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ON THE SHARPNESS OF THE STOLZ APPROACH

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Abstract. We study the sharpness of the Stolz approach for the a.e. convergence of functions in the Hardy spaces in the unit disc, first settled in the rotation invariant case by J. E. Littlewood in 1927 and later examined, under less stringent, quantitative hypothesis, by H. Aikawa in 1991. We introduce a new regularity condition, of a qualitative type, under which we prove a version of Littlewood's theorem for tangential approach whose shape may vary from point to point. Our regularity condition can be extended in those contexts where no group is involved, such as NTA domains in $\mathbf{R}^n$. We show exactly in what sense our regularity condition is sharp.

1. Overview of our results

Let H^∞ be the space of bounded holomorphic functions in the unit disc $\mathbf{D}$ in $\mathbf{C}$. How sharp is the Stolz (nontangential) approach

$$\Gamma_\alpha(e^{i\theta}) = \bigl\{ z \in \mathbf{D} : |z - e^{i\theta}| < (1+\alpha)(1-|z|) \bigr\} \tag{1.1}$$

for the a.e. boundary convergence of H^∞ functions? A family $\gamma = \{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of $\mathbf{D}$, called an *approach*, may have the following properties:

(c) each $\gamma(\theta)$ is a curve in $\mathbf{D}$ ending at $e^{i\theta}$;
(tg) each $\gamma(\theta)$ ends tangentially at $e^{i\theta}$;
(aecv) each $h \in \mathrm{H}^\infty$ converges a.e. along $\gamma(\theta)$ to its Stolz boundary values.

The *Strong Sharpness Statement* is the following claim.

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(SSS) *There is no approach* γ *satisfying* (c) & (tg) & (aecv).

This claim is coherent with a principle— implicit in [10]—whose first rendition is found in [15], who showed that there is no *rotation invariant* approach γ satisfying (c) & (tg) & (aecv). Another rendition of this principle (with stronger conclusions) has been given by [2], who proved that, if (u) is the condition:

(u) the curves $\{\gamma(\theta)\}_\theta$ are uniformly bi-Lipschitz equivalent;

then there is no approach γ satisfying (u) and (c) & (tg) & (aecv).

See also [1].

Our first result [1] is a theorem of Littlewood type where the tangential curve is allowed to vary its shape, and we do not require uniformity in the order of tangency. Moreover, we show that, in a precise sense, Theorem 1.1 is sharp.

Theorem 1.1 (A sharp Littlewood type theorem). *Let* $\gamma\colon [0,2\pi) \to 2^{\mathbf{D}}$ *such that*

(c$\star$) *for each* $\theta \in [0,2\pi)$, *the set* $\{e^{i\theta}\} \cup \gamma(\theta)$ *is connected;*

(tg) *for each* $\alpha > 0$ *and* $\theta \in [0,2\pi)$ *there exists* $\delta > 0$ *such that if* $z \in \gamma(\theta) \cap \Gamma_\alpha(e^{i\theta})$ *then* $|z - e^{i\theta}| > \delta$;

(reg) *for each open subset* O *of* $\mathbf{D}$ *the set*

$$\{\theta \in [0,2\pi) : \gamma(\theta) \cap O \neq \emptyset\}$$

is a measurable subset of $[0,2\pi)$.

Then there exists $h \in \mathrm{H}^\infty$ *with the property that, for almost every* $\theta \in [0,2\pi)$, *the limit of* $h(z)$ *as* $z \to e^{i\theta}$ *and* $z \in \gamma(\theta)$ *does not exist.*

– Condition (c$\star$) is strictly weaker than (c) but it *cannot* be relaxed to the minimal condition one may ask for:

(apprch) $e^{i\theta}$ belongs to the closure of $\gamma(\theta)$ for all θ

since Nagel and Stein [18] showed that there is a rotation invariant approach γ satisfying (apprch) and (tg) & (aecv). This discovery disproved a conjecture of Rudin [19], prompted by his construction of a highly oscillating inner function in $\mathbf{D}$. See also [20]. Thus, (c$\star$) identifies the property of *curves* relevant to a theorem of Littlewood type.

– It is not easy to see (reg) fail. The images of radii by an inner function satisfy (reg): this example prompted Rudin [19] to ask about the truth value of (SSS). Observe that (reg) is a qualitative condition, while (u) is quantitative. The former is perhaps more commonly met than the latter. They are independent of each other.

– Since our hypotheses do not impose any smoothness, neither on $\gamma(\theta)$ nor on the domain, a version of our theorem can be formulated, and proved as well, for domains with rough boundary, such as NTA domains in $\mathbf{R}^n$; see Theorem 1.3.

[1] A preliminary version of this result was announced in [8].

– Is it possible to prove Theorem 1.1 without assuming (reg)? Several theorems in analysis do fail if we omit some regularity conditions, while others (typically those involving null sets) remain valid without 'regularity' hypothesis[2]. This question brings us back to the truth value of (SSS), and we prove the following result.

