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# PARAMETRIZATIONS OF ALGEBRAIC CURVES

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**Abstract.** We show that if a pair of meromorphic functions parametrize an algebraic curve then they have a common right factor, and we use this to derive a variety of results on algebraic curves.

## 1. Introduction

We show that any parametrization of an affine algebraic curve by meromorphic functions factorizes in a certain way, and we use this to derive several classical results about the parametrizations of such curves. The distinction between the affine curve, the projective curve, and the associated compact Riemann surface, is not always clear in the literature. Here we start with a careful statement of the Desingularization Theorem (whose proof is now available in accordance with modern standards) but, apart from this, we use only basic complex analysis. Some of our results can be found in the literature on algebraic geometry, but we hope that this partly expository paper will be of interest to complex analysts.

We use $\mathbf{C}$ to denote the complex plane, and $\mathbf{P}^1$ and $\mathbf{P}^2$ to denote the complex projective spaces of dimension one (the extended complex plane) and two, respectively. Let $P(u,v)$ be an irreducible complex polynomial in two variables. Then the *affine algebraic curve* associated with $P$ is

$$\mathscr{C} = \{(u,v) \in \mathbf{C} \times \mathbf{C} : P(u,v) = 0\}, \tag{1.1}$$

and (because $P$ is irreducible) $\mathscr{C}$ determines $P$ to within a non-zero scalar multiple. Given $P$ we can also form the corresponding homogeneous polynomial $\widetilde{P}$, and hence construct the projective curve $\widetilde{\mathscr{C}}$ in $\mathbf{P}^2$, and a compact Riemann surface $\mathscr{R}$, which is called the *desingularization* of $\mathscr{C}$, and which (given $P$) is unique up to a conformal map. We shall use $P$, $\widetilde{P}$, $\mathscr{C}$, $\widetilde{\mathscr{C}}$ and $\mathscr{R}$ in this sense throughout the

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paper and without further discussion (the details and formal definitions appear in Section 2).

The triple $(f,g,D)$ is a *meromorphic parametrization* of an affine curve $\mathscr{C}$ on the domain $D$ if

(i) $D$ is a simply connected subdomain of $\mathbf{P}^1$;
(ii) $f$ and $g$ are non-constant and meromorphic in $D$;
(iii) if $f(z) \neq \infty$ and $g(z) \neq \infty$, then $\big(f(z), g(z)\big) \in \mathscr{C}$;
(iv) with finitely many exceptions, every point of $\mathscr{C}$ is of the form $\big(f(z), g(z)\big)$ for some $z$ in $D$.

We also say that $(f,g)$ is a
(a) *rational parametrization* if $D = \mathbf{P}^1$, and $f$ and $g$ are rational maps;
(b) *polynomial parametrization* if $D = \mathbf{C}$, and $f$ and $g$ are polynomials;
(c) *entire parametrization* if $D = \mathbf{C}$, and $f$ and $g$ are entire functions.

The parametrization of an affine curve is closely related to the uniformization of an algebraic curve. For example, in [3] Bers gives the following definition of uniformization. Let $\Sigma$ be a set in $\mathbf{C} \times \mathbf{C}$ (for example, an affine algebraic curve). Suppose that $f$ and $g$ are meromorphic in a domain $D$ in $\mathbf{C}$, and let $\Phi = (f,g)$ and $D_0 = \{z \in D : f(z),\ g(z) \neq \infty\}$. Then $f$ and $g$ *uniformize* $\Sigma$ if

(i) $\Phi(D_0)$ is a dense subset of $\Sigma$, and
(ii) there is a discrete group of holomorphic self-maps of $D$ such that $\Phi(z_1) = \Phi(z_2)$ if and only if $z_2 = g(z_1)$ for some $g$ in $G$ (so that $D_0/G$ can be identified with $\Phi(D_0)$).

Throughout, we use juxtaposition to denote the composition of maps; for example, $fU(z) = f\big(U(z)\big)$. We now state our main result (the terms $\pi$, $\widetilde{S}$ and $\mathbf{L}$ in this will be defined shortly).

**Theorem 1.** *Suppose that $(f,g,D)$ is a meromorphic parametrization of an affine algebraic curve. Then $f = Uh$ and $g = Vh$ for some $h$, $U$ and $V$, where*
(a) *$h\colon D \to \mathscr{R}$ is holomorphic and non-constant*;
(b) *$U$ and $V$ are non-constant and meromorphic on $\mathscr{R}$*;
(c) *$(U,V)\colon \mathscr{R} \to \mathbf{P}^1 \times \mathbf{P}^1$ is injective on $\mathscr{R} \backslash E$ where $E$ is the finite set $\pi^{-1}(\widetilde{S} \cup \mathbf{L})$.*

Theorem 1 implies that the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "(f,g)"'] & \mathscr{R} \arrow[d, "(U,V)"] \\ \mathscr{C} \arrow[r, "I"] & \mathbf{P}^1 \times \mathbf{P}^1, \end{tikzcd}$$

where $I$ is the identity (or inclusion) map. We shall use Theorem 1 to prove the following known results.

