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HIGHER ORDER VARIATIONAL PROBLEMS ON TWO-DIMENSIONAL DOMAINS

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Abstract. Let $u\colon \mathbf{R}^2 \supset \Omega \to \mathbf{R}^M$ denote a local minimizer of $J[w] = \int_\Omega f(\nabla^k w)\,\mathrm{d}x$, where $k \geq 2$ and $\nabla^k w$ is the tensor of all k^{th} order (weak) partial derivatives. Assuming rather general growth and ellipticity conditions for f, we prove that u actually belongs to the class $C^{k,\alpha}(\Omega;\mathbf{R}^M)$ by the way extending the result of [BF2] to the higher order case by using different methods. A major tool is a lemma on the higher integrability of functions established in [BFZ].

1. Introduction

Let Ω denote a bounded domain in $\mathbf{R}^2$ and consider a function $u\colon \Omega \to \mathbf{R}^M$ which locally minimizes the variational integral

$$J[w,\Omega] = \int_\Omega f(\nabla^k w)\,\mathrm{d}x,$$

where $\nabla^k w$ represents the tensor of all k^{th} order (weak) partial derivatives. Our main concern is the investigation of the smoothness properties of such local minimizers under suitable assumptions on the energy density f. For the first order case (i.e. $k = 1$) we have rather general results which can be found for example in the textbooks of Morrey [Mo], Ladyzhenskaya and Ural'tseva [LU], Gilbarg and Trudinger [GT] or Giaquinta [Gi], for an update of the history including recent contributions we refer to [Bi]. In order to keep our exposition simple (and only for this reason) we consider the scalar case (i.e. $M = 1$) and restrict ourselves to variational problems involving the second (generalized) derivative. Then our variational problem is related to the theory of plates: one may think of $u\colon \Omega \to \mathbf{R}$ as the displacement in vertical direction from the flat state of an elastic plate. The classical case of a potential f with quadratic growth is discussed in the monographs of Ciarlet and Rabier [CR], Necǎs and Hlávácek [NH], Chudinovich and Constanda [CC] or Friedman [Fr], further references are contained in Zeidler's book [Ze]. We also like to remark that plates with other hardening laws (logarithmic and power growth case) together with an additional obstacle have been studied in the papers [BF1] and [FLM] but not with optimal regularity results. The purpose of this

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note is to present a rather satisfying regularity theory for a quite large class of potentials allowing even anisotropic growth.

To be precise let $\mathbf{M}$ denote the space of all (2×2)-matrices and suppose that we are given a function $f\colon \mathbf{M} \to [0,\infty)$ of class C^2 which satisfies with exponents $1 < p \le q < \infty$ the anisotropic ellipticity estimate

$$\lambda(1+|\xi|^2)^{(p-2)/2}|\sigma|^2 \le D^2 f(\xi)(\sigma,\sigma) \le \Lambda(1+|\xi|^2)^{(q-2)/2}|\sigma|^2 \tag{1.1}$$

for all ξ, $\sigma \in \mathbf{M}$ with positive constants λ, Λ. Note that (1.1) implies the growth condition

$$a|\xi|^p - b \le f(\xi) \le A|\xi|^q + B \tag{1.2}$$

with suitable constants a, $A > 0$, b, $B \ge 0$. Let

$$J[w,\Omega] = \int_\Omega f(\nabla^2 w)\,\mathrm{d}x, \quad \nabla^2 w = (\partial_\alpha \partial_\beta w)_{1\le\alpha,\beta\le 2}.$$

We say that a function $u \in W^2_{p,\mathrm{loc}}(\Omega)$ is a local J-minimizer if and only if $J[u,\Omega'] < \infty$ for any subdomain $\Omega' \Subset \Omega$ and

$$J[u,\Omega'] \le J[v,\Omega']$$

for all $v \in W^2_{p,\mathrm{loc}}(\Omega)$ such that $u - v \in \overset{\circ}{W}{}^2_p(\Omega')$ (here $W^k_{p,\mathrm{loc}}(\Omega)$ etc. denote the standard Sobolev spaces, see [Ad]). Note that (1.1) implies the strict convexity of f. Therefore, given a function $u_0 \in W^2_q(\Omega)$, the direct method ensures the existence of a unique J-minimizer u in the class

$$\big\{ v \in W^2_p(\Omega) : J[v,\Omega] < \infty,\ v - u_0 \in \overset{\circ}{W}{}^2_p(\Omega) \big\}$$

which motivates the discussion of local J-minimizers. Our main result reads as follows:

Theorem 1.1. *Let u denote a local J-minimizer under condition* (1.1). *Assume further that*

$$q < \min(2p, p+2) \tag{1.3}$$

holds. Then u is of class $C^{2,\alpha}(\Omega)$ for any $0 < \alpha < 1$.

