}_0(\Omega)$ be nonnegative, and let $\varphi' = \varphi\chi_{\Omega\setminus E}$. Then $\varphi' \in N^{1,p}_0(\Omega \setminus E)$. Since U is a Q-quasisuperminimizer in $\Omega \setminus E$, we see that (using characterization (c) in Proposition 3.2),

$$\int_{\varphi\neq 0} g_U^p \, d\mu = \int_{\varphi'\neq 0} g_U^p \, d\mu \le Q \int_{\varphi'\neq 0} g_{U+\varphi'}^p \, d\mu = Q \int_{\varphi\neq 0} g_{U+\varphi}^p \, d\mu.$$

Thus U is a Q-quasisuperminimizer in Ω, and since U is lower semicontinuously regularized, U is Q-quasisuperharmonic in Ω.

That (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (c) hold are clear.

Let us consider the following condition:

(e) u is bounded.

We next want to show that (e) $\Rightarrow$ (c). Assume therefore that (e) holds. Let $2B \Subset \Omega$ and $\eta \in \operatorname{Lip}_c(2B)$ be such that $0 \le \eta \le 1$ in Ω and $\eta = 1$ in B. Since $E \cap \overline{2B}$ is compact and has zero capacity, there exists a sequence $\psi_j \in \operatorname{Lip}_c(X)$ such that $\|\psi_j\|_{N^{1,p}(X)} \to 0$, $0 \le \psi_j \le 1$ in X and $\psi_j = 1$ on $E \cap \overline{2B}$, see Theorem 1.1 in Kallunki–Shanmugalingam [17] and Proposition 4.4 in Heinonen–Koskela [13]. We may also assume that $\psi_j \to 0$ μ-a.e. Let $\eta_j = \eta(1-2\psi_j)_+$. Then $\eta_j \in \operatorname{Lip}_c(\Omega \setminus E)$, $0 \le \eta_j \le 1$ and $\eta_j \to \eta$ in $N^{1,p}(X)$. Let $v = -u$. Without loss of generality $\frac{1}{2} \le v \le 1$. Using the Caccioppoli type estimate (Theorem 4.1) we see that

$$\int_{B\setminus E} g_u^p \eta_j^p \, d\mu \le \int_{\Omega\setminus E} g_v^p \eta_j^p \, d\mu \le c \int_{\Omega\setminus E} v^p g_{\eta_j}^p \, d\mu \le c \int_{\Omega\setminus E} g_{\eta_j}^p \, d\mu.$$

By Fatou's lemma

$$\int_{B\setminus E} g_u^p \, d\mu \le \liminf_{j\to\infty} \int_{B\setminus E} g_u^p \eta_j^p \, d\mu \le c \liminf_{j\to\infty} \int_{\Omega\setminus E} g_{\eta_j}^p \, d\mu = c \int_{\Omega\setminus E} g_\eta^p \, d\mu < \infty.$$

Since u is bounded we conclude that $u \in N^{1,p}(B \setminus E)$, and thus (c) holds.

Assume now that (d) holds. As we have shown that E is uniquely removable for bounded Q-quasisuperharmonic functions on $\Omega \setminus E$, it follows from Proposition 6.4 that u has a Q-quasisuperharmonic extension to Ω, which must equal U. □

In order to obtain Theorem 6.2 from Theorem 6.3 we formulate the following result. In view of Problem 9.2 we make it more general than what is actually needed in this section.

Proposition 6.5. *Assume that $\mu(E)=0$ and that u is a Q-quasiharmonic function in $\Omega\setminus E$, which has a Q-quasisuperharmonic extension U and a Q-quasisubharmonic extension V to Ω. Then $U=V$ is both unique and Q-quasiharmonic in Ω. If u is bounded, then U is also bounded, and if $u\in N^{1,p}(\Omega\setminus E)$, then $U\in N^{1,p}(\Omega)$.*

Here V is Q-*quasisubharmonic* if $-V$ is Q-quasisuperharmonic.

Proof. Since U is lower semicontinuously regularized and $\mu(E)=0$, we have

$$U(x)=\operatorname*{ess\,lim\,inf}_{\Omega\ni y\to x} U(y)=\operatorname*{ess\,lim\,inf}_{\Omega\setminus E\ni y\to x} u(y), \quad x\in\Omega.$$

Thus U is unique. Moreover if u is bounded, then U is bounded.