**Theorem 2.** *If $\mathscr{C}$ has a meromorphic parametrization on $\mathbf{C}$ then $\mathscr{R}$ has genus zero or one.*

**Theorem 3.** *Suppose that $\mathscr{C}$ has an entire parametrization $(f,g)$. Then $\mathscr{R}$ has genus zero and, in Theorem 1, we may assume that $\mathscr{R}=\mathbf{P}^1$, $U$ and $V$ are rational, and $h$ is entire.*

**Theorem 4.** *An affine curve $\mathscr{C}$ has a rational parametrization if and only if $\mathscr{R}$ has genus zero. If this is so then, in Theorem 1, we may assume that $\mathscr{R}=\mathbf{P}^1$, and $U$ and $V$ are rational.*

**Theorem 5.** *Suppose that an affine algebraic curve $\mathscr{C}$ has a rational parametrization. Then it has a polynomial parametrization if and only if it has exactly one place at infinity. If this is so then, in Theorem 1, we may assume that $\mathscr{R}=\mathbf{P}^1$, and $U$ and $V$ are polynomials.*

**Theorem 6.** *An algebraic curve of genus $g$ can be parametrized by rational functions if $g=0$, by elliptic functions if $g=1$, and by functions that are automorphic with respect to some Fuchsian group if $g\geqslant 2$.*

**Theorem 7.** *Suppose that $(f,g,D)$ is a meromorphic parametrization of an affine algebraic curve, let $\Gamma_f$ be the group of conformal automorphisms $\gamma$ of $D$ such that $f\gamma=f$, and similarly for $\Gamma_g$. Then there exists a positive integer $N$ with the property that, for every $\gamma$ in $\Gamma_f$, there is an integer $n$ with $0\leqslant n\leqslant N$ and $\gamma^n\in\Gamma_g$. A similar statement holds with $f$ and $g$ interchanged.*

We give some examples to illustrate these theorems. First,

$$(f,g)=\left(\frac{2z}{1+z^2},\frac{1-z^2}{1+z^2}\right)$$

is a rational parametrization of the algebraic curve $\mathscr{C}$ given by $u^2+v^2=1$ so, by Theorem 4, $\mathscr{R}$ has genus zero (and so, from the theory of Riemann surfaces, it is conformally equivalent to $\mathbf{P}^1$). Now $(f,g)=(\sin z,\cos z)$ is an entire parametrization of $\mathscr{C}$, and this illustrates Theorem 3 with $h(z)=e^{iz}$. Note that $f$ and $g$ have the same set of periods. On the other hand, the curve given by $u^2=v^3$ has an entire parametrization $\left(e^{3\pi iz},e^{2\pi iz}\right)$ where these two parametrizing functions have different, but common, periods (see Theorem 7). If $\wp$ is the Weierstrass elliptic function (with respect to some lattice), then $\wp'(z)^2$ is a cubic polynomial in $\wp(z)$, and this shows that the genus one case can arise in Theorem 2.

Theorem 2 was proved by Picard [13], but see also [7], [10], [11] (where a proof based on Nevanlinna theory can be found on p. 232), and (for a historical comment) [5, p. 16]. Theorem 3 is stated in [8], where only a sketch of a proof is offered. Theorem 4 (due to Lüroth) is proved algebraically in [16, p. 151]; Theorem 5 is stated without proof in [1]. Theorem 6 is classical, and the statements here are not reversible in the sense that every algebraic curve can be uniformized by functions defined in the upper-half-plane $\{x+iy : y>0\}$ (see [3, p. 259]). Theorem 7 is a generalization of Theorem 1, [15]. We conjecture that $\Gamma_f \cap \Gamma_g$ is of finite index in $\Gamma_f$ and $\Gamma_g$, but we are unable to prove this. It is easy to verify this when $D = \mathbf{C}$ and $\Gamma_f$ and $\Gamma_g$ are lattices.

## 2. Preliminary results

An irreducible complex polynomial $P(u,v)$ gives rise to the *affine curve* $\mathscr{C}$ in $\mathbf{C}\times\mathbf{C}$ given in (1.1). If we write the equation $P(u,v)=0$ in homogeneous co-ordinates we obtain an equation $\widetilde{P}(u,v,w)=0$ which gives rise to the *projective curve*

$$\widetilde{\mathscr{C}} = \{[u,v,w]\in\mathbf{P}^2 : \widetilde{P}(u,v,w)=0\},$$

where we use the notation $[u,v,w]$ for points in projective space $\mathbf{P}^2$. We shall denote the line at infinity in $\mathbf{P}^2$ by $\mathbf{L}$; this is given by $w=0$. Then $\widetilde{\mathscr{C}}$ can be expressed as the disjoint union $\widetilde{\mathscr{C}} = (\widetilde{\mathscr{C}}\backslash\mathbf{L})\cup(\widetilde{\mathscr{C}}\cap\mathbf{L})$, where $\widetilde{\mathscr{C}}\cap\mathbf{L}$ is finite, and the map

$$\alpha\colon \mathscr{C}\to\widetilde{\mathscr{C}}\backslash\mathbf{L},\quad (u,v)\mapsto[u,v,1], \tag{2.1}$$

is bijective. Let $S$ and $\widetilde{S}$ denote the set of singular points in $\mathscr{C}$ and $\widetilde{C}$, respectively. Then $S$ and $\widetilde{S}$ are finite sets, and if we give $\mathscr{C}\backslash S$ and $\widetilde{\mathscr{C}}\backslash\widetilde{S}$ the usual (and natural) conformal structure that is described in the general theory of algebraic curves, we find that the restriction

$$\alpha\colon \mathscr{C}\backslash S\to\widetilde{\mathscr{C}}\backslash(\widetilde{S}\cup\mathbf{L}) \tag{2.2}$$

is biholomorphic.