Remark 1.1. (i) Clearly the result of Theorem 1.1 extends to local minimizers of the variational integral

$$I[w,\Omega] = \int_\Omega f(\nabla^2 w)\,\mathrm{d}x + \int_\Omega g(\nabla w)\,\mathrm{d}x,$$

where f is as before and where g denotes a density of class C^2 satisfying

$$0 \le D^2 g(\xi)(\eta,\eta) \le c(1+|\xi|^2)^{(s-2)/2}|\eta|^2$$

for some suitable exponent s. In case $p \ge 2$ any finite number is admissible for s, in case $p < 2$ we require the bound $s \le 2p/(2-p)$. The details are left to the reader.

(ii) Without loss of generality we may assume that $q \ge 2$: if (1.1) holds with some exponent $q < 2$, then of course (1.1) is true with q replaced by $\bar{q} := 2$ and (1.3) continues to hold for the new exponent.

(iii) If we consider the higher order variational integral $\int_\Omega f(\nabla^k w)\,\mathrm{d}x$ with $k \ge 2$ and f satisfying (1.1), then (1.3) implies that local minimizers $u \in W^k_{p,\mathrm{loc}}(\Omega)$ actually belong to the space $C^{k,\alpha}(\Omega)$.

(iv) The degree of smoothness of u can be improved by standard arguments provided f is sufficiently regular.

(v) A typical example of an energy J satisfying the assumptions of Theorem 1.1 is given by

$$J[w,\Omega] = \int_\Omega |\nabla^2 w|^2\,\mathrm{d}x + \int_\Omega (1+|\partial_1\partial_2 w|^2)^{q/2}\,\mathrm{d}x$$

with some exponent $q \in (2,4)$.

(vi) Our arguments can easily be adjusted to prove $C^{k,\alpha}$-regularity of local minimizers $u \in W^k_{p(x),\mathrm{loc}}(\Omega)$ of the energy $\int_\Omega (1+|\nabla^k w|^2)^{p(x)/2}\,\mathrm{d}x$ provided that $1 < p_* \le p(x) \le p^* < \infty$ for some numbers p_*, p^* and if $p(x)$ is sufficiently smooth. Another possible extension concerns the logarithmic case, i.e. we now consider the variational integral $\int_\Omega |\nabla^k w| \ln(1+|\nabla^k w|)\,\mathrm{d}x$ and its local minimizers which have to be taken from the corresponding higher order Orlicz–Sobolev space.

The proof of Theorem 1.1 is organized as follows: we first introduce some suitable regularization and then prove the existence of higher order weak derivatives for this approximating sequence in Step 2. Here we also derive a Caccioppoli-type inequality using difference quotient methods. In a third step we deduce uniform higher integrability of the second generalized derivatives for any finite exponent. From this together with a lemma established in [BFZ] we finally obtain our regularity result in the last two steps.

2. Proof of Theorem 1.1

Step 1. *Approximation.* Let us fix some open domains $\Omega_1 \Subset \Omega_2 \Subset \Omega$ and denote by $\bar{u}_m$ the mollification of u with radius $1/m$, in particular

$$\|\bar{u}_m - u\|_{W^2_p(\Omega_2)} \overset{m\to\infty}{\longrightarrow} 0.$$

Jensen's inequality implies

$$J[\bar{u}_m, \Omega_2] \leq J[u, \Omega_2] + \tau_m,$$

where $\tau_m \to 0$ as $m \to \infty$. This, together with the lower semicontinuity of the functional J, shows that

$$J[\bar{u}_m, \Omega_2] \overset{m\to\infty}{\longrightarrow} J[u, \Omega_2]. \tag{2.1}$$

Next let

$$\varrho_m := \|\bar{u}_m - u\|_{W_p^2(\Omega_2)} \left[\int_{\Omega_2} (1 + |\nabla^2 \bar{u}_m|^2)^{q/2} \, \mathrm{d}x \right]^{-1},$$

which obviously tends to 0 as $m \to \infty$. With these preliminaries we introduce the regularized functional

$$J_m[w, \Omega_2] := \varrho_m \int_{\Omega_2} (1 + |\nabla^2 w|^2)^{q/2} \, \mathrm{d}x + J[w, \Omega_2]$$

and the corresponding regularizing sequence $\{u_m\}$ as the sequence of the unique solutions to the problems

$$J_m[\,\cdot\,, \Omega_2] \to \min \quad \text{in } \bar{u}_m + \overset{\circ}{W}{}_q^2(\Omega_2). \tag{2.2}$$