Theorem 1.2. *It is neither possible to prove the Strong Sharpness Statement, nor to disprove it.*

The proof uses a combination of methods of modern logic (developed after 1929) and harmonic analysis, based upon an insight about the location of the link that makes the combination possible. See Theorem 2.2, Theorem 2.3 and Theorem 2.4.

Let h^∞ be the space of bounded harmonic functions on a bounded domain $D \subset \mathbf{R}^n$. Assume that D is NTA—as defined by [13]. How sharp is the so-called *corkscrew approach*

$$\Gamma_\alpha(w) \stackrel{\text{def}}{=} \{z \in D : |z-w| < (1+\alpha)\,\mathrm{dist}\,(z, \partial D)\} \tag{1.2}$$

for the boundary convergence for h^∞ functions, a.e. relative to harmonic measure?

Observe that D may be *twisting* a.e. relative to harmonic measure. In this case, the 'corkscrew' approach (1.2) does not look like a sectorial angle at all.

Theorem 1.1 lends itself to the task of formulating[3] the appropriate sharpness statement for NTA domains, without any further restrictions on the domain.

Theorem 1.3. *If D is an NTA domain in $\mathbf{R}^n$ and $\gamma = \{\gamma(w)\}_{w \in \partial D}$ is a family of subsets of D such that*

(c⋆) *for each $w \in \partial D$, $\gamma(w) \cup \{w\}$ is connected*;

(tg) *for each $\alpha > 0$ and $w \in \partial D$ there exists $\delta > 0$ such that if $z \in \gamma(w) \cap \Gamma_\alpha(w)$ then $|z-w| > \delta$*;

(reg) *for each open subset O of D the set*

$$\{w \in \partial D : \gamma(w) \cap O \neq \emptyset\}$$

is a measurable subset of ∂D (i.e. its characteristic function is resolutive);

then there exists $h \in \mathrm{h}^\infty$ such that for almost every $w \in \partial D$, with respect to harmonic measure, the limit of $h(z)$ as $z \to w$ and $z \in \gamma(w)$ does not exist.

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[2] A regularity hypothesis in a theorem is one which is not (formally) necessary to give meaning to the conclusion of the theorem. A priori it is not clear which theorems belong to which group. Egorov's theorem on pointwise convergence belongs to the first; see [5, p. 198]. One example in the second group can be found in [21, p. 251].

[3] In formulating (and proving) our Theorem 1.3 we also had this goal in mind. The proof of Theorem 1.3, due to F. Di Biase and O. Svensson, will appear elsewhere.

– A condition such as rotation invariance, in place of (reg), would have no meaning, since in this context there is no group suitably acting, not even locally.

– Observe that (c$\star$) *cannot* be relaxed to the condition

$$w \text{ belongs to the closure of } \gamma(w) \tag{1.3}$$

(the minimal one needed to take boundary values). Indeed, the first-named author showed the existence, for NTA domains in $\mathbf{R}^n$, of an approach γ, satisfying (1.3) and (tg), along which all h^∞ functions converge to their boundary values taken along (1.2), a.e. relative to harmonic measure[4].

2. Notation and other results

The core of the problem belongs to harmonic analysis, so we restrict ourselves, without loss of generality, to the space h^∞ of bounded harmonic functions on $\mathbf{D}$.

The boundary of $\mathbf{D}$, denoted by $\partial\mathbf{D}$, is naturally identified to the quotient group $\mathbf{R}/2\pi\mathbf{Z}$, from which it inherits the *Lebesgue measure* m; thus, $m(\partial\mathbf{D}) = 2\pi$.

If $h \in \mathrm{h}^\infty$, the *Fatou set of* h, denoted by $\mathscr{F}(h) \subset \partial\mathbf{D}$, is the set of points $w \in \partial\mathbf{D}$, such that *the limit of* $h(z)$ *as* $z \to w$ *and* $z \in \Gamma_\alpha(w)$ *exists for all* $\alpha > 0$; this limit is denoted $h_\flat(w)$. Now, $m(\mathscr{F}(h)) = 2\pi$ and $h_\flat \in \mathrm{L}^\infty(\partial\mathbf{D})$; see [10].

The *Poisson extension* P: $\mathrm{L}^\infty(\partial\mathbf{D}) \to \mathrm{h}^\infty$ recaptures h from $h_\flat$, since $h = P[h_\flat]$.

If γ is a subset of $\mathbf{D} \times \partial\mathbf{D}$ and $w \in \partial\mathbf{D}$, *the shape of* γ *at* w is the set

$$\gamma(w) \overset{\text{def}}{=} \{z \in \mathbf{D} : (z, w) \in \gamma\} \subset \mathbf{D}.$$

An *approach* is a subset γ of $\mathbf{D} \times \partial\mathbf{D}$ such that (apprch) holds for all θ. One may think of γ as a family $\{\gamma(\theta)\}_{\theta\in[0,2\pi)}$ of subsets of $\mathbf{D}$. If $h \in \mathrm{h}^\infty$ and γ is an approach, then define the following two subsets of $\partial\mathbf{D}$:

$$\mathrm{C}(h,\gamma) \overset{\text{def}}{=} \big\{w \in \mathscr{F}(h)\,;\, h(z) \text{ converges to } h_\flat(w) \text{ as } z \to w \text{ and } z \in \gamma(w)\big\},$$

$$D(h,\gamma) \overset{\text{def}}{=} \big\{w \in \partial\mathbf{D}\,;\, h(z) \text{ does not have any limit as } z \to w \text{ and } z \in \gamma(w)\big\}.$$

If γ is an approach and u: $D \to \mathbf{R}$ a function on D, the function on $\partial\mathbf{D}$ given by

$$\gamma^\star(u)(w) \overset{\text{def}}{=} \sup\{|u(z)| : z \in \gamma(w)\}$$

is called the *maximal function of* u *along* γ at $w \in \partial\mathbf{D}$.