The proofs of Theorems 2–7 are based on the construction of $\mathscr{R}$ from the polynomial $P$ via the affine and projective curves $\mathscr{C}$ and $\widetilde{\mathscr{C}}$. The distinction between these spaces is often blurred in the literature, but in any rigorous argument it is essential to distinguish carefully between them. The fact is that $\mathscr{C}$ is not compact, and to remedy this we embed $\mathscr{C}$ in the projective curve $\widetilde{\mathscr{C}}$ which is compact. However, $\widetilde{\mathscr{C}}$ *need not be a Riemann surface.* To overcome this, we construct the Riemann surface $\mathscr{R}$ (without any direct reference to $\mathscr{C}$ or $\widetilde{\mathscr{C}}$) as the space of germs derived from the polynomial $P$. The link between $\mathscr{R}$ and $\widetilde{\mathscr{C}}$ is then given by the Desingularization Theorem which is stated below. Explicitly, $\mathscr{R}$ is constructed as follows. A function element is a pair $(f,D)$, where $f$ is meromorphic in $D$, and two function elements $(f,D)$ and $(g,\Delta)$ are equivalent at a point $\zeta$ in $D\cap\Delta$ if $f=g$ near $\zeta$. A *germ* at $\zeta$ is an equivalence class of function elements $(f,D)$, where $\zeta\in D$, and we denote this germ by $[f]_\zeta$. Finally, $\mathscr{R}$ is (essentially) the space of germs $[w]_\zeta$ as $\zeta$ varies, where the function $w(z)$ satisfies $P\bigl(z,w(z)\bigr)=0$ in some neighbourhood of $\zeta$.

We can summarize all this (informally) by saying that whereas $\mathscr{R}$ is the space of germs, the affine curve $\mathscr{C}$ is the set of points $(z, w)$, where $w$ is the *value* of the germ at $z$. As there may be different germs which take the same value at a given point, we see that $\mathscr{R}$ and $\mathscr{C}$ are essentially different objects. Consider, for example, the affine curve $\mathscr{C}$ given by $v^2 = u^3 + u^2$, and let

$$\sigma(z) = \sqrt{1+z} = 1 + z/2 + \cdots$$

(which is single-valued and holomorphic in the unit disc $\mathbf{D}$). Let $w_1(z) = z\sigma(z)$ and $w_2(z) = -z\sigma(z)$; then for all $z$ in $\mathbf{D}$, $P\big(z, w_1(z)\big) = P\big(z, w_2(z)\big) = 0$. As $w_1(0) = 0 = w_2(0)$, we see that $\big(0, w_1(0)\big)$ and $\big(0, w_2(0)\big)$ are the same point of $\mathscr{C}$, namely $(0,0)$. However, as $w_1$ and $w_2$ have different Taylor expansions at the origin, the germs $[w_1]_0$ and $[w_2]_0$ are distinct points of $\mathscr{R}$. In conclusion, the distinct points $[w_1]_0$ and $[w_2]_0$ of $\mathscr{R}$ correspond to the single point $(0,0)$ in $\mathscr{C}$; see Figure 1. Note (see Theorems 4 and 5) that $\mathscr{C}$ has a polynomial parametrization, namely

$$u = f(z) = 2z + z^2, \quad v = g(z) = 2z + 3z^2 + z^3,$$

so that in this case, $\mathscr{R}$ is conformally equivalent to $\mathbf{P}^1$.

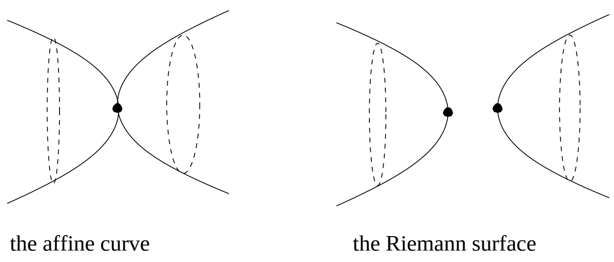


Figure 1.

Every compact surface has a genus, and it is known that if a Riemann surface has genus zero, then it is conformally equivalent to $\mathbf{P}^1$. The genus of the curve $\mathscr{C}$, and of $\widetilde{\mathscr{C}}$, is defined to be the genus of the corresponding Riemann surface $\mathscr{R}$. Thus if $\mathscr{C}$ has genus zero, then we may assume that $\mathscr{R} = \mathbf{P}^1$. We shall need other results on Riemann surfaces (for example, the Uniformization Theorem and the Riemann–Hurwitz Formula) as well as the basic theory of algebraic curves, and we refer the reader to [2], [4], [6], [9], [12], [16] and [17] for more details.

Recall that $S$ and $\widetilde{S}$ are the sets of singular points of $\mathscr{C}$ and $\widetilde{\mathscr{C}}$, respectively. The link between $\mathscr{R}$ and $\widetilde{\mathscr{C}}$ is given in the following fundamental result (modern proofs of which can be found in, for example, [6, p. 169], [9, pp. 66–82] and [12, p. 192]).

**The Desingularization Theorem I.** *Given $\mathscr{C}$, there exists a compact Riemann surface $\mathscr{R}$, and a surjective map $\pi\colon \mathscr{R} \to \widetilde{\mathscr{C}}$ such that $\pi^{-1}(\widetilde{S})$ is a finite set, and $\pi\colon \mathscr{R}\backslash\pi^{-1}(\widetilde{S}) \to \widetilde{\mathscr{C}}\backslash\widetilde{S}$ is biholomorphic.*

The compact Riemann surface $\mathscr{R}$ is called the *desingularization* or *normalization* of the projective curve $\widetilde{\mathscr{C}}$, and it is unique up to a conformal map.