By (2.1) and (2.2) we have

$$\begin{aligned} J_m[u_m, \Omega_2] &\leq J_m[\bar{u}_m, \Omega_2] \\ &= \|\bar{u}_m - u\|_{W_p^2(\Omega_2)} + J[\bar{u}_m, \Omega_2] \overset{m\to\infty}{\longrightarrow} J[u, \Omega_2], \end{aligned}$$

hence one gets

$$\limsup_{m\to\infty} J_m[u_m, \Omega_2] \leq J[u, \Omega_2]. \tag{2.3}$$

On account of (2.3) and the growth of f we may assume

$$u_m \overset{m\to\infty}{\rightharpoondown} : \hat{u} \quad \text{in } W_p^2(\Omega_2).$$

Moreover, lower semicontinuity gives

$$J[\hat{u}, \Omega_2] \leq \liminf_{m\to\infty} J[u_m, \Omega_2],$$

which together with (2.3) and the strict convexity of f implies $\hat{u} = u$ (here we also note that $\hat{u} - u \in \mathring{W}^2_p(\Omega_2)$). Summarizing the results it is shown up to now that (as $m \to \infty$)

$$
\begin{aligned}
u_m &\rightharpoondown u \quad \text{in } W^2_p(\Omega_2),\\
J_m[u_m, \Omega_2] &\to J[u, \Omega_2].
\end{aligned}
\tag{2.4}
$$

Step 2. Existence of higher order weak derivatives. In this second step we will prove that ($f_m(\xi) := \varrho_m(1+|\xi|^2)^{q/2} + f(\xi)$)

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x\\
&\quad \le c(\|\nabla\eta\|^2_\infty + \|\nabla^2\eta\|^2_\infty)\int_{\operatorname{spt}\nabla\eta} |D^2 f_m(\nabla^2 u_m)|\big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\big]\,\mathrm{d}x,
\end{aligned}
\tag{2.5}
$$

where $\eta \in C^\infty_0(\Omega_2)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on Ω_1 and where we take the sum over repeated indices. To this purpose let us recall the Euler equation

$$
\int_{\Omega_2} Df_m(\nabla^2 u_m) : \nabla^2\varphi = 0 \quad \text{for all } \varphi \in \mathring{W}^2_q(\Omega_2).
\tag{2.6}
$$

If Δ_h denotes the difference quotient in the coordinate direction e_α, $\alpha = 1, 2$, then the test function $\Delta_{-h}(\eta^6 \Delta_h u_m)$ is admissible in (2.6) with the result

$$
\int_{\Omega_2} \Delta_h\{Df_m(\nabla^2 u_m)\} : \nabla^2(\eta^6 \Delta_h u_m)\,\mathrm{d}x = 0.
\tag{2.7}
$$

Now denote by $\mathscr{B}_x$ the bilinear form

$$
\mathscr{B}_x = \int_0^1 D^2 f_m\big(\nabla^2 u_m(x) + th\nabla^2(\Delta_h u_m)(x)\big)\,\mathrm{d}t,
$$

and observe that

$$
\begin{aligned}
\Delta_h\{Df_m(\nabla^2 u_m)\}(x) &= \frac{1}{h}\int_0^1 \frac{d}{dt} Df_m\Big(\nabla^2 u_m(x)\\
&\qquad + t[\nabla^2 u_m(x + he_\alpha) - \nabla^2 u_m(x)]\Big)\,\mathrm{d}t\\
&= \frac{1}{h}\int_0^1 \frac{d}{dt} Df_m\big(\nabla^2 u_m(x) + ht\nabla^2(\Delta_h u_m)(x)\big)\,\mathrm{d}t\\
&= \mathscr{B}_x\big(\nabla^2(\Delta_h u_m)(x), \cdot\big),
\end{aligned}
$$

hence (2.7) can be written as

$$\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\eta^6 \Delta_h u_m)\big)\,\mathrm{d}x = 0,$$

which means that we have

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\qquad = -\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2\eta^6 \Delta_h u_m\big)\,\mathrm{d}x \\
&\qquad\quad - 2\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla\eta^6 \odot \nabla(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\qquad =: -T_1 - 2T_2.
\end{aligned} \tag{2.8}$$