[4] In [7], the existence is showed by reducing the problem to the discrete setting of a (not-necessarily-homogeneous) tree, rather than on the action of a group on the space. In general, in this context, there is no group suitably acting on the space.

Lemma 2.1. *The following properties of an approach* γ *are equivalent*:

(a) $\gamma^\star$ *maps all continuous functions* (*on* D) *to measurable functions* (*on* $\partial\mathbf{D}$);

(b) *for every open* $Z \subset \mathbf{D}$, *the boundary subset*

$$\gamma^\downarrow(Z) \overset{\text{def}}{=} \{w \in \partial\mathbf{D} : Z \cap \gamma(w) \neq \emptyset\} \subset \partial\mathbf{D}$$

is a measurable subset of $\partial\mathbf{D}$.

The subset in (b) is called the *shadow projected by* Z *along* γ. The proof of Lemma 2.1 is left to the reader[5]. The approach γ is called: *regular* if it satisfies (a) or (b) in Lemma 2.1; *rotation invariant* if $(z, w) \in \gamma$ implies $(e^{i\theta}z, e^{i\theta}w) \in \gamma$ for all θ, z, w. A rotation invariant approach is regular. If h: $\mathbf{D} \to \mathbf{D}$ is an inner function, then the set

$$\big\{(z, w) \in \mathbf{D} \times \partial\mathbf{D}\,;\, z = f(ru) \text{ for some } u \in \mathscr{F}(h), h_\flat(u) = w,\ 0 \leq r < 1\big\}$$

is a (not necessarily rotation invariant) regular approach whose shape, given by the images of radii by h, may be empty over a null set only; see [19].

2.1. The independence theorem. Modern logic gives us tools that show that some statements can be neither proved nor disproved. The basic idea is familiar: if different models (or 'concrete' representations) of some axioms exhibit different properties, then these properties do not follow from those axioms. For example, the existence of a single, 'concrete' non commutative group shows that commutativity cannot be derived from the group axioms, and the existence of different models of geometry shows that Euclid's Fifth Postulate does not follow from the others. Since the currently adopted system of axioms for Mathematics is ZFC[6], to prove a theorem amounts to deducing the statement from ZFC. A *model* of ZFC stands to ZFC as, say, a 'concrete' group stands to the axioms of groups. If ZFC is consistent, then it has several, different models. K. Gödel showed, in his *completeness theorem*, that a statement can be deduced from ZFC if and only if it holds in every model of ZFC; in particular, if it holds in some models but not in others, then it follows that it *can be neither proved nor disproved.* The tangential boundary behaviour of h^∞ functions is radically different in different models of ZFC[7].

Theorem 2.2. *There is a model of ZFC in which there exists an approach* γ *satisfying* (c) *and* (tg) *and such that* $C(h, \gamma)$ *has measure equal to* 2π *for every* $h \in \mathrm{h}^\infty$.

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5 This circle of ideas is based on the work of E. M. Stein. Cf. [11].

6 Acronym for Zermelo, Fraenkel and the Axiom of Choice. See [6], [9], [12], [14].

7 Since an approach is a fairly arbitrary subset of $\mathbf{D} \times \partial\mathbf{D}$, in retrospect this result can be rationalized, but other examples in analysis show that this rationalization is not a priori infallible.

Theorem 2.3. *There is a model of ZFC in which for every approach satisfying* (c$\star$) *and* (tg) *there exists* $h \in \mathrm{h}^\infty$ *such that* $D(h,\gamma)$ *has outer measure equal to* 2π.

The following result shows that Theorem 2.3 cannot be improved[8].

Theorem 2.4 (A theorem in ZFC). *There exists an approach* γ *satisfying* (c) *and* (tg) *such that for each* $h \in \mathrm{h}^\infty$, *the set* $C(h,\gamma)$ *has outer measure equal to* 2π.

3. Proofs in ZFC

Observe that h^∞ has the same cardinality as $\partial\mathbf{D}$.

Lemma 3.1 ([17]). *There is a collection* $\{G_u\}_{u\in(0,1)}$ *of mutually disjoint subsets of* $\partial\mathbf{D}$, *such that* (a) *for each* $u \in (0,1)$, *the set* G_u *has outer measure equal to* 2π; (b) $\partial\mathbf{D} = \bigcup_{u\in(0,1)} G_u$.

The following (qualitative) consequence of the theorem of Fatou can also be derived from Theorem 3.4. The proof is omitted.