As Theorem 1 is concerned with maps from $\mathscr{R}$ into $\mathbf{C}\times\mathbf{C}$ (instead of maps into $\mathbf{P}^2$), we want to convert the map $\pi\colon \mathscr{R} \to \widetilde{\mathscr{C}}$ into a map $\beta\colon \mathscr{R} \to \mathscr{C}$ and study this. To do this we recall that $\widetilde{\mathscr{C}}\cap\mathbf{L}$ is finite. As $\pi$ is holomorphic on the compact surface $\mathscr{R}$, we see that $\pi^{-1}(\mathbf{L})$ is a finite subset of $\mathscr{R}$. Now let

$$E = \pi^{-1}(\widetilde{S}) \cup \pi^{-1}(\mathbf{L}) = \pi^{-1}(\widetilde{S}\cup\mathbf{L}).$$

It follows that $E$ is a finite subset of $\mathscr{R}$ and, from (2.1), that the composition

$$\beta\colon \mathscr{R}\backslash E \xrightarrow{\pi} \widetilde{\mathscr{C}}\backslash\big(\widetilde{\mathscr{S}}\cup\mathbf{L}\big) \xrightarrow{\alpha^{-1}} \mathscr{C}\backslash\mathscr{S} \tag{2.3}$$

is biholomorphic. As $\mathscr{C}\subset\mathbf{C}\times\mathbf{C}$, we can write $\beta=(U,V)$, where $U\colon \mathscr{R}\backslash E\to\mathbf{C}$ and $V\colon \mathscr{R}\backslash E\to\mathbf{C}$ are holomorphic. This yields the following corollary.

**The Desingularization Theorem II.** *Given $\mathscr{C}$, there exists a compact Riemann surface $\mathscr{R}$, and a biholomorphic map $\beta=(U,V)$ of $\mathscr{R}\backslash E$ onto $\mathscr{C}\backslash\mathscr{S}$, where $E=\pi^{-1}(\widetilde{S}\cup\mathbf{L})$ is a finite subset of $\mathscr{R}$.*

According to [16, p. 96], the concept of a *place* is the algebraic counterpart of a *branch* of a curve over $\mathbf{C}$. We say that $\widetilde{\mathscr{C}}$ (or $\mathscr{C}$) has *one place at infinity* if $\pi^{-1}(\widetilde{\mathscr{C}}\cap\mathbf{L})$ contains exactly one point. In other words, $\widetilde{\mathscr{C}}$ has one place at infinity if it has exactly one point on the line at infinity, and if this point is the image under $\pi$ of exactly one point of $\mathscr{R}$. For example, the affine curve given by $w^2=z^4-3z^3$ has two places at infinity (see [14]) and so cannot be parametrized by polynomials. In this case, $\widetilde{\mathscr{C}}\cap\mathbf{L}$ is the single point $[0,1,0]$, and the two places arise from the two sets of germs $\{[w_1]_z : |z|>3\}$ and $\{[w_2]_z : |z|>3\}$, where $w_1(z)=z(1-3/z)^{1/2}$ and $w_2(z)=-z(1-3/z)^{1/2}$, and where $(1-3/z)^{1/2}$ is the single-valued function on $|z|>3$ that takes the value 1 at $\infty$. By contrast, the affine curve given by $(8z+1)z^2=9w^2$ has a polynomial parametrization, namely $(f,g)$, where

$$f(z)=\tfrac12 z(z+1),\quad g(z)=\tfrac16 z(z+1)(2z+1).$$

The corresponding homogeneous curve is given by $(8u+w)u^2=9v^2w$, and this meets $\mathbf{L}$ at the single point $[0,1,0]$. For sufficiently large $|u|$, the germs of $v$ (as a function of $u$; equivalently, the points on $\mathscr{R}$) are given by

$$v(u)=\pm\big(u\sqrt{u}\,\sqrt{8+1/u}\,\big)/3,$$

where we are taking (for $|u|>1/8$) the single valued choice of $\sqrt{8+1/u}$ that takes the value $2\sqrt2$ at $\infty$. As these germs are converted into each other by analytic continuation around $\infty$ we see (as we already knew from Theorem 5) that this curve has only one place at infinity.

## 3. The proof of Theorem 1

The idea behind our proof is to consider the maps

$$\begin{aligned}
&f, g\colon D \to \mathbf{P}^1, \qquad \varphi = (f,g)\colon D \to \mathscr{C},\\
&U, V\colon \mathscr{R} \to \mathbf{P}^1, \quad \beta = (U,V) : \mathscr{R} \to \mathscr{C},\\
&h = \beta^{-1}\varphi\colon D \to \mathscr{R},\\
&\pi\colon \mathscr{R} \to \widetilde{\mathscr{C}},
\end{aligned}$$

which act as indicated in the following commutative diagram except that *in each case there may be some finite exceptional set* on which the function may not yet be defined.

$$\begin{tikzcd}
 & \mathscr{C} & & \\
D \arrow[ur, "\varphi"] \arrow[rr, "h"] \arrow[dr, "{f,g}"'] & & \mathscr{R} \arrow[ul, "\beta"'] \arrow[dl, "{U,V}"] \arrow[r, "\pi"] & \widetilde{\mathscr{C}}. \\
 & \mathbf{P}^1 & &
\end{tikzcd}$$

Our proof depends on giving a complete description of what is happening at these exceptional points, and for this we must show that certain isolated singularities of a meromorphic function are removable. It is known that any isolated singularity of an injective analytic map is removable, and we shall need the following mild extension of this result.