To handle T_1 we just observe $\partial_\alpha\partial_\beta\eta^6 = 30\partial_\alpha\eta\partial_\beta\eta\eta^4 + 6\partial_\alpha\partial_\beta\eta\eta^5$, for T_2 we use $\nabla\eta^6 = 6\eta^5\nabla\eta$. The Cauchy–Schwarz inequality for the bilinear form $\mathscr{B}_x$ implies

$$\begin{aligned}
|T_2| &= 6\left|\int_{\Omega_2} \mathscr{B}_x\big(\eta^3\nabla^2(\Delta_h u_m), \eta^2\nabla\eta \odot \nabla(\Delta_h u_m)\big)\,\mathrm{d}x\right| \\
&\le 6\left[\int_{\Omega_2} \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\eta^6\,\mathrm{d}x\right]^{1/2} \\
&\quad \times \left[\int_{\Omega_2} \mathscr{B}_x\big(\nabla\eta \odot \nabla(\Delta_h u_m), \nabla\eta \odot \nabla(\Delta_h u_m)\big)\eta^4\,\mathrm{d}x\right]^{1/2},
\end{aligned}$$

an analogous estimate being valid for T_1. Absorbing terms, (2.8) turns into

$$\begin{aligned}
&\int_{\Omega_2} \eta^6 \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\qquad \le c(\|\nabla\eta\|_\infty^2 + \|\nabla^2\eta\|_\infty^2) \int_{\mathrm{spt}\nabla\eta} |\mathscr{B}_x|\big(|\nabla(\Delta_h u_m)|^2 + |\Delta_h u_m|^2\big)\,\mathrm{d}x.
\end{aligned} \tag{2.9}$$

Next we estimate (note that in the following calculations we always assume, without loss of generality, $q \ge 2$, compare Remark 1.1(ii)) for h sufficiently small

$$\begin{aligned}
&\int_{\mathrm{spt}\nabla\eta} |\mathscr{B}_x|\,|\nabla(\Delta_h u_m)|^2\,\mathrm{d}x \\
&\qquad \le \int_{\mathrm{spt}\nabla\eta} \big(1 + |\nabla^2 u_m|^2 + h^2|\nabla^2(\Delta_h u_m)|^2\big)^{(q-2)/2} |\nabla(\Delta_h u_m)|^2\,\mathrm{d}x \\
&\qquad \le c\left[\int_{\mathrm{spt}\nabla\eta} |\nabla(\Delta_h u_m)|^{q/2}\,\mathrm{d}x\right. \\
&\qquad\quad \left. + \int_{\mathrm{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2 + h^2|\nabla^2(\Delta_h u_m)|^2)^{q/2}\,\mathrm{d}x\right] \\
&\qquad \le c\int_{\mathrm{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x.
\end{aligned}$$

In a similar way we estimate $\int_{\mathrm{spt}\nabla\eta} |\mathscr{B}_x|\,|\Delta_h u_m|^2\,\mathrm{d}x$ and end up with

$$
\begin{aligned}
&\limsup_{h\to 0} \int_{\Omega_2} \eta^6 \mathscr{B}_x\big(\nabla^2(\Delta_h u_m), \nabla^2(\Delta_h u_m)\big)\,\mathrm{d}x \\
&\qquad \le c(\|\nabla\eta\|_\infty^2 + \|\nabla^2\eta\|_\infty^2) \int_{\mathrm{spt}\nabla\eta} (1+|\nabla u_m|^2 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x.
\end{aligned}
\tag{2.10}
$$

Since $q \ge 2$ is assumed, (2.10) implies that $\nabla^2 u_m \in W^1_{2,\mathrm{loc}}(\Omega_2)$ and

$$
\Delta_h(\nabla^2 u_m) \overset{h\to 0}{\longrightarrow} \partial_\alpha(\nabla^2 u_m) \quad \text{in } L^2_{\mathrm{loc}}(\Omega_2) \text{ and a.e.}
$$

Remark 2.1. With (2.10) we have

$$
|\Delta_h\{Df_m(\nabla^2 u_m)\}|^{q/(q-1)} \in L^1_{\mathrm{loc}}(\Omega_2) \text{ uniformly with regard to } h,
$$

and, as a consequence,

$$
Df_m(\nabla^2 u_m) \in W^1_{q/(q-1),\mathrm{loc}}(\Omega_2).
$$

This follows exactly as outlined in the calculations after (3.12) of [BF3].

With the above convergences and Fatou's lemma we find the lower bound

$$
\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x
$$

for the left-hand side of (2.10) which gives using (1.1)

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 (1+|\nabla^2 u_m|^2)^{(p-2)/2} |\nabla^3 u_m|^2\,\mathrm{d}x \\
&\qquad \le c(\|\nabla\eta\|_\infty^2 + \|\nabla^2\eta\|_\infty^2) \int_{\mathrm{spt}\nabla\eta} (1+|\nabla u_m|^2 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x < \infty,
\end{aligned}
$$

in particular

$$
h_m := (1+|\nabla^2 u_m|^2)^{p/4} \in W^1_{2,\mathrm{loc}}(\Omega_2). \tag{2.11}
$$