Lemma 3.2. *For each* $h \in h^\infty(\mathbf{D})$ *there exists an approach* γ_h *satisfying* (c) *and* (tg) *and such that* $\mathrm{C}(h,\gamma_h) = \mathscr{F}(h)$; *therefore,* $m\big(\mathrm{C}(h,\gamma_h)\big) = 2\pi$.

If $h \in \mathrm{h}^\infty$, $s \in \mathbf{R}$, $\theta > 0$ and $\mathrm{v} \in \mathbf{R}$ we define

$$h^*(s,\theta;\mathrm{v}) \overset{\mathrm{def}}{=} \sup_{0<|t|\le\theta} \left| \frac{1}{t} \int_s^{s+t} \big(h_\flat(e^{iu}) - \mathrm{v}\big)\, du \right|.$$

The limit of

$$\frac{1}{t} \int_s^{s+t} h_\flat(e^{iu})\, du$$

as $t \to 0$ exists and is equal to v if and only if

$$\lim_{\theta\downarrow 0} h^*(s,\theta;\mathrm{v}) = 0.$$

Observe that $h^*(s,\theta;\mathrm{v})$ is an increasing function of θ.

Proposition 3.3 ([10] and [16]). *Let* $h \in \mathrm{h}^\infty$ *and* $s \in \mathbf{R}$. *Then the following conditions are equivalent.*

(i) $e^{is} \in \mathscr{F}(h)$ *and* $h_\flat(e^{is}) = \mathrm{v}$;
(ii) $\lim_{\theta\downarrow 0} h^*(s,\theta;\mathrm{v}) = 0$.

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[8] Theorem 2.4 in itself does not say whether (SSS) can be proved or not.

Let c be a continuous function $c\colon [0,\infty) \to \mathbf{D}$ ending at e^{is} and assume that c can written in the form $c(\tau) = |c(\tau)| e^{is} e^{i\theta(\tau)}$ where $\theta = \theta(\tau) > 0$ is a continuous function of τ such that $\lim_{\tau\to\infty} \theta(\tau) = 0$ and

$$\lim_{\tau\to\infty} \frac{\theta(\tau)}{1 - |c(\tau)|} = +\infty.$$

Then c is called an *upper tangential curve ending at* e^{is}. The function $\theta = \theta(\tau)$ (uniquely determined by c) is called the *angle of* c with respect to e^{is}.

Theorem 3.4 ([4]). *Let $h \in \mathrm{h}^\infty$, $e^{is} \in \mathscr{F}(h)$ and $\mathrm{v} = h_\flat(e^{is})$. Let c be an upper tangential curve ending at e^{is} and let θ be the angle of c with respect to e^{is}. If*

$$\lim_{\tau\to\infty} \frac{\theta(\tau)}{1 - |c(\tau)|} h^*(s, 2\theta(\tau); \mathrm{v}) = 0 \tag{3.1}$$

then

$$\lim_{\tau\to\infty} h\big(c(\tau)\big) = h_\flat(e^{is}).$$

Thus, h converges to $h_\flat(e^{is})$ along c as long as c is not too tangential. If $B \subset \partial\mathbf{D}$, let $1_B\colon \partial\mathbf{D} \to \{0,1\}$ be the function equal to 1 on B and 0 on $\partial\mathbf{D} \setminus B$.

Lemma 3.5. *Assume that $B \subset \partial\mathbf{D}$ is open and that $m(\partial\mathbf{D} \setminus B) > 0$. Let γ be an approach satisfying* (c$\star$). *Then*

$$\liminf_{\substack{z\in\gamma(w)\\ z\to w}} P[1_B](z) = 0 \qquad \textit{for a.e. } w \in \partial\mathbf{D} \setminus B. \tag{3.2}$$

Proof. (Cf. [23]). Fatou's theorem implies that

$$\lim_{r\uparrow 1} P[1_B](rw) = 0 \qquad \text{for a.e. } w \in \partial\mathbf{D} \setminus B. \tag{3.3}$$

An application of Egorov's theorem shows that for each $\varepsilon > 0$ there is a perfect subset A of $\partial\mathbf{D} \setminus B$ such that the limit in (3.3) is uniform for $w \in A$ and $m(A) > 2\pi - m(B) - \varepsilon$. We may assume that each $w \in A$ is a limit point of a sequence $we^{i\theta_n} \in A$ where $\theta_n \to 0$ and $\theta_n > 0$ for n even, $\theta_n < 0$ for n odd. It follows that (3.2) holds at each point $w \in A$, since $\{w\} \cup \gamma(w)$ is connected, and, therefore, $\gamma(w)$ intersects the radii ending at $we^{i\theta_n}$ for an appropriate subsequence of n's, close enough to the boundary. The conclusion follows because ε is arbitrary. □

The subset of $\partial\mathbf{D}$ given by $\{e^{is} : \theta - r < s < \theta + r\}$ is called the *arc* of *center* $e^{i\theta}$ and *radius* $r > 0$. Fix the value of α at $\alpha = 1/10$. If J is an arc in $\partial\mathbf{D}$, define

$$\triangle(J) \overset{\text{def}}{=} \big\{z \in \mathbf{D} : (\Gamma_\alpha)^\downarrow(\{z\}) \subset J\big\}.$$