Let $B\Subset\Omega$. As V is upper semicontinuous in Ω it is bounded from above in B. It follows that u is bounded from above in $B\setminus E$, and hence also U is bounded from above in B. By Theorem 7.3 in Kinnunen–Martio [23], U is a Q-quasisuperminimizer in Ω and $U\in N^{1,p}_{\mathrm{loc}}(\Omega)$. Similarly $V\in N^{1,p}_{\mathrm{loc}}(\Omega)$ is a Q-quasisubminimizer in Ω. Since $U=V$ μ-a.e. in Ω, Corollary 3.3 in Shanmugalingam [30] shows that $U=V$ q.e. in Ω. Thus U is also a Q-quasisubminimizer in Ω. Hence U is a Q-quasiminimizer in Ω, and there exists a Q-quasiharmonic function W such that $W=U=u$ μ-a.e. in Ω. Since both W and u are continuous in $\Omega\setminus E$, they coincide in $\Omega\setminus E$. By uniqueness, $W=U=V$ in Ω.

Since $\Omega\setminus E$ is open, the minimal p-weak upper gradient g_U of U in Ω is also minimal as a p-weak upper gradient in $\Omega\setminus E$. Hence $\|U\|_{N^{1,p}(\Omega)}=\|u\|_{N^{1,p}(\Omega\setminus E)}$. □

Proof of Theorem 6.2. This now follows directly by combining Theorem 6.3 and Proposition 6.5. □

We end this section by observing that removability for quasi(super)harmonic functions is the same as removability for quasi(super)minimizers. By saying that removability is unique for Q-quasi(super)minimizers we mean that the extensions are unique up to capacity zero.

Proposition 6.6. *The set E is* (*uniquely*) *removable for bounded Q-quasi-*(*super*)*minimizers in $\Omega\setminus E$ if and only if E is* (*uniquely*) *removable for bounded Q-quasi*(*super*)*harmonic functions in $\Omega\setminus E$.*

This result can be combined with Proposition 6.4. There is also a similar result showing that removability for Q-quasi(super)minimizers in $N^{1,p}$ is the same as removability for Q-quasi(super)harmonic functions in $N^{1,p}$.

Proof. Assume first that E is removable for bounded Q-quasi(super)minimizers in $\Omega \setminus E$, and that u is a bounded Q-quasi(super)harmonic function in $\Omega \setminus E$. Then u is a Q-quasi(super)minimizer in $\Omega \setminus E$, and has a bounded Q-quasi(super)minimizer extension U to Ω. Let U^* be the lower semicontinuous regularization of U, a Q-quasi(super)harmonic function in Ω. Since both U^* and u are lower semicontinuously regularized in $\Omega \setminus E$ and $U^* = u$ q.e. in $\Omega \setminus E$, we see that $U^* = u$ everywhere in $\Omega \setminus E$.

Conversely, assume that E is removable for bounded Q-quasi(super)harmonic functions in $\Omega \setminus E$, and that u is a bounded Q-quasi(super)minimizer in $\Omega \setminus E$. Let u^* be the lower semicontinuous regularization of u in $\Omega \setminus E$. Then u^* is Q-quasi(super)harmonic in $\Omega \setminus E$ and has a bounded Q-quasi(super)harmonic extension U^* to Ω. Thus

$$U = \begin{cases} U^*, & \text{in } E, \\ u, & \text{in } \Omega \setminus E, \end{cases}$$

is a Q-quasi(super)minimizer extending u to Ω.

If the removability is not unique in one case, then the constructions above can be used to obtain two extensions which are different on a set of positive capacity from which the nonuniqueness of the other case follows. □

7. Nonremovability

Let from now on $K \subset \Omega$ be compact, and recall that $E \subsetneq \Omega$ is assumed to be relatively closed and such that no component of Ω is completely contained in E.

For unbounded Ω the Dirichlet problem needs to be further studied (with or without a boundary condition at ∞). Without such a theory we are able to give one nonremovability result which holds for unbounded domains.

Proposition 7.1. *Assume that for some component $G \subset \Omega$ the set $G \setminus K$ is disconnected. Then there is a bounded p-harmonic function in $\Omega \setminus K$ with no quasisuperharmonic extension to Ω.*

Note that by Example 10.2 there are relatively closed removable sets $E \subset \Omega$ for which Ω is connected and $\Omega \setminus E$ is disconnected.

Proof. Let $u \equiv 1$ in one component of $G \setminus K$ and 0 otherwise. Then u is a bounded p-harmonic function in $\Omega \setminus K$. Assume that it has a quasisuperharmonic extension U to Ω. Since U is lower semicontinuous there is $x \in G \cap K$ such that $U(x) = \inf_{G \cap K} U$. Thus

$$\inf_G U = \min\Big\{U(x), \inf_{G \setminus K} u\Big\} = \min\{U(x), 0\}$$

is attained at some point in G. By the minimum principle, U is constant in G. But this contradicts the fact that u is nonconstant in $G \setminus K$. □

In the rest of this section we will consider bounded open sets Ω.