But (2.11) implies $h_m \in L^r_{\mathrm{loc}}(\Omega_2)$ for any $r < \infty$, i.e.

$$
\nabla^2 u_m \in L^t_{\mathrm{loc}}(\Omega_2) \quad \text{for any } t < \infty. \tag{2.12}
$$

Using Fatou's lemma again we obtain from (2.8)

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\
(2.13)\quad &\le \liminf_{h\to 0} \int_{\Omega_2} \eta^6 \Delta_h\{Df_m(\nabla^2 u_m)\} : \nabla^2(\Delta_h u_m)\,\mathrm{d}x \\
&= \liminf_{h\to 0} -\int_{\Omega_2} \Delta_h\{Df_m(\nabla^2 u_m)\} : [\nabla^2\eta^6 \Delta_h u_m + 2\nabla\eta^6 \odot \nabla(\Delta_h u_m)]\,\mathrm{d}x.
\end{aligned}
$$

On account of (2.12), Remark 2.1 and Vitali's convergence theorem we may pass to the limit $h \to 0$ on the right-hand side of (2.13) and obtain

$$
\begin{aligned}
&\int_{\Omega_2} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\
&\qquad \le -\int_{\Omega_2} D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \nabla^2\eta^6 \partial_\alpha u_m + 2\nabla\eta^6 \odot \nabla\partial_\alpha u_m)\,\mathrm{d}x.
\end{aligned}
$$

This immediately gives (2.5) by repeating the calculations leading from (2.8) to (2.9).

Step 3. Uniform higher integrability of $\nabla^2 u_m$. Let χ denote any real number satisfying $\chi > p/(2p-q)$, moreover we set $\alpha = \chi p/2$. For all discs $B_r \Subset B_R \Subset \Omega_2$ any $\eta \in C_0^\infty(B_R)$, $\eta \equiv 1$ on B_r, $|\nabla^k \eta| \le c/(R-r)^k$, $k = 1, 2$, we have by Sobolev's inequality

$$
\int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha\,\mathrm{d}x \le \int_{B_R} (\eta^3 h_m)^{2\chi}\,\mathrm{d}x \le c\Big[\int_{B_R} |\nabla(\eta^3 h_m)|^t\,\mathrm{d}x\Big]^{2\chi/t},
$$

where $t \in (1,2)$ satisfies $2\chi = 2t/(2-t)$. Hölder's inequality implies

$$
\begin{aligned}
\int_{B_r} (1 + |\nabla^2 u_m|^2)^\alpha\,\mathrm{d}x &\le c(r,R)\Big[\int_{B_R} |\nabla(\eta^3 h_m)|^2\,\mathrm{d}x\Big]^\chi \\
&\le c(r,R)\Big[\int_{B_R} \eta^6 |\nabla h_m|^2\,\mathrm{d}x + \int_{\mathrm{spt}\nabla\eta} |\nabla\eta^3|^2 h_m^2\,\mathrm{d}x\Big]^\chi.
\end{aligned}
$$

Observing that obviously

$$
\int_{\mathrm{spt}\nabla\eta} |\nabla\eta^3|^2 h_m^2\,\mathrm{d}x \le c(r,R) \int_{\mathrm{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{p/2}\,\mathrm{d}x
$$

and that by (2.5)

$$
\begin{aligned}
\int_{B_R} \eta^6 |\nabla h_m|^2\,\mathrm{d}x &\le c(r,R) \int_{\mathrm{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{(q-2)/2}\big[|\nabla^2 u_m|^2 + |\nabla u_m|^2\big]\,\mathrm{d}x \\
&\le c(r,R)\Big[\int_{\mathrm{spt}\nabla\eta} (1 + |\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x + \int_{\mathrm{spt}\nabla\eta} |\nabla u_m|^q\,\mathrm{d}x\Big],
\end{aligned}
$$

we deduce

$$\int_{B_r}(1+|\nabla^2 u_m|^2)^\alpha \,\mathrm{d}x \le c(r,R)\Big[\int_{\operatorname{spt}\nabla\eta}(1+|\nabla^2 u_m|^2)^{q/2}\,\mathrm{d}x + \int_{\operatorname{spt}\nabla\eta}|\nabla u_m|^q\,\mathrm{d}x\Big]^\chi, \tag{2.14}$$