Lemma 3.6. *There is a number $c_1 > 0$ such that $P[1_J](z) \geq c_1$ for each arc $J \subset \partial\mathbf{D}$ and each $z \in \triangle(J)$.*

Proof. Since $P[1_J](z) \geq P\big[1_{(\Gamma_\alpha)^\downarrow(\{z\})}\big](z)$ for each $z \in \triangle(J)$, it suffices to show that

$$\inf_{z\in\mathbf{D}} P\big[1_{\Gamma^\downarrow(\{z\})}\big](z) > 0. \tag{3.4}$$

The proof of (3.4) is left to the reader. □

If $B \subset \partial\mathbf{D}$ is open and γ is an approach, we define $Z_\gamma(B)$ as follows[9]: $w \in Z_\gamma(B)$ if and only if $w \in \partial\mathbf{D} \setminus \{B\}$ and there is a sequence $\{J_k\}_{k\in\mathbf{N}}$ of arcs contained in B such that for all $k \in \mathbf{N}$, $\gamma(w) \cap \triangle(J_k) \neq \emptyset$ and, moreover, for each $\varepsilon > 0$ there is n_ε such that the set J_k is within ε distance from w for all $k \geq n_\varepsilon$. Let us see why we shall construct B of small measure and such that $Z_\gamma(B)$ is appropriately large.

Lemma 3.7. *If γ is an approach and $B \subset \partial\mathbf{D}$ is open then, for all $w \in Z_\gamma(B)$, $\limsup_{\substack{z\to w\\ z\in\gamma(w)}} P[1_B](z) \geq c_1$.*

Proof. It follows from Lemma 3.6, since if $J \subset B$ then $P[1_J] \leq P[1_B]$. □

3.1. Proof of Theorem 1.1. Define $\tau\colon \partial\mathbf{D} \times \mathbf{D} \to (0, 1]$ by

$$\tau(w, z) \stackrel{\text{def}}{=} \frac{1 - |z|}{|w - z|}.$$

Consider the sequence of everywhere defined functions $f_n\colon \partial\mathbf{D} \to (0, \infty)$ gauging the order of tangency at the various points:

$$f_n(w) \stackrel{\text{def}}{=} \sup\big\{\tau(w, z) : z \in \gamma(w),\ |z - w| < 2\pi/n\big\}. \tag{3.5}$$

Observe that $1 \geq f_n(w) \geq f_{n+1}(w)$ and that $\lim_{n\to\infty} f_n(w) = 0$ for each $w \in \partial\mathbf{D}$, since γ is tangential. Since γ is regular, the functions f_n are measurable. If $N \in \mathbf{N}$ then there is a set $C_N \subset \partial\mathbf{D}$ whose Lebesgue measure is greater than $2\pi - 1/2^N$ and such that the sequence $\{f_n\}$ converges uniformly to 0 on C_N. We may and will assume that $C_N \subset C_{N+1}$ for all $N \in \mathbf{N}$. Thus, there is an element $\phi_N \in \mathbf{N}^{\mathbf{N}}$ such that if $l \in \mathbf{N}$ and $n \geq \phi_N(l)$ then $\sup_{w\in C_N} f_n(w) < 2^{-l}$. Define a strictly increasing sequence $\phi \in \mathbf{N}^{\mathbf{N}}$ dominating each ϕ_N, as follows. Let $\phi(1) \geq \phi_1(1)$, $\phi(2) \geq \max\{\phi_1(2), \phi_2(2)\}$, $\phi(3) \geq \max\{\phi_1(3), \phi_2(3), \phi_3(3)\}$, and so on. Then $\phi(i) \geq \phi_N(i)$ for all $i \geq N$. It follows that

$$c(k) \stackrel{\text{def}}{=} \sup_{w\in C_k} f_{\phi(k)}(w) < 2^{-k}.$$

9 Cf. [23].

If $J \subset \partial\mathbf{D}$ is the arc $\{e^{is} : \theta - r < s < \theta + r\}$ of center $e^{i\theta}$ and radius $r >$ and $0 < c \le 1$, we denote $cJ \overset{\text{def}}{=} \{e^{is} : \theta - cr < s < \theta + cr\}$ the arc of center $e^{i\theta}$ and radius cr. Thus, $m(cJ) = cm(J)$. For $n, p \in \mathbf{N}$ and $1 \le p \le n$ define

$$J(n,p) \overset{\text{def}}{=} \left\{ e^{is} : (p-1)\frac{2\pi}{n} < s < p\frac{2\pi}{n} \right\} \subset \partial\mathbf{D}.$$

Define

$$I_k \overset{\text{def}}{=} \bigcup_{p=1}^{\phi(k)} c(k) J\big(\phi(k), p\big).$$

Then $m(I_k) \le 2\pi\, c(k) < 2\pi\, 2^{-k}$. Define $B(l) \overset{\text{def}}{=} \bigcup_{k=l}^{\infty} I_k$. Let $D \overset{\text{def}}{=} \bigcup_1^{\infty} C_N$. Then the measure of D is equal to 2π.