**Lemma 8.** *Suppose that $X$ is a finite subset of a Riemann surface $R$, and that $F\colon R\backslash X \to \mathbf{C}$ is holomorphic. Suppose also that there is an integer $M$ such that for all $a$ in $R\backslash X$, the equation $F(z) = F(a)$ has (counting multiplicities) at most $M$ solutions in $R\backslash X$. Then $F$ extends to a meromorphic function on $R$.*

*Proof.* We may assume that $M$ is minimal; thus there is some $a$ in $R\backslash X$ such that the solutions of $F(z) = F(a)$ are, say, $a_1, \dots, a_q$, where $a_1 = a$, and where the valency of $F$ at $a_j$ is $k_j$, and $\sum_j k_j = M$. It is well known that we can, for each $j$, construct mutually disjoint open neighbourhoods $N_j$ of $a_j$ so that $N_j$ lies in a parametric disc at $a_j$, and such that the restriction of $F$ to $N_j$ is, up to a conformal change of coordinates, the map $z \mapsto z^{k_j}$. Let $N = \bigcap_j F(N_j)$. Then $N$ is an open neighbourhood of $F(a)$, and because each point in $N$ has exactly $M$ pre-images in $\bigcup_j N_j$, it follows that $F^{-1}(N) \subset \bigcup_j N_j$.

Now choose a point $\zeta$ in $X$. We may assume that the closures of the $N_j$ chosen above do not contain $\zeta$, so we can find a neighbourhood $\mathscr{N}$ of $\zeta$ that is disjoint from $\bigcup_j N_j$, and hence that $F(\mathscr{N}) \cap N = \emptyset$. We may assume that $\mathscr{N}$

lies in a parametric disc at $\zeta$, and if we consider the restriction of $F$ to $\mathscr{N}\backslash\{\zeta\}$, and then apply the Weierstrass–Casorati Theorem, we see that $\zeta$ is a removable singularity of $F$. □

We now prove Theorem 1.

*The proof of Theorem* 1. First, define $\varphi\colon D \to \mathbf{P}^1 \times \mathbf{P}^1$ by $\varphi(z) = \big(f(z), g(z)\big)$. Next, let $L$ be the set of $(u,v)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ such that $u = \infty$ or $v = \infty$. If $z \in D\backslash\varphi^{-1}(S \cup L)$ then $f(z) \neq \infty$, $g(z) \neq \infty$ and $\big(f(z), g(z)\big)$ is a non-singular point of $\mathscr{C}$; thus $\varphi$ maps $D\backslash\varphi^{-1}(S \cup L)$ into $\mathscr{C}\backslash S$. We let $K = \varphi^{-1}(S \cup L)$; then, as $S$ is finite, and the poles of $f$ and $g$ are isolated, we see that $K$ is a discrete subset of $D$.

Next, we define a holomorphic mapping $h\colon D\backslash K \to \mathscr{R}\backslash E$ as the composition of maps given by

$$h\colon D\backslash K \xrightarrow{\varphi} \mathscr{C}\backslash S \xrightarrow{\beta^{-1}} \mathscr{R}\backslash E.$$

Also, as $h = \beta^{-1}\varphi$ we have

$$(f,g) = \varphi = \beta h = (Uh, Vh) \tag{3.1}$$

on $D\backslash K$, where $\beta = (U,V)$, and $U$ and $V$ are maps of $\mathscr{R}\backslash E$ into $\mathbf{C}$. These maps are illustrated in the following commutative diagram in which

(1) $\alpha$, $\beta$ and $\pi$ are biholomorphic, and
(2) $h$, $U$ and $V$ are holomorphic;
see (2.2) and (2.3).

$$\begin{tikzcd}
 & \mathscr{C}\backslash S \arrow[rr, "\alpha"] & & \widetilde{\mathscr{C}}\backslash(\widetilde{S} \cup \mathbf{L}) & \\
D\backslash K \arrow[ur, "\varphi"] \arrow[rr, "h"] & & \mathscr{R}\backslash E \arrow[ul, "\beta"'] \arrow[ur, "\pi"'] \arrow[rr, "U{,}V"] & & \mathbf{C}.
\end{tikzcd}$$

Our first task is to show that $U$ and $V$ are meromorphic on $\mathscr{R}$; that is, that $U$ and $V$ are holomorphic maps of $\mathscr{R}$ onto $\mathbf{P}^1$. Note that once $U$ and $V$ are holomorphic on $\mathscr{R}$ they are (as maps between compact Riemann surfaces) necessarily surjective. Let $\operatorname{card}(X)$ denote the cardinality of a set $X$. Now take any $a$ in $\mathscr{R}\backslash E$. As $\beta$ is biholomorphic, and $\beta = (U,V)$, we have

$$\begin{aligned}
\operatorname{card}\big(\{z \in \mathscr{R}\backslash E : U(z) = U(a)\}\big) &= \operatorname{card}\big(\beta\{z \in \mathscr{R}\backslash E : U(z) = U(a)\}\big)\\
&\leqslant \operatorname{card}\big(\{(u,v) \in \mathscr{C} : u = U(a)\}\big)\\
&= \operatorname{card}\big(\{\big(U(a), v\big) \in \mathscr{C}\}\big)\\
&\leqslant d,
\end{aligned}$$

where $d$ is the degree of $v$ in $P(u,v)$. Thus Lemma 8 is applicable, and $U$ extends to a meromorphic function on $\mathscr{R}$. Clearly the same argument applies to $V$. As $\beta$ is biholomorphic on $\mathscr{R}\backslash E$, Theorem 1(b) and Theorem 1(c) hold.

It remains to show that Theorem 1(a) holds. Now $h = \beta^{-1}\varphi$, $\beta$ is biholomorphic, and $\varphi = (f,g)$ is not constant, so that $h$ is not constant. As $h$ is holomorphic on $D\backslash K$, where $K$ is a discrete subset of $D$, it remains to show that the (isolated) points of $K$ are removable singularities of $h$.