where $c(r,R) = c(R-r)^{-\beta}$ for some suitable $\beta > 0$. For discussing (2.14) we first note that the term $\int_{\operatorname{spt}\nabla\eta}|\nabla u_m|^q\,\mathrm{d}x$ causes no problems. In fact, since $\|u_m\|_{W^2_p(\Omega_2)} \le c < \infty$ we know that $\nabla u_m \in L^t_{\mathrm{loc}}(\Omega_2)$ for any $t < \infty$ in case $p \ge 2$. If $p < 2$, then we have local L^t-integrability of ∇u_m provided that $t < 2p/(2-p)$, but $q < 2p < 2p/(2-p)$ on account of (1.3). As a consequence, we may argue exactly as in [ELM] or [Bi, p. 60], to derive from (2.14) by interpolation and hole-filling (here $q < 2p$ enters in an essential way)

$$\nabla^2 u_m \in L^t_{\mathrm{loc}}(\Omega_2) \quad \text{for any } t<\infty \text{ and uniformly with regard to } m. \tag{2.15}$$

Note that (2.15) implies with Step 2 the uniform bound

$$\int_{\Omega_2}\eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha\nabla^2 u_m, \partial_\alpha\nabla^2 u_m)\,\mathrm{d}x \le c(\eta) < \infty, \tag{2.16}$$

in particular (2.16) shows

$$h_m \in W^1_{2,\mathrm{loc}}(\Omega_2) \quad \text{uniformly with regard to } m. \tag{2.17}$$

Remark 2.2. (i) If u is a local J-minimizer subject to an additional constraint of the form $u \ge \psi$ a.e. on Ω for a sufficiently regular function $\psi\colon \Omega \to \mathbf{R}$, then it is an easy exercise to adjust the technique used in [BF1] to the present situation which means that we still have (2.15) so that (recall (2.4)) $u \in W^2_{t,\mathrm{loc}}(\Omega)$ for any $t < \infty$, hence $u \in C^{1,\alpha}(\Omega)$ for all $0 < \alpha < 1$. In [Fr, Theorem 10.6, p. 98], it is shown for the special case $f(w) = |\Delta w|^2$ that actually $u \in C^2(\Omega)$ is true, and it would be interesting to see if this result also holds for the energy densities discussed here.

(ii) We remark that the proof of (2.15) just needs the inequality $q < 2p$, whereas the additional assumption $q < p+2$ enters in the next step.

Step 4. C^2-regularity. Now we consider an arbitrary disc $B_{2R} \Subset \Omega_1$ and $\eta \in C^\infty_0(B_{2R})$ satisfying $\eta \equiv 1$ on B_R and $|\nabla\eta| \le c/R$, $|\nabla^2\eta| \le c/R^2$. Moreover we denote by T_{2R} the annulus $T_{2R} := B_{2R} - B_R$ and by P_m a polynomial function of degree less than or equal to 2. Exactly as in Step 2 (replacing u_m by $u_m - P_m$) we obtain

$$\begin{aligned}\int_{B_{2R}} \eta^6 D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\,\mathrm{d}x \\ \leq -\int_{T_{2R}} D^2 f_m(\nabla^2 u_m)\Big(\partial_\alpha \nabla^2 u_m, \nabla^2 \eta^6 \partial_\alpha [u_m - P_m] \\ + 2\nabla \eta^6 \odot \nabla \partial_\alpha (u_m - P_m)\Big)\,\mathrm{d}x.\end{aligned}$$

With the notation

$$H_m := \Big[D^2 f_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m)\Big]^{1/2}, \quad \sigma_m := D f_m(\nabla^2 u_m)$$

we therefore have

$$\int_{B_{2R}} \eta^6 H_m^2\,\mathrm{d}x \leq c \int_{T_{2R}} |\nabla \sigma_m| \big[|\nabla^2 \eta^6|\,|\nabla u_m - \nabla P_m| + |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\big]\,\mathrm{d}x.$$

Moreover, by the Cauchy–Schwarz inequality and (1.1)

$$|\nabla \sigma_m|^2 \leq H_m \big[D^2 f_m(\nabla^2 u_m)(\partial_\alpha \sigma_m, \partial_\alpha \sigma_m)\big]^{1/2} \leq H_m |\nabla \sigma_m| \Gamma_m^{(q-2)/4},$$

where $\Gamma_m := 1 + |\nabla^2 u_m|^2$. Finally we let

$$\tilde{h}_m := \max\big[\Gamma_m^{(q-2)/4}, \Gamma_m^{(2-p)/4}\big]$$

and obtain

$$|\nabla \sigma_m| \leq c H_m \Gamma_m^{(q-2)/4} \leq c H_m \tilde{h}_m,$$

hence

$$\begin{aligned}\int_{B_{2R}} \eta^6 H_m^2\,\mathrm{d}x \leq c \int_{T_{2R}} H_m \tilde{h}_m \big[|\nabla^2 \eta^6|\,|\nabla u_m - \nabla P_m| \\ + |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\big]\,\mathrm{d}x.\end{aligned} \tag{2.18}$$