Claim. *If* $l_0 \in \mathbf{N}$ *then* $D \setminus B(l_0) \subset Z_\gamma(B(l_0))$.

If $h \in \mathrm{h}^\infty$ and $w \in \partial\mathbf{D}$, we define

$$\operatorname{osc}(h; w) \overset{\text{def}}{=} \limsup_{\substack{z \to w \\ z \in \gamma(w)}} h(z) - \liminf_{\substack{z \to w \\ z \in \gamma(w)}} h(z).$$

Consider $\mathbf{1}_{B(l)} \in \mathrm{L}^\infty(\partial\mathbf{D})$ and its Poisson integral $P[\mathbf{1}_{B(l)}] \in \mathrm{h}^\infty$. Lemmas 3.5, 3.7 and the Claim imply that there is a set $N(l)$ of Lebesgue measure zero such that if $w \in \big(D \setminus B(l)\big) \setminus N(l)$ then $\operatorname{osc}\big(P[\mathbf{1}_{B(l)}]; w\big) \ge c_1$. For $q > 1$ to be determined later, we define, following [23], $g \overset{\text{def}}{=} \sum_{l=1}^{\infty} q^{-l} \mathbf{1}_{B(l)}$. It follows that $P[g] = \sum_{l=1}^{\infty} q^{-l} P[\mathbf{1}_{B(l)}]$. Define $N \overset{\text{def}}{=} \bigcup_1^{\infty} N(l)$. Then $m(N) = 0$. Define $B \overset{\text{def}}{=} \bigcap_1^{\infty} B(l)$. Then $m(B) = 0$. We now show that if $w \in (D \setminus B) \setminus N$ then $\operatorname{osc}(P[g]; w) > 0$. Indeed, let l be the smallest integer n such that $w \notin B(n)$. Then w belongs to the open set

$$\bigcap_{k=1}^{l-1} B(k). \tag{3.6}$$

For $k = 1, 2, \dots, l-1$, the function $\mathbf{1}_{B(k)}$ is equal to 1 on the set (3.6); since this set is open, it follows that $\operatorname{osc}\big(P[\mathbf{1}_{B(k)}]; w\big) = 0$ for each $k = 1, 2, \dots, l-1$. On the other hand, $\operatorname{osc}\big(q^{-l} P[\mathbf{1}_{B(l)}]; w\big) \ge q^{-l} c_1$ and

$$\operatorname{osc}\left(\sum_{k=l+1} q^{-k} P[\mathbf{1}_{B(k)}]; w \right) \le \sum_{k=l+1}^{\infty} q^{-k} \le q^{-l} \frac{1}{q-1}.$$

It follows that

$$\operatorname{osc}(P[g]; w) \ge q^{-l} c_1 - q^{-l} \frac{1}{q-1} > 0$$

if q is chosen greater than $(1 + c_1)/c_1$. Since the set $(D \setminus B) \setminus N$ has measure equal to 2π, the proof is completed. □

Proof of the Claim. Assume that $w_0 \in D \setminus B(l_0)$. The set $\gamma(w_0)$ contains a branch ending tangentially at w_0 from one side. Assume it ends at w_0, say, from the right. Let $N_0 \in \mathbf{N}$ be such that $w_0 \in C_{N_0}$. Let $\varrho_0 > 0$ be such that if $z \in \gamma(w_0)$ and $|z - w_0| < \varrho_0$ then $\tau(w_0, z) < 2^{-10}$. Choose $z_0 \in \gamma(w_0)$ such that $|z_0 - w_0| < \varrho_0$. The role of z_0 will be to make sure that our final choice is not empty, exploting the fact that each approach region in the approach is connected. Indeed, it may happen that each different approach region in the approach starts from a different distance from the boundary. Choose $l_1 \in \mathbf{N}$ such that $l_1 \geq l_0$, $l_1 \geq N_0$ and

$$\frac{2\pi}{\phi(l_1)} < 2^{-10}|w_0 - z_0|.$$

Let $l \geq l_1$. Then $w_0 \notin B(l)$. Let $k \geq l$. Then $w_0 \notin I_k$. Let $p \in \{1, 2, \ldots, \phi(k)\}$ be such that the arc $J_k \stackrel{\text{def}}{=} c(k) J\big(\phi(k), p\big)$ is closer to w from the right. We know that $w_0 \in C_k$, since $k \geq N_0$. Thus,

$$\sup\left\{\tau(w_0, z) : z \in \gamma(w_0),\ |z - w_0| < \frac{2\pi}{\phi(k)}\right\} \leq c(k).$$

Let w_1 be the center of the arc J_k. Then there is a point $z_1 \in \gamma(w_0)$ such that $|z_1 - w_0| = |w_1 - w_0|$ and z_1 is located on the same side as $\gamma(w_0)$. Observe that $|w_1 - w_0| < 2\pi/\phi(k)$. It follows that $\tau(w_0, z_1) \leq c(k)$. Thus, $z_1 \in \triangle(J_k)$. ▫