Proposition 7.2. *Assume that Ω is bounded, $C_p(X \setminus \Omega) > 0$ and $\mu(E) = 0 < C_p(K \cup E)$. Then there is a bounded p-harmonic function in $N^{1,p}\bigl(\Omega \setminus (K \cup E)\bigr)$ with no quasisuperharmonic extension to Ω.*

When we just consider quasiharmonic extensions we can be slightly more general.

Proposition 7.3. *Assume that Ω is bounded, $C_p(X \setminus \Omega) > 0$, E has empty interior and $C_p(K \cup E) > 0$. Then there is a bounded p-harmonic function in $N^{1,p}\bigl(\Omega \setminus (K \cup E)\bigr)$ with no quasiharmonic extension to Ω.*

Note that in neither Proposition 7.2 nor 7.3 the requirement $C_p(X \setminus \Omega) > 0$ can be omitted, not even when $E = \emptyset$, see Example 9.3.

In the case when Ω is bounded and $C_p(X \setminus \Omega) = 0$ we have the following result. (Observe that in this case $X = \overline{\Omega}$ is bounded and Ω is connected.)

Proposition 7.4. *Assume that Ω is bounded, $C_p(X \setminus \Omega) = 0$ and $C_p(E) > 0$. Assume that either*

(a) *the capacity of $\Omega \cap \partial E$ is not concentrated to one point, i.e. that for every $x \in \Omega \cap \partial E$ we have $C_p\bigl((\Omega \cap \partial E) \setminus \{x\}\bigr) > 0$, or*
(b) *$\Omega \setminus E$ is disconnected.*

Then there exists a bounded p-harmonic function in $N^{1,p}(\Omega \setminus E)$ with no bounded quasisuperharmonic extension to Ω, nor any quasisuperharmonic extension in $N^{1,p}(\Omega)$.

When the (positive) capacity of E is concentrated to one point and $\Omega \setminus E$ is connected, it is possible both to have removability and nonremovability for quasiharmonic functions, see Examples 7.7 and 9.3.

When singleton sets have zero capacity we can say more.

Proposition 7.5. *Assume that Ω is bounded and $C_p(E) > 0$. If $C_p(\{x\}) = 0$ for all $x \in \Omega \cap \partial E$, then there is a bounded p-harmonic function in $N^{1,p}(\Omega \setminus E)$ that has no quasiharmonic extension to Ω.*

To prove these propositions we need the following lemma.

Lemma 7.6 (Björn–Björn–Shanmugalingam [10, Lemma 8.6]). *If Ω is a connected set and $C_p(E) > 0$, then $C_p(\Omega \cap \partial E) > 0$.*

A standing assumption in [10] was that Ω is a nonempty bounded connected open set with $C_p(X \setminus \Omega) > 0$. However, the proof of this lemma in [10] does not use the boundedness nor the assumption $C_p(X \setminus \Omega) > 0$, and thus the lemma holds as stated here.

Proof of Proposition 7.2. Since $C_p(K \cup E) > 0$ we can find a compact set K' such that $K \subset K' \subset K \cup E$ and $C_p(K') > 0$. Observe that $f(x) = \min\{\operatorname{dist}(K',x)/\operatorname{dist}(K',X \setminus \Omega),1\}$ is a Lipschitz function. Let $u = H_{\Omega \setminus K'} f$. Then u is p-harmonic in $\Omega \setminus K'$, $u = 0$ on K' and $u = 1$ on $X \setminus \Omega$. Note that $0 \le u \le 1$.

Suppose there is a quasisuperharmonic function U in Ω such that $U = u$ in the open set $\Omega \setminus (K \cup E)$. The continuity of u and the lower semicontinuous regularity of U together with the condition $\mu(E) = 0$ imply that $U = u$ in $\Omega \setminus K \supset \Omega \setminus K'$.

Since $u \ge 0$, it follows that $U \ge 0$ in Ω, by the minimum principle. Lemma 7.6 implies that $C_p(\partial K') > 0$ and by the Kellogg property (Theorem 5.3), there exists a regular point $x_0 \in \partial K'$, i.e.

$$\lim_{\Omega \setminus K' \ni y \to x_0} U(y) = \lim_{\Omega \setminus K' \ni y \to x_0} u(y) = u(x_0) = 0.$$

By the lower semicontinuity of U, $U(x_0) = 0$. The minimum principle implies that U is constant in the component $G \subset \Omega$ containing x_0. Since $C_p(X \setminus G) \ge C_p(X \setminus \Omega) > 0$, Lemma 7.6 and the Kellogg property (Theorem 5.3) however imply that U is nonconstant in G, a contradiction. $\square$