Letting $\gamma = 4/3$ we discuss the right-hand side of (2.18):

$$\begin{aligned}\int_{T_{2R}} H_m \tilde{h}_m |\nabla \eta^6|\,|\nabla^2 u_m - \nabla^2 P_m|\,\mathrm{d}x \\ \leq \frac{c}{R} \Big[\int_{B_{2R}} (H_m \tilde{h}_m)^\gamma\,\mathrm{d}x\Big]^{1/\gamma} \Big[\int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4\,\mathrm{d}x\Big]^{1/4}.\end{aligned}$$

Next the choice of P_m is made more precise by the requirement

$$\nabla^2 P_m = \fint_{B_{2R}} \nabla^2 u_m \, \mathrm{d}x. \tag{2.19}$$

Then Sobolev–Poincaré's inequality together with the definition of $\tilde{h}_m$ gives

$$\begin{aligned}\left[\int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \, \mathrm{d}x\right]^{1/4} &\le c\left[\int_{B_{2R}} |\nabla^3 u_m|^\gamma \, \mathrm{d}x\right]^{1/\gamma}\\ &\le c\left[\int_{B_{2R}} (H_m\tilde{h}_m)^\gamma \, \mathrm{d}x\right]^{1/\gamma},\end{aligned}$$

hence

$$\int_{T_{2R}} H_m\tilde{h}_m|\nabla\eta^6|\,|\nabla^2 u_m - \nabla^2 P_m| \, \mathrm{d}x \le \frac{c}{R}\left[\int_{B_{2R}} (H_m\tilde{h}_m)^\gamma \, \mathrm{d}x\right]^{2/\gamma}. \tag{2.20}$$

To handle the remaining term on the right-hand side of (2.18) we need in addition to (2.19)

$$\fint_{B_{2R}} (\nabla u_m - \nabla P_m) \, \mathrm{d}x = 0,$$

which can be achieved by adjusting the linear part of P_m. Then we have by Poincaré's inequality

$$\begin{aligned}&\int_{B_{2R}} H_m\tilde{h}_m|\nabla^2\eta^6|\,|\nabla u_m - \nabla P_m| \, \mathrm{d}x\\ &\quad\le \frac{c}{R^2}\left[\int_{B_{2R}} (H_m\tilde{h}_m)^\gamma \, \mathrm{d}x\right]^{1/\gamma}\left[\int_{B_{2R}} |\nabla u_m - \nabla P_m|^4 \, \mathrm{d}x\right]^{1/4}\\ &\quad\le \frac{c}{R}\left[\int_{B_{2R}} (H_m\tilde{h}_m)^\gamma \, \mathrm{d}x\right]^{1/\gamma}\left[\int_{B_{2R}} |\nabla^2 u_m - \nabla^2 P_m|^4 \, \mathrm{d}x\right]^{1/4},\end{aligned}$$

and the right-hand side is bounded by the right-hand side of (2.20). Hence, recalling (2.18) and (2.20), we have established the inequality

$$\left[\fint_{B_R} H_m^2 \, \mathrm{d}x\right]^{\gamma/2} \le c\fint_{B_{2R}} (H_m\tilde{h}_m)^\gamma \, \mathrm{d}x. \tag{2.21}$$

Given this starting inequality we like to apply the following lemma which is proved in [BFZ].

Lemma 2.1. *Let $d > 1$, $\beta > 0$ be two constants. With a slight abuse of notation let f, g, h now denote any non-negative functions on $\Omega \subset \mathbf{R}^n$ satisfying*

$$f \in L^d_{\mathrm{loc}}(\Omega), \quad \exp(\beta g^d) \in L^1_{\mathrm{loc}}(\Omega), \quad h \in L^d_{\mathrm{loc}}(\Omega).$$

Suppose that there is a constant $C > 0$ such that

$$\left[\fint_B f^d \,\mathrm{d}x \right]^{1/d} \leq C \fint_{2B} f g \,\mathrm{d}x + C \left[\fint_{2B} h^d \,\mathrm{d}x \right]^{1/d}$$

holds for all balls $B = B_r(x)$ with $2B = B_{2r}(x) \Subset \Omega$. Then there is a real number $c_0 = c_0(n, d, C)$ such that if $h^d \log^{c_0 \beta}(e+h) \in L^1_{\mathrm{loc}}(\Omega)$, then the same is true for f. Moreover, for all balls B as above we have

$$\fint_B f^d \log^{c_0\beta} \left[e + \frac{f}{\|f\|_{d,2B}} \right] \mathrm{d}x \leq c \left[\fint_{2B} \exp(\beta g^d) \,\mathrm{d}x \right] \left[\fint_{2B} f^d \,\mathrm{d}x \right] + c \fint_{2B} h^d \log^{c_0\beta} \left[e + \frac{h}{\|f\|_{d,2B}} \right] \mathrm{d}x,$$

where $c = c(n, d, \beta, C) > 0$ and $\|f\|_{d,2B} = (\fint_{2B} f^d \,\mathrm{d}x)^{1/d}$.