Select any $z_0$ in $K$. Now $f(z_0) \in \mathbf{P}^1$ so that $U^{-1}\big(f(z_0)\big) \subset \mathscr{R}$. We have seen above that $U$ has degree at most $d$, so that we can let $w_1, \ldots, w_q$ be the distinct points of $U^{-1}\big(f(z_0)\big)$, where $q \leqslant d$. Now choose a positive $r$ such that the compact neighbourhood $N = \{z : |z - z_0| \leqslant r\}$ of $z_0$ has the following properties:

(a) $N$ does not contain any point of $K$ other than $z_0$;
(b) the components of $U^{-1}\big(f(N)\big)$ are $\mathscr{N}_j$, $j = 1, \ldots, q$, where, for each $j$, $\mathscr{N}_j$ is a compact neighbourhood of $w_j$;
(c) each $\mathscr{N}_j$ lies in some co-ordinate chart of $\mathscr{R}$.

Note that

(d) $h$ is holomorphic in $N_0 = \{z : 0 < |z - z_0| < r\}$, and
(e) the $\mathscr{N}_j$ are pairwise disjoint and compact.

As $(f,g) = (Uh, Vh)$ on $D\backslash K$ we see that $f(N_0) = Uh(N_0)$, so that

$$h(N_0) \subset U^{-1}\big(f(N_0)\big) = \bigcup_j \mathscr{N}_j.$$

As $h(N_0)$ is connected, it lies in some $\mathscr{N}_j$, and it follows from (c) above that $z_0$ is a removable singularity of $h$. The proof of Theorem 1 is complete. □

## 4. The proof of Theorem 2

The proof mimics the usual proof of Picard's Little Theorem. Let $(f,g)$ be a meromorphic parametrization of $\mathscr{C}$ on $\mathbf{C}$. Then, by Theorem 1, there exists a non-constant holomorphic map $h\colon \mathbf{C} \to \mathscr{R}$. Now let $\widehat{\mathscr{R}}$ be the universal cover of $\mathscr{R}$. As $\mathbf{C}$ is simply connected, we can lift $h$ to a non-constant holomorphic map $\hat{h}\colon \mathbf{C} \to \widehat{\mathscr{R}}$. Liouville's Theorem implies that $\widehat{\mathscr{R}}$ cannot be the unit disc; thus $\widehat{\mathscr{R}}$ is either $\mathbf{C}$ or $\mathbf{P}^1$, so that $\mathscr{R}$ has genus zero or one. □

## 5. The proof of Theorem 3

By Theorem 2, $\mathscr{R}$ is of genus zero or one. First, we suppose that $\mathscr{R}$ is of genus one and reach a contradiction. As $\mathscr{R}$ has genus one the universal covering space of $\mathscr{R}$ is $\mathbf{C}$, and we let $\pi_0\colon \mathbf{C} \to \mathscr{R}$ be a universal covering map. By Theorem 1, there exists a non-constant holomorphic map $h\colon \mathbf{C} \to \mathscr{R}$ and non-constant meromorphic maps $U$ and $V$ on $\mathscr{R}$ such that $(f,g) = (Uh, Vh)$ on $\mathbf{C}$. Now lift $h$ to a holomorphic map $h_0\colon \mathbf{C} \to \mathbf{C}$ such that $\pi_0 h_0 = h$. As $\mathscr{R}$ is compact, $U$ and $V$ are surjective, so there is some point $a$ with $U(a) = \infty$. Now

$h(z) \neq a$ in $\mathbf{C}$, for otherwise there is some $z$ with $f(z) = Uh(z) = U(a) = \infty$, and this contradicts the fact that $f$ is entire. As $\pi_0 h_0 = h$, it follows that $h_0$ cannot take any value in the infinite set $\pi_0^{-1}(\{a\})$, and this contradicts Picard's Little Theorem. Therefore, $\mathscr{R}$ has genus zero and so we may assume that $\mathscr{R} = \mathbf{P}^1$. It follows from this that $U$ and $V$ are rational functions.

Finally, we must show that $h$ can be taken to be entire. If $a = \infty$, then (because $h \neq a$) $h$ is entire. If $a \neq \infty$, let $L(z) = 1/(z-a)$ and $U_1 = UL^{-1}$, $V_1 = VL^{-1}$ and $h_1 = Lh$. Then $f = U_1 h_1$, $g = V_1 h_1$, $U_1$ and $V_1$ are rational and $h_1$ is entire. □

## 6. The proof of Theorem 4

(i) Suppose first that $\mathscr{R}$ has genus zero; thus, by applying a conformal map, we may assume that $\mathscr{R} = \mathbf{P}^1$. Then $U$ and $V$ are meromorphic maps from $\mathbf{P}^1$ to itself and so are rational. In this case, $(U, V)$ is a rational parametrization of $\mathscr{C}$.

(ii) Now suppose that $(f, g)$ is a rational parametrization of $\mathscr{C}$. Then, in Theorem 1, $D = \mathbf{P}^1$ so there exists a non-constant holomorphic map $h\colon \mathbf{P}^1 \to \mathscr{R}$, and non-constant meromorphic maps $U\colon \mathscr{R} \to \mathbf{P}^1$, and $V\colon \mathscr{R} \to \mathbf{P}^1$, such that $(f, g) = (Uh, Vh)$. The Riemann–Hurwitz Formula applied to $h$ implies that the genus of $\mathscr{R}$ is at most the genus of $\mathbf{P}^1$, so that $\mathscr{R}$ has genus zero. We may now assume that $\mathscr{R} = \mathbf{P}^1$, and this implies that $U$ and $V$ are rational functions. □

## 7. The proof of Theorem 5

Throughout this proof we consider an affine algebraic curve $\mathscr{C}$ with a rational parametrization. We have to show that $\mathscr{C}$ *has a polynomial parametrization if and only if* $\pi^{-1}(\mathbf{L})$ *contains exactly one point* of $\mathscr{R}$. We must also show that, under these circumstances, $U$ and $V$ may be taken to be polynomials. Theorem 4 implies that we may take $\mathscr{R} = \mathbf{P}^1$, and that there exist rational functions $U$ and $V$, and a finite set $E$, such that $P\big(U(z), V(z)\big) = 0$ on $\mathbf{C}\backslash E$, and $\beta(z) = \big(U(z), V(z)\big)$ is injective on $\mathbf{C}\backslash E$. We recall that $S \subset \alpha^{-1}(\widetilde{S})$, where $S$ and $\widetilde{S}$ are the sets of singular points of $\mathscr{C}$ and $\widetilde{\mathscr{C}}$, respectively.