Proof of Proposition 7.3. This proof is almost identical to the proof of Proposition 7.2, we only need to modify the second paragraph a little as follows: Suppose there is a quasiharmonic function U in Ω such that $U = u$ in the open set $\Omega \setminus (K \cup E)$. The continuity of u and U together with the condition that E has no interior imply that $U = u$ in $\Omega \setminus K \supset \Omega \setminus K'$. The rest of the proof is identical to the proof of Proposition 7.2. $\square$

Proof of Proposition 7.4. In case (i), it follows that we can find two disjoint compact sets $K_1, K_2 \subset \Omega \cap \partial E$ with positive capacity. Let $f = 1$ on K_1, $f = 0$ on K_2 and let $u = H_{X \setminus (K_1 \cup K_2)} f$. By the Kellogg property there are regular points $x_1 \in \partial K_1$ and $x_2 \in \partial K_2$. It follows that u is nonconstant.

In case (ii), we let $u \equiv 1$ in one component of $\Omega \setminus E$ and $u \equiv 0$ in all other components of $\Omega \setminus E$.

Thus in both cases we have a nonconstant p-harmonic function u in $\Omega \setminus E$. Assume that u has a quasisuperharmonic extension U to Ω, which is either bounded or belongs to $N^{1,p}(\Omega)$. Then, by Theorem 6.3, there is a quasisuperharmonic function V on X which is an extension of U and hence of u. By Corollary 3.5, V is constant, which contradicts the fact that u is nonconstant. $\square$

Proof of Proposition 7.5. There is a component G of Ω such that $C_p(G \cap E) > 0$. Since by Lemma 7.6 we have $C_p(G \cap \partial E) > 0$, there exists $\tau > 0$ so that $C_p(\overline{G}_\tau \cap \partial E) > 0$, where $G_\tau := \{x \in G : \operatorname{dist}(x, X \setminus G) > \tau\}$ (if $X = G$ we set $G_\tau = G$).

By the Kellogg property (Theorem 5.3) and by the fact that finite subsets of $G \cap \partial E$ have zero capacity, there exists a sequence $\{x_n\}_{n=1}^\infty$ of points in $\overline{G}_\tau \cap \partial E$ that are regular for the open set $G \setminus E$. Since $\overline{G}_\tau$ is compact, without loss of generality we may assume that this sequence converges to a point $x_\infty \in G \cap \partial E$, has no other limit points, and moreover consists of distinct points. For each x_n in this sequence, let $B_n = B(x_n, r_n)$ be a ball so that $\overline{B}_n \subset G$. We can also choose the balls B_n to be pairwise disjoint.

It follows from Theorem 1.1 in Kallunki–Shanmugalingam [17] and Proposition 4.4 in Heinonen–Koskela [13], that we can find $\varphi_n \in \operatorname{Lip}_c(B_n)$ so that $\varphi_n(x_n) = 1$, $0 \le \varphi_n \le 1$ and $\|g_{\varphi_n}\|_{L^p} < 2^{-j}$. Let

$$\Phi(x) = \sum_{n=1}^\infty \varphi_{2n}(x).$$

It is easy to see that $\Phi \in N^{1,p}(X)$ is a bounded lower semicontinuous function which is continuous apart from at x_∞. Let $u = H_{\Omega \setminus E}\Phi$. Clearly u is a bounded p-harmonic function in $N^{1,p}(\Omega \setminus E)$. We will show that u has no quasiharmonic extension to Ω.

Since Φ is continuous at x_n, we see by Corollary 7.2 in Björn–Björn–Shanmugalingam [10] that

$$\lim_{\Omega \setminus E \ni y \to x_n} u(y) = \Phi(x_n).$$

Note that $\Phi(x_n) = 1$ if n is even and $\Phi(x_n) = 0$ if n is odd. Hence as x_∞ is the limit point of the sequence $\{x_n\}_{n=1}^\infty$, we obtain a sequence $\{y_n\}_{n=1}^\infty$ in $\Omega \setminus E$ which converges to x_∞ and is such that $u(y_n) \ge \frac{3}{4}$ if n is even and $u(y_n) \le \frac{1}{4}$ if n is odd. Thus $u|_{\Omega \setminus E}$ has no continuous extension to the point $x_\infty \in \Omega \cap \partial E$. Since quasiharmonic functions are continuous, u has no quasiharmonic extension to Ω. □

Example 7.7. Let $X \subset \mathbf{R}^2$ be the unit circle with the Euclidean distance and the one-dimensional Lebesgue measure. Let $K = \{(1,0)\}$. Observe that $C_p(K) > 0$. By considering $(1,0)$ as two points we can find a nonconstant p-harmonic function u in $X \setminus K$ such that

$$\lim_{\substack{(x,y)\to(1,0)\\ y<0}} u(x,y) = 0 \quad \text{and} \quad \lim_{\substack{(x,y)\to(1,0)\\ y>0}} u(x,y) = 1.$$

Since quasisuperharmonic functions on X are constant, by Corollary 3.5, u has no quasisuperharmonic extension to X, and K is not removable.