3.2. Proof of Theorem 2.4. A decomposition $\partial\mathbf{D} = \bigcup_{h \in \mathrm{h}^\infty(\mathbf{D})} G(h)$, where each set $G(h)$ has full outer measure and sets indexed by different functions are disjoint, exists by Lemma 3.1. Let γ_h be the approach associated to h in Lemma 3.2. For $w \in G(h) \cap \mathscr{F}(h)$ define $\gamma(w) \stackrel{\text{def}}{=} \gamma_h(w)$. For $w \in G(h) \setminus \mathscr{F}(h)$ define $\gamma(w)$ any tangential way you like. Then, for each $h \in \mathrm{h}^\infty$ the set $\mathrm{C}(h, \gamma)$ has outer measure equal to 2π. Indeed, it suffices to observe that $\mathrm{C}(h, \gamma)$ contains $G(h) \cap \mathscr{F}(h)$.

4. Model dependent statements

4.1. A new model dependent property. We were led to formulate[10] the Generalized Egorov Property as we gained insight on its role in the truth value of (SSS).

> (GEP) For each $\varepsilon > 0$, every sequence of not-necessarily-measurable real valued functions on $[0, 1]$, converging pointwise to zero, has a subsequence converging uniformly on a subset of $[0, 1]$ whose outer measure is greater than $1 - \varepsilon$.

[10] We could not find GEP in the literature. I. Recław (private communication) has noticed that, to show that GEP holds in some models of ZFC, the following property, holding in the iterated Laver real model, can be used; see [3]: the cardinality of the smallest subset of $[0, 1]$ of full outer measure is smaller that the cardinality of the smallest unbounded family in the Baire space. The original proof given in [22] is different.

Theorem 4.1 ([22]). *GEP is independent of ZFC.*

4.1.1. *Known model dependent properties.* A set has *small cardinality* if its cardinality is stricly less than the cardinality of the continuum. The *Baire space* $\mathbf{N}^{\mathbf{N}}$ is the collection of all sequences of natural numbers. The *dominating order* $\leq_*$ in the Baire space is an order relation defined as follows: $f \leq_* g$ if and only if there exists an integer m such that $f(n) \leq g(n)$ for each $n \geq m$. A model of ZFC has *Property D* if and only if for each $S \subset \mathbf{N}^{\mathbf{N}}$ of small cardinality there is a $g \in \mathbf{N}^{\mathbf{N}}$ such that $f \leq_* g$ for every $f \in S$. A model of ZFC has *Property* $\mathrm{Unif}(\mathscr{N}) = \mathfrak{c}$ if and only if every subset of $\partial\mathbf{D}$ of small cardinality has Lebesgue measure zero. If ZFC is consistent, then there are models of ZFC where both these properties hold but the Continuum Hypothesis does not. Cf. [3].

4.2. Proof of Theorem 2.2. Assume that ZFC is consistent and choose a model of ZFC where Properties D and $\mathrm{Unif}(\mathscr{N}) = \mathfrak{c}$ hold. We claim that in this model, the conclusion of Theorem 2.2 holds. Let I be a set having the cardinality of the continuum and let $\prec$ be a well-ordering of I such that all its initial segments have small cardinality. Let $\{h_\alpha\}_{\alpha\in I}$ be a list of all bounded harmonic functions in $\mathbf{D}$ and let $\{w_\beta\}_{\beta\in I}$ be a list of all points in $\partial\mathbf{D}$. If $\beta \in I$ then the set

$$T(\beta) \stackrel{\text{def}}{=} \{\alpha \in I : \alpha \prec \beta \text{ and } w_\beta \in \mathscr{F}(h_\alpha)\}$$

has small cardinality. We claim that Theorem 3.4, and Property D imply that there exists a continuous curve $c_\beta\colon [0,\infty) \to \mathbf{D}$ in $\mathbf{D}$ ending tangentially at w_β and such that if $\alpha \in T(\beta)$ then

$$\lim_{s\to\infty} h_\alpha\big(c_\beta(s)\big) = (h_\alpha)_\flat(w_\beta) \tag{4.1}$$

holds. Indeed, write $w_\beta = e^{is}$, and, for each $\alpha \in T(\beta)$, let $v_\alpha = (h_\alpha)_\flat(w_\beta)$ and define $f_\alpha \in \mathbf{N}^{\mathbf{N}}$ by letting $f_\alpha(n)$ be the smallest integer k such that

$$(h_\alpha)^*(s, 2e^{-l}; v_\alpha) \leq \frac{1}{2^{n+n}}$$

for all $l \geq k$. Then the family $\{f_\alpha\}_{\alpha\in T(\beta)} \subset \mathbf{N}^{\mathbf{N}}$ has small cardinality. Property D implies that there is an element $f \in \mathbf{N}^{\mathbf{N}}$ such that $f_\alpha \leq_* f$ for each $\alpha \in T(\beta)$. We may always assume that f is strictly increasing. The upper tangential curve $c = c_\beta$ ending at w_β with angle $\theta(\tau) = e^{-\tau}$ and such that $\theta(\tau)/\big(1-|c(\tau)|\big)$ interpolates linearly between 2^n and 2^{n+1} when τ is between $f(n)$ and $f(n+1)$ has the required property, by Theorem 3.4. Indeed, if $\alpha \in T(\beta)$ then there is a k such that if $n \geq k$ then $f_\alpha(n) \leq f(n)$. Thus, if $n \geq k$ and $f(n) \leq \tau < f(n+1)$ then

$$\frac{\theta(\tau)}{1-|c(\tau)|}(h_\alpha)^*(s, 2e^{-\tau}; v_\alpha) \leq \frac{2}{2^n}.$$