(i) *First we show that if* $\mathscr{C}$ *has only one place at* $\infty$ *then it has a polynomial parametrization.* Let $\mathscr{P}$ be the set of poles of $U$ or $V$. As $U$ and $V$ are rational, $\mathscr{P}$ is a finite non-empty set. If $\mathscr{P} = \{\infty\}$, then $U$ and $V$ are polynomials and $(U, V)$ is a polynomial parametrization of the curve. Thus we may now assume that $\mathscr{P}$ contains a point $z_0$ in $\mathbf{C}$. We are now going to show that $\mathscr{P} = \{z_0\}$.

As $z_0 \in \mathscr{P}$, we can write

$$U(z) = \frac{U_1(z)}{(z - z_0)^m}, \quad V(z) = \frac{V_1(z)}{(z - z_0)^n},$$

where $U_1$, $V_1$ are rational functions, $U_1(z_0)$ and $V_1(z_0)$ are nonzero, $m \geqslant 0$, $n \geqslant 0$, and either $m > 0$ or $n > 0$. Without loss of generality, we may assume that $m \geqslant n$, $m \geqslant 1$ and $n \geqslant 0$.

Now consider the following mapping diagram where $\alpha$ and $\beta$ are injective and holomorphic, and $\pi$ is biholomorphic.

$$\begin{tikzcd}
\mathbf{C}\backslash\{\beta^{-1}(\alpha^{-1}(\widetilde{S}))\cup E\} \arrow[r, "\Phi"] \arrow[d, "\beta"'] & \mathscr{R}\backslash\pi^{-1}(\widetilde{S}) \arrow[d, "\pi"] \\
\mathscr{C}\backslash\alpha^{-1}(\widetilde{S}) \arrow[r, "\alpha"'] & \widetilde{\mathscr{C}}\backslash\widetilde{S}.
\end{tikzcd}$$

Let $\Phi = \pi^{-1}\alpha\beta$; clearly, this is injective and analytic except possibly at isolated points. Also,

$$\begin{aligned}
\lim_{z\to z_0}\Phi(z) &= \lim_{z\to z_0}\pi^{-1}\alpha\beta(z)\\
&= \lim_{z\to z_0}\pi^{-1}\alpha\big(U(z),V(z)\big)\\
&= \lim_{z\to z_0}\pi^{-1}\left(\left[\frac{U_1(z)}{(z-z_0)^m},\frac{V_1(z)}{(z-z_0)^n},1\right]\right)\\
&= \lim_{z\to z_0}\pi^{-1}\big(\big[U_1(z),V_1(z)(z-z_0)^{m-n},(z-z_0)^m\big]\big)\\
&= \pi^{-1}\big([a,b,0]\big),
\end{aligned}$$

for some complex numbers $a$, $b$ (not both zero). This shows that $z_0$ is a removable singularity of $\Phi$. Moreover, since $\widetilde{\mathscr{C}}$ is closed, $[a,b,0]\in\widetilde{\mathscr{C}}\cap\mathbf{L}$, and as $\pi^{-1}(\widetilde{\mathscr{C}}\cap\mathbf{L})$ contains exactly one point, say $p$, we must have $\Phi(z_0)=p$. Clearly, this argument holds for any $z_0$ in $\mathscr{P}\cap\mathbf{C}$. However, as $\Phi$ is holomorphic and injective on the complement of a finite set, any holomorphic extension to isolated points must also lead to an injective map. It follows that if $z_1\in\mathscr{P}\cap\mathbf{C}$, then $\Phi(z_1)=p$ so that $z_1=z_0$. We conclude that $\mathscr{P}\cap\mathbf{C}=\{z_0\}$ and, consequently, that

$$U(z)=\frac{U_1(z)}{(z-z_0)^m},\quad V(z)=\frac{V_1(z)}{(z-z_0)^n},$$

where $U_1$ and $V_1$ are *polynomials* with $U_1(z_0)\neq 0$, $V_1(z_0)\neq 0$, $m\geqslant 1$ and $n\geqslant 0$.

We now claim that $\deg(U_1)\leqslant m$ and $\deg(V_1)\leqslant n$ (equivalently, that $U(\infty)$ and $V(\infty)$ are finite). We assume the contrary and let $d=\max\{m,n\}$ and

$$k=\max\{\deg(U_1)-m+d,\deg(V_1)-n+d,d\}.$$

Then $k > d$. Note that as $z \to \infty$,

$$\begin{aligned}\left[\frac{U_1(z)}{(z-z_0)^m}, \frac{V_1(z)}{(z-z_0)^n}, 1\right] &= \left[U_1(z)(z-z_0)^{d-m}, V_1(z)(z-z_0)^{d-n}, (z-z_0)^d\right] \\ &= \left[\frac{U_1(z)(z-z_0)^{d-m}}{z^k}, \frac{V_1(z)(z-z_0)^{d-n}}{z^k}, \frac{(z-z_0)^d}{z^k}\right] \\ &\to [c_0, d_0, 0],\end{aligned}$$

where $c_0$ and $d_0$ are complex numbers, not both zero. As before we get $\Phi(\infty) = p$, which is again a contradiction. Thus, finally, $U_1$ and $V_1$ are polynomials with $\deg(U_1) \leqslant m$ and $\deg(V_1) \leqslant n$. Now let $s = 1/(z - z_0)$, then both $U(z) = U_1\bigl((s^{-1}+z_0)s^m\bigr)$ and $V(z) = V_1\bigl((s^{-1}+z_0)s^n\bigr)$ are polynomials in $s$ and we have obtained a polynomial parametrization of $\mathscr{C}$.