Note that in this case no (nonempty) set is removable.

8. Characterizations of removability

Combining Theorems 6.2, 6.3 and Proposition 7.2, gives the following consequence. (Note that the condition $C_p(X \setminus \Omega)$ is essential, see Example 9.3.)

Corollary 8.1. *Assume that* Ω *is bounded and* $C_p(X \setminus \Omega) > 0$. *Then the following are equivalent*:

(a) $C_p(K) = 0$;
(b) K *is removable for bounded* Q*-quasiharmonic functions in* $\Omega \setminus K$;
(c) K *is removable for* Q*-quasiharmonic functions in* $N^{1,p}(\Omega \setminus K)$;
(d) K *is removable for bounded* Q*-quasisuperharmonic functions in* $\Omega \setminus K$;
(e) K *is removable for* Q*-quasisuperharmonic functions in* $N^{1,p}(\Omega \setminus K)$.

The following characterization is an immediate consequence of Theorem 6.2 and Proposition 7.5.

Corollary 8.2. *Assume that* Ω *is bounded and that* $C_p(\{x\}) = 0$ *for each* $x \in \Omega \cap \partial E$. *Then the following are equivalent*:

(a) $C_p(E) = 0$;
(b) E *is removable for bounded* Q*-quasiharmonic functions in* $\Omega \setminus E$;
(c) E *is removable for* Q*-quasiharmonic functions in* $N^{1,p}(\Omega \setminus E)$.

(Recall that if some component of Ω is contained in E, then E is not considered removable, by Definition 6.1.)

The following is a consequence of Theorems 6.2, 6.3 and Propositions 7.2 and 7.3.

Corollary 8.3. *Assume that* Ω *is bounded,* $C_p(X\setminus\Omega) > 0$ *and* E *has empty interior. Then the following are equivalent*:

(a) $C_p(K \cup E) = 0$;
(b) $K \cup E$ *is removable for bounded* Q*-quasiharmonic functions in* $\Omega \setminus (K \cup E)$;
(c) $K \cup E$ *is removable for* Q*-quasiharmonic functions in* $N^{1,p}\bigl(\Omega \setminus (K \cup E)\bigr)$.

If moreover $\mu(E) = 0$, *then also the following statements are equivalent to those above*:

(d) $K \cup E$ *is removable for bounded* Q*-quasisuperharmonic functions in* $\Omega\setminus (K \cup E)$;
(e) $K\cup E$ *is removable for* Q*-quasisuperharmonic functions in* $N^{1,p}\bigl(\Omega\setminus(K\cup E)\bigr)$.

9. Removable sets with positive capacity

In this section we construct some examples of sets with positive capacity which are removable for bounded p-harmonic functions. Example 9.4 is a little simpler than our first example, but we prefer to start with the following example.

Example 9.1. Let $X = \mathbf{R}$ (equipped with the Euclidean distance and the one-dimensional Lebesgue measure). In this one-dimensional setting it is easy to see that p-harmonic functions are exactly the linear functions, i.e. of the form $x \mapsto ax + b$, moreover this is true simultaneously for all p. (Observe that on bounded sets all p-harmonic functions are automatically bounded in this case.) Let $\Omega = (0, 2)$ and $E = [1, 2)$. Note that $C_p(E) > 0$ and even $\mu(E) > 0$. Thus if $f\colon \Omega \setminus E \to \mathbf{R}$ is a p-harmonic function it is linear and thus directly extends to a p-harmonic function in Ω, i.e. E is removable for p-harmonic functions. Note that the extension is unique.

Let us now look at bounded superharmonic functions. In this setting superharmonic functions are exactly the concave functions (again simultaneously for all p). Thus $f(x) = \sqrt{1 - x^2}$ is a bounded superharmonic function in $\Omega \setminus E$, and since $\lim_{x \to 1-} f'(x) = -\infty$ we cannot find a superharmonic (i.e. concave) extension of f to all of Ω. Thus E is not removable for bounded superharmonic functions.

This example shows that the sets removable for bounded p-harmonic functions do not coincide with the sets removable for bounded superharmonic functions. A natural question to ask is the following question.

Problem 9.2. If E is removable for bounded superharmonic functions in $\Omega \setminus E$, does it then follow that E is removable for bounded p-harmonic functions in $\Omega \setminus E$?

If $\mu(E) = 0$, then this follows from Proposition 6.5. Otherwise, if f is a bounded p-harmonic function in $\Omega \setminus E$, then it, of course, has a subharmonic extension and a superharmonic extension to all of Ω, but it is not clear whether these can always be made to coincide.