Define $\gamma(w_\beta)\overset{\mathrm{def}}{=}c_\beta(0,\infty)$. We claim that for each $\alpha\in I$ the set $\mathrm{C}(h_\alpha,\gamma)$ is measurable and it has measure equal to 2π. Indeed, consider the subset $S(\alpha)\overset{\mathrm{def}}{=}\{w_\beta : \alpha\prec\beta \text{ and } w_\beta\in\mathscr{F}(h_\alpha)\}$ of $\mathscr{F}(h_\alpha)$, obtained by removing a certain set of small cardinality (thus a null set, in our model). Thus, $S(\alpha)$ is measurable and it has measure 2π. We claim that $S(\alpha)\subset\mathrm{C}(h_\alpha,\gamma)$. Indeed, if $w\in S(\alpha)$ then $w=w_\beta$ for some $\beta\in I$ such that $\alpha\prec\beta$ and $w_\beta\in\mathscr{F}(h_\alpha)$. Thus, $\alpha\in T(\beta)$ and therefore (4.1) holds, i.e. $w=w_\beta\in\mathrm{C}(h_\alpha,\gamma)$.

4.3. Proof of Theorem 2.3. Choose a model of ZFC where GEP holds. We claim that in this model, the conclusion of Theorem 2.3 holds. Indeed, it suffices to repeat the proof of Theorem 1.1 replacing every occurence of 'measure' by 'outer measure'.

References

[1] Aikawa, H.: Harmonic functions having no tangential limits. - Proc. Amer. Math. Soc. 108, 1990, 457–464.

[2] Aikawa, H.: Harmonic functions and Green potentials having no tangential limits. - J. London Math. Soc. (2) 43, 1991, 125–136.

[3] Bartoszyński, T., and H. Judah: Set Theory. On the Structure of the Real Line. - A. K. Peters, 1995.

[4] Boehme, T. K., and M. L. Weiss: Extensions of Fatou's theorem to tangential asymptotic values. - Proc. Amer. Math. Soc. 27, 1971, 289–298.

[5] Bourbaki, N.: Intégration, Chapters I–IV. - Herman, 1952.

[6] Cohen, P. J.: Set Theory and the Continuum Hypothesis. - Benjamin, 1966.

[7] Di Biase, F.: Fatou Type Theorems. Maximal Functions and Approach Regions. - Birkhäuser, 1998.

[8] Di Biase, F., A. Stokolos, O. Svensson, and T. Weiss: Tangential boundary behaviour of bounded harmonic functions in the unit disc. - Geometry Seminars 1996–97, 1998, 63–68.

[9] Drake, F. R.: Set Theory. - North-Holland, 1974.

[10] Fatou, P.: Séries trigonométriques et séries de Taylor. - Acta Math. 30, 1906, 335–400.

[11] Fefferman, C., and E. M. Stein: Some maximal inequalities. - Amer. J. Math. 93, 1971, 107–115.

[12] Jech, T.: Set Theory. - Academic Press, 1978.

[13] Jerison, D., and C. Kenig: Boundary behaviour of harmonic functions in non-tangentially accessible domains. - Adv. Math. 46, 1982, 80–147.

[14] Kunen, K.: Set Theory. An Introduction to Independence Proofs. - North-Holland, 1980.

[15] Littlewood, J. E.: Mathematical notes (4): On a theorem of Fatou. - J. London Math. Soc. 2, 1927, 172–176.

[16] Loomis, L. H.: The converse of the Fatou theorem for positive harmonic functions. - Trans. Amer. Math. Soc. 53, 1943, 239–250.

[17] Lusin, N., and W. Sierpiński: Sur une dècomposition d'un intervalle en une infinitè non dènombrable d'ensembles non mesurables. - C. R. Acad. Sci. 165, 1917, 422–424.

[18] Nagel, A., and E. M. Stein: On certain maximal functions and approach regions. - Adv. Math. 54, 1984, 83–106.

[19] Rudin, W.: Inner function images of radii. - Math. Proc. Cambridge Philos. Soc. 85(2), 1979, 357–360.

[20] Rudin, W.: Tangential H^∞-images of boundary curves. - Math. Proc. Cambridge Philos. Soc. 104(1), 1988, 115–118.

[21] Stein, E. M.: Singular Integrals and Differentiability Properties of Functions. - Princeton Univ. Press, Princeton, N.J., 1970.

[22] Weiss, T.: A note on generalized Egorov's theorem. - Preprint, 2003.

[23] Zygmund, A.: On a theorem of Littlewood. - Summa Brasil. Math. 2(5), 1949, 51–57.

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