The appropriate choices in the setting at hand are $d = 2/\gamma = 3/2$, $f = H_m^\gamma$, $g = \tilde{h}_m^\gamma$, $h \equiv 0$. We claim that

$$\fint_{B_{2R}} \exp(\tilde{h}_m^2 \beta) \,\mathrm{d}x \leq c \quad \text{and} \quad \fint_{B_{2R}} H_m^2 \,\mathrm{d}x \leq c$$

for a constant being uniform in m. The uniform bound of the second integral follows from (2.16); thus let us discuss the first one. By (2.17) and Trudinger's inequality (see e.g. Theorem 7.15 of [GT]) we know that for any disc $B_\varrho \Subset \Omega_1$

$$\int_{B_\varrho} \exp(\beta_0 h_m^2) \,\mathrm{d}x \leq c(\varrho) < \infty,$$

where β_0 just depends on the uniformly bounded quantities $\|h_m\|_{W^1_2(\Omega_1)}$. This implies for any $\beta > 0$ and $\kappa \in (0, 1)$

$$\int_{B_\varrho} \exp(\beta h_m^{2-\kappa}) \,\mathrm{d}x \leq c(\varrho, \beta, \kappa) < \infty.$$

Moreover, on account of $q < p + 2$ we have

$$\Gamma_m^{(q-2)/2} \leq h_m^{2-\kappa} \quad \text{and clearly} \quad \Gamma_m^{(2-p)/2} \leq h_m^{2-\kappa}$$

for κ sufficiently small, which gives our claim and we may indeed apply the lemma with the result

$$\fint_{B_\varrho} H_m^2 \log^{c_0\beta}(e + H_m)\,\mathrm{d}x \le c(\beta,\varrho) < \infty$$

for all discs $B_\varrho \subset \Omega_1$ and all $\beta > 0$. Thus we have established the counterparts of (2.7) and (2.10) in [BFZ], and exactly the same arguments as given there lead to (2.11) from [BFZ]. Thus we deduce the uniform continuity of the sequence $\{\sigma_m\}$ (see again [BFZ], end of Section 2), hence we have uniform convergence $\sigma_m \to: \sigma$ for some continuous tensor σ. In order to identify σ with $Df(\nabla^2 u)$, we recall the weak convergence stated in (2.4) and also observe that $\nabla^2 u_m \to \nabla^2 u$ a.e. which can be deduced along the same lines as in Lemma 4.5c) of [BF3], we also refer to Proposition 3.29 iii) of [Bi]. Therefore $Df(\nabla^2 u)$ is a continuous function, i.e. $\nabla^2 u$ is of class C^0, and finally $u \in C^2(\Omega)$ follows.

Step 5. *$C^{2,\alpha}$-regularity of u.* To finish the proof of Theorem 1.1 we observe that with Step 4 we get from (2.5) the estimate

$$\int_{\Omega_1} |\nabla^3 u_m|^2\,\mathrm{d}x \le c(\Omega_1) < \infty,$$

in particular one has for $\alpha = 1, 2$

$$U := \partial_\alpha u \in W^2_{2,\mathrm{loc}}(\Omega).$$

Moreover we have

$$\int_\Omega D^2 f_m(\nabla^2 u_m)(\nabla^2 \partial_\alpha u_m, \nabla^2 \varphi)\,\mathrm{d}x = 0 \quad \text{for any } \varphi \in C_0^\infty(\Omega).$$

Together with the convergences (as $m \to \infty$)

$$D^2 f_m(\nabla^2 u_m) \to D^2 f(\nabla^2 u) \quad \text{in } L^\infty_{\mathrm{loc}}(\Omega),$$
$$\nabla^2 \partial_\alpha u_m \rightharpoonup \nabla^2 U \quad \text{in } L^2_{\mathrm{loc}}(\Omega)$$

we therefore arrive at the limit equation

$$\int_\Omega D^2 f(\nabla^2 u)(\nabla^2 U, \nabla^2 \varphi)\,\mathrm{d}x = 0.$$

Hence U is a weak solution of an equation with continous coefficients and $u \in C^{2,\alpha}(\Omega)$ for any $0 < \alpha < 1$ follows from [GM, Theorem 4.1]. ▫

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