In the special case when $\Omega = X$ is bounded, we can find a compact set with positive capacity which is removable for quasiharmonic functions.

Example 9.3. Let $X = \Omega = [0, 1]$ with the Euclidean distance and the one-dimensional Lebesgue measure. Let $K = \{1\}$. Observe that $C_p(K) > 0$. Let u be a quasiharmonic function in $X \setminus K$. Assume that $u(a) = u(b)$ for some $0 \le a < b < 1$. Then in order to quasiminimize in (a, b), u must be constant in $[a, b]$. It follows that u is a continuous monotone function. Without loss of generality, assume that u is nondecreasing. Then $u(0) = \inf_{X \setminus K} u$, and by the minimum principle, u is constant, and thus extendible to all of X as a p-harmonic function. Thus K is removable for Q-quasiharmonic functions in $X \setminus K$.

Superharmonic functions in $X \setminus K$ are exactly concave nonincreasing functions, whereas all quasisuperharmonic functions in X are constant, by Corollary 3.5. Thus K is not removable for bounded Q-quasisuperharmonic functions in $X \setminus K$.

In this case the sets E removable for bounded Q-quasiharmonic functions in $X \setminus E$ are all the sets $[0,a]$ and $[a,1]$ for $0 \le a \le 1$, except for X itself.

Example 9.4. Let $X = [0,\infty)$ with the Euclidean distance and the one-dimensional Lebesgue measure. Let $\Omega = [0,1)$. Then by the arguments in the previous example, a relatively closed set $E \subset \Omega$ is removable for bounded Q-quasiharmonic functions in $\Omega \setminus E$ if and only if $E = [a,1)$ for some $0 < a < 1$.

10. Nonunique removability

Next we will give an example of a set which is removable for p-harmonic functions, but in which the extensions are not unique. We will consider p-harmonic functions on a graph. Such p-harmonic functions were considered by Holopainen–Soardi [15], [16] and Shanmugalingam [32]. For the reader's convenience we give a brief explanation of how p-harmonic functions are defined on graphs.

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be a connected finite or infinite graph, where $\mathscr{V}$ stands for the set of vertices and $\mathscr{E}$ the set of edges. If x and y are endpoints of an edge we say that they are *neighbours* and write $x \sim y$. Consider an edge as a geodesic open ray of length 1 between its endpoints, and let $X = \mathscr{V} \cup \bigcup_{e \in \mathscr{E}} e$ be the metric graph equipped with the one-dimensional Hausdorff measure μ.

Let $\Omega \Subset X$ be a domain and assume for simplicity that $\partial\Omega \subset \mathscr{V}$. Then u is a p-harmonic function in Ω if and only if it is linear on each edge in Ω and satisfies

$$\sum_{y \sim x} |u(y) - u(x)|^{p-2}\bigl(u(y) - u(x)\bigr) = 0 \quad \text{for all } x \in \mathscr{V} \cap \Omega. \tag{10.1}$$

Example 10.1. Consider the graph $\mathscr{G} = \bigl(\{1,2,3,4\}, \{(1,2),(1,3),(1,4)\}\bigr)$, let X be the corresponding metric graph, $\Omega = X \setminus \{2,3,4\}$ and $E = (1,3) \cup (1,4) \cup \{1\}$, i.e. $\Omega \setminus E$ is just the open edge $(1,2)$.

A p-harmonic function u in $\Omega \setminus E$ is linear and thus can be described by giving its boundary values $u(1)$ and $u(2)$. Let now

$$\begin{aligned} U_1(1) &= U_2(1) = u(1), \\ U_1(2) &= U_2(2) = u(2), \\ U_1(3) &= U_2(4) = u(1), \\ U_1(4) &= U_2(3) = 2u(1) - u(2) \end{aligned}$$

and let U_1 and U_2 be linear on the edges. It is then easy to check that U_1 and U_2 are p-harmonic in Ω (simultaneously for all p). Thus E is removable for

p-harmonic functions in $\Omega \setminus E$ (which are automatically bounded). However, the extensions are not unique in this case.

Finally note that the same argument as in Example 9.1 shows that E is not removable for bounded superharmonic functions.

In the next example we show that even if E disconnects Ω it is possible to have removability.

Example 10.2. Consider the graph

$$\mathscr{G} = \bigl(\{1, \ldots, 6\}, \{(1,2), (1,3), (1,4), (4,5), (4,6)\}\bigr),$$

let X be the corresponding metric graph, $\Omega = X \setminus \{2,3,5,6\}$ and

$$E = \{1,4\} \cup (1,3) \cup (1,4) \cup (4,6).$$

Then $\Omega \setminus E$ is disconnected and consists of the two edges $(1,2)$ and $(4,5)$.