(ii) *We show that if $\mathscr{C}$ has a polynomial parametrization then it has only one place at $\infty$*. Let $(f, g)$ be a polynomial parametrization of $\mathscr{C}$. Then, by Theorem 3, we may assume that $\mathscr{R} = \mathbf{P}^1$, and that $f = Uh$, $g = Vh$, where $U$, $V$ are rational, and $h$ is a polynomial. Since $f$ is a polynomial, $f^{-1}(\infty) = \{\infty\}$. Hence $h^{-1}U^{-1}(\infty) = \{\infty\}$. It follows that $U^{-1}(\infty) = \{h(\infty)\} = \{\infty\}$ and therefore $U$ is a polynomial. Similarly, $V$ is also a polynomial.

Now

$$\Phi = \pi^{-1}\alpha\beta\colon \mathbf{P}^1\backslash E_1 \to \mathbf{P}^1\backslash E_2$$

is injective, where $E_1$ and $E_2$ are some finite sets, and it follows from this that any point in $E_1$ is a removable singularity of $\Phi$. Thus we can extend $\Phi$ to the whole of $\mathbf{P}^1$, and this extension will be a map of $\mathbf{P}^1$ onto itself. Next, $\pi\Phi = \alpha\beta$ on the dense subset $\mathbf{C}\backslash E_1$ of $\mathbf{C}$. Since both $\pi\Phi$ and $\alpha\beta$ are continuous on $\mathbf{C}$, we now see that $\pi\Phi = \alpha\beta$ on $\mathbf{C}$.

Now suppose that $\pi^{-1}(\widetilde{\mathscr{C}} \cap \mathbf{L})$ contains two different points $p$ and $q$. Then there are distinct points $p_0$ and $q_0$ in $\mathbf{P}^1$ such that $\Phi(p_0) = p$ and $\Phi(q_0) = q$. One of them, say $p_0$, is in $\mathbf{C}$, and hence $\alpha\beta(p_0) \in \alpha(\mathscr{C})$. On the other hand, $\pi\Phi(p_0) \in \widetilde{\mathscr{C}} \cap \mathbf{L}$, and this contradicts the fact that $\pi\Phi = \alpha\beta$ on $\mathbf{C}$. The proof is complete. □

## 8. The proof of Theorem 6

We have seen that if $\mathscr{R}$ is of genus zero, then we have a rational parametrization of $\mathscr{C}$. If $\mathscr{R}$ is of genus one then, by composing $U$ and $V$ with the universal covering map of $\mathscr{R}$, we conclude that $\mathscr{C}$ can be parametrized by elliptic functions. Now suppose that $\mathscr{R}$ has genus greater than one. Then, by the Uniformization Theorem, $\mathscr{R} = \mathbf{D}/\Gamma$ for some Fuchsian group $\Gamma$ acting on the unit disc $\mathbf{D}$ without elliptic elements. Each point of $\mathscr{R}$ is then a $\Gamma$-orbit $[z]_\Gamma$ of a point $z$ in $\mathbf{D}$. Now define $U_1$ and $V_1$ by $U_1(z) = U([z]_\Gamma)$ and $V_1(z) = V([z]_\Gamma)$. Then $U_1$ and $V_1$ are automorphic functions invariant under $\Gamma$, and $(U_1, V_1)$ parametrizes $\mathscr{C}$. □

## 9. The proof of Theorem 7

We suppose that $(f, g, D)$ is a meromorphic parametrization of $P(u, v) = 0$, and we write

$$P(u, v) = a_0(u) + a_1(u)v + \cdots + a_m(u)v^m,$$

where $a_m(z)$ is not identically zero. Choose any complex number $z_0$ in $D$ such that the orbit $\Gamma_f(z_0)$ does not contain a pole of $f$ or $g$, or a zero of $a_m(u)$. Then the polynomial $p$ defined by $p(t) = P\big(f(z_0), t\big)$ is not constant and, for each $\gamma$ in $\Gamma_f$, and each $n = 1, 2, \ldots, m+1$,

$$p\big(g\gamma^n(z_0)\big) = P\big(f\gamma^n(z_0), g\gamma^n(z_0)\big) = 0.$$

As $p$ has exactly $m$ zeros, there must be some distinct $s$ and $t$ (which may depend on $z_0$) such that $1 \leqslant s < t \leqslant m+1$ and $g\gamma^s(z_0) = g\gamma^t(z_0)$. As there is an uncountable number of choices of $z_0$ here, there must be an uncountable set $U$ of $z$ for which the integers $s$ and $t$ are independent of $z_0$ in $U$. As any uncountable subset of $\mathbf{C}$ (for example, $U$) has an uncountable set of accumulation points, we deduce that $g\gamma^s(z) = g\gamma^t(z)$ for all $z$ in $D$ or, equivalently, that $\gamma^{s-t} \in \Gamma_g$. This completes the proof. □

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