Let u be any p-harmonic function in $\Omega \setminus E$, which can be described by its boundary values. We want to extend it to a p-harmonic function U in Ω. Again U is described by its boundary values. Apart from being linear at each edge we only need to require U to satisfy (10.1) at the internal edges, and this holds if and only if

$$\begin{aligned} 0 &= \varphi\bigl(U(2) - U(1)\bigr) + \varphi\bigl(U(3) - U(1)\bigr) + \varphi\bigl(U(4) - U(1)\bigr) \\ &= \varphi\bigl(U(1) - U(4)\bigr) + \varphi\bigl(U(5) - U(4)\bigr) + \varphi\bigl(U(6) - U(4)\bigr), \end{aligned}$$

where $\varphi(t) = |t|^{p-2}t$. Since $\varphi\colon \mathbf{R} \to \mathbf{R}$ is onto and we can freely choose $U(3)$ and $U(6)$ this can always be achieved.

We can modify this construction to get a set E that disconnects Ω and is nonuniquely removable.

Example 10.3. Consider the graph

$$\mathscr{G} = \bigl(\{1, \ldots, 7\}, \{(1,2), (1,3), (1,4), (4,5), (4,6), (4,7)\}\bigr),$$

let X be the corresponding metric graph, $\Omega = X \setminus \{2,3,5,6,7\}$ and

$$E = \{1,4\} \cup (1,3) \cup (1,4) \cup (4,6) \cup (4,7).$$

(Thus we have added the extra edge $(4,7)$ to Example 10.2.)

Arguing as in Example 10.2 we see that every p-harmonic function in $\Omega \setminus E$ is extendible to Ω. Moreover, since we now have the freedom to choose both $U(6)$ and $U(7)$ we can actually prescribe one of them arbitrarily, which shows that the removability is nonunique.

11. Properties of removable singularities

Let $E = E_1$, $E_2, \dots$, be sets of capacity zero, relatively closed in the nonempty open sets $\Omega = \Omega_1$, $\Omega_2, \dots$, respectively. Then E_j is (uniquely) removable for bounded p-harmonic functions in $\Omega_j \setminus E_j$. Moreover, we have the following properties: (Let Ω' be a nonempty open set.)

(a) (Removability is independent of the surrounding set.) If E is relatively closed in Ω', then E is uniquely removable for bounded p-harmonic functions in $\Omega' \setminus E$.

(b) (A subset of a removable set is removable.) If $E' \subset E$ and $E' \subset \Omega$ is relatively closed, then E' is uniquely removable for bounded p-harmonic functions in $\Omega \setminus E'$.

(c) (A countable union of removable sets is removable.) If $E' := \bigcup_{j=1}^\infty E_j \subset \Omega$ is relatively closed, then E' is uniquely removable for bounded p-harmonic functions in $\Omega \setminus E'$.

(d) (A generalization of (a)–(c).) If $E' \subset \bigcup_{j=1}^\infty E_j$ and $E' \subset \Omega'$ is relatively closed, then E' is uniquely removable for bounded p-harmonic functions in $\Omega' \setminus E'$.

This follows directly from the countable subadditivity of the capacity together with Theorem 6.2. The corresponding statements for Q-quasi(super)harmonic functions that are bounded or in $N^{1,p}$ follow in the same way.

For removable singularities with positive capacity the situation is quite different, even when we restrict our attention to compact removable singularities.

Let $X = [0,1]$ with the Euclidean distance and the one-dimensional Lebesgue measure, as in Example 9.3. Recall that in this case the sets E removable for bounded Q-quasiharmonic functions in $X \setminus E$ are all the sets $[0,a]$ and $[a,1]$ for $0 \le a \le 1$, except for X itself.

We get the following counterexamples corresponding to (a)–(c) above:

(a′) Let $K = \{0\}$ and $\Omega' = [0,1)$, then K is removable for bounded Q-quasiharmonic functions in $X \setminus K$, but not for bounded Q-quasiharmonic functions in $\Omega' \setminus K$.

(b′) Let $K = \left[0, \frac{1}{2}\right]$ and $K' = \left\{\frac{1}{2}\right\} \subset K$, then K is removable for bounded Q-quasiharmonic functions in $X \setminus K$, but K' is not removable for bounded Q-quasiharmonic functions in $X \setminus K'$.

(c′) Let $K_1 = \{0\}$ and $K_2 = \{1\}$, then K_j is removable for bounded Q-quasiharmonic functions in $X \setminus K_j$, $j = 1, 2$, but $K_1 \cup K_2$ is not removable for bounded Q-quasiharmonic functions in $X \setminus (K_1 \cup K_2)$.

There is however one similar property that holds also for removable singularities of positive capacity.

Proposition 11.1. *Let $E_1, E_2 \subset \Omega$ be relatively closed and such that no component of Ω is contained in $E_1 \cup E_2$. Assume that E_1 and E_2 are* *(uniquely) removable for bounded Q-quasi(super)harmonic functions in $\Omega \setminus E_1$ and $\Omega \setminus (E_1 \cup E_2)$, respectively. Then $E_1 \cup E_2$ is (uniquely) removable for bounded Q-quasi(super)harmonic functions in $\Omega \setminus (E_1 \cup E_2)$.*

It is implicitly assumed that $\Omega' := (\Omega \setminus E_1) \cup E_2$ is open.

Proof. Let u be a bounded Q-quasi(super)harmonic function in $\Omega' \setminus E_2 = \Omega \setminus (E_1 \cup E_2)$. By assumption there is a bounded Q-quasi(super)harmonic extension U' to Ω'. Thus $U'' := U'|_{\Omega \setminus E_1}$ is a bounded Q-quasi(super)harmonic extension of u to $\Omega \setminus E_1$. Since E_1 is removable, there exists a bounded Q-quasi(super)harmonic extension U of U'' to Ω. It is immediate that U is a bounded Q-quasi(super)harmonic extension of u to Ω.

In the case when E_1 and E_2 are uniquely removable we see that $U|_{\Omega'}$ is a Q-quasisuperharmonic extension of u to Ω', and hence is unique (and equal to U'). Therefore $U|_{\Omega \setminus E_1}$ is also unique (and equal to U''). Using that E_1 is uniquely removable, the uniqueness of U follows. □

It is not known if being p-harmonic is a local property, and in particular if p-harmonic functions form sheaves, i.e. p-harmonicity in both Ω_1 and Ω_2 implies p-harmonicity in $\Omega_1 \cup \Omega_2$. When p-(super)harmonic functions are defined using Cheeger's definition, see, e.g., Björn–Björn–Shanmugalingam [10], then the sheaf property is known to hold (this follows using the p-Laplace equation and a partition of unity argument, see J. Björn [11]). This means that, e.g., on graphs and in Euclidean $\mathbf{R}^n$ the sheaf property holds.

On the other hand Q-quasiharmonic functions do not form sheaves.

Proposition 11.2. *Assume that X is such that p-(super)harmonic functions form a sheaf. Assume further that Ω' is open, $E \subset \Omega' \subset \Omega$ and that E is (uniquely) removable for bounded p-(super)harmonic functions on $\Omega' \setminus E$. Then E is (uniquely) removable for bounded p-(super)harmonic functions on $\Omega \setminus E$.*

Recall that $\Omega \setminus E$ is assumed to be open.

Proof. Let u be a bounded p-harmonic function in $\Omega \setminus E$. Then u is bounded and p-harmonic in $\Omega' \setminus E$, and hence has a bounded p-harmonic extension U' to Ω'. Let now

$$U = \begin{cases} u, & \text{in } \Omega \setminus E, \\ U', & \text{in } \Omega'. \end{cases}$$

Then U is p-harmonic in Ω' as well as in $\Omega \setminus E$, and hence by the sheaf property U is p-harmonic in Ω.

If U' is unique, then any bounded p-harmonic extension U of u to Ω, must satisfy $U|_{\Omega'} = U'$, and hence U is also unique.

The corresponding result for superharmonic functions is proved similarly. □

Corollary 11.3. *Assume that X is such that p-(super)harmonic functions form a sheaf. Let $E_1, E_2 \subset \Omega$ be relatively closed and such that no component of Ω is contained in $E_1 \cup E_2$. Assume further that E_1 and E_2 are (uniquely) removable for bounded p-(super)harmonic functions on $\Omega \setminus (E_1 \cup E_2)$. Then $E_1 \cup E_2$ is (uniquely) removable for bounded p-(super)harmonic functions on $\Omega \setminus (E_1 \cup E_2)$.*

It is implicitly assumed that $(\Omega \setminus E_1) \cup E_2$ and $(\Omega \setminus E_2) \cup E_1$ are open.

Proof. By Proposition 11.2, E_1 is (uniquely) removable for bounded p-(super)harmonic functions on $\Omega \setminus E_1$. The result then follows from Proposition 11.1. □

The results corresponding to Propositions 11.1, 11.2 and Corollary 11.3 for functions in $N^{1,p}$ can be proved analogously. Propositions 11.1, 11.2 and Corollary 11.3 can also be combined with Proposition 6.4.

The author has constructed examples of removable singularities for various classes of analytic functions with a similar character to (a′)–(c′), see [1], [2] and [6], see also Hejhal [14, Example 1, p. 19].

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