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VERY WEAK SOLUTIONS OF NONLINEAR SUBELLIPTIC EQUATIONS

Anna Zatorska-Goldstein

University of Warsaw, Institute of Applied Mathematics and Mechanics
Banacha 2, PL-02-097 Warszawa, Poland; azator@mimuw.edu.pl

Abstract. We prove a generalization of a theorem of Iwaniec, Sbordone and Lewis on higher integrability of very weak solutions of the A-harmonic equation onto a case of subelliptic operators defined by a family of vector fields satisfying the Hörmander condition. The main tool is a form of the Gehring Lemma formulated and proved in an arbitrary metric space with a doubling measure. This result might be of special interest, as the Gehring Lemma is a vital tool in many applications.

1. Introduction

Our aim is to study properties of the so-called very weak solutions to nonlinear subelliptic equations in the form

$$X^*A(x,u,Xu)+B(x,u,Xu)=0. \tag{1.1}$$

Here x belongs to a bounded region $\Omega\subset\mathbf{R}^n$ and $X=(X_1,\dots,X_k)$ is a family of smooth vector fields in $\mathbf{R}^n$ defined on a neighborhood of Ω, satisfying the Hörmander condition, and $X^*=(X_1^*,\dots,X_k^*)$ is a family of operators formal adjoint to X_i in L^2. We will call the equation a subelliptic A-harmonic equation. In the classical situation

$$X=\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\dots,\frac{\partial}{\partial x_n}\right)$$

we obtain the familiar A-harmonic equation. The vector fields X_i also satisfy some additional assumptions which are described in Section 2. A and B are both Carathéodory functions and satisfy standard growth conditions, i.e. $A(\,\cdot\,,\cdot\,,\xi)\approx|\xi|^{p-2}\xi$. The precise statement of the conditions is given in Section 4.

We say that u is a weak solution of the equation (1.1) if for every $\phi\in C_0^\infty(\Omega)$

$$\int_\Omega A(x,u,Xu)\cdot X\phi(x)\,\mathrm{d}x+\int_\Omega B(x,u,Xu)\phi(x)\,\mathrm{d}x=0 \tag{1.2}$$

and the function u belongs to the Sobolev space $W^{1,p}$. The last assumption comes from the variational formulation of the problem. If the function A satisfies

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standard growth conditions, i.e. $|A(x,s,\xi)| \approx |\xi|^{p-1}$, then the L^p-integrability condition on u and its derivatives allows us to take as a test function an appropriate power of u multiplied by a smooth cut-off function (or another local construction of a test function based on u). In such a way we can obtain better properties of solutions (e.g. Hölder continuity).

On the other hand, the integrals in (1.2) are well defined for $|Xu|^{p-1} \in L^1$. It is natural to ask, if one can work with weaker regularity assumptions for weak solutions. In the classical situation where $X = \nabla$, T. Iwaniec and C. Sbordone [15] proved that if u satisfies (1.2) but its derivatives are à priori integrable with some exponent strictly lower than the natural exponent p, then in fact they are integrable with the exponent p and therefore u belongs to Sobolev space $W^{1,p}$.

Definition 1.1. A function u is called a very weak solution of (1.1) if u satisfies (1.2) but belongs to the Sobolev space $W^{1,r}$, where the exponent r is strictly lower than the natural exponent p.

Assume that functions A and B satisfy conditions (4.23) and the set $\Omega \subset \mathbf{R}^n$ is open and bounded. Let $X_1, \ldots, X_k$ be vector fields on a neighborhood of Ω, with real, C^∞ smooth and globally Lipschitz coefficients satisfying the Hörmander condition.

Theorem 1.2. *There exists $\delta > 0$, such that if u is a very weak solution of* (1.1), $u \in W^{1,p-\delta}_{X,\mathrm{loc}}(\Omega)$, *then* $u \in W^{1,p+\tilde{\delta}}_{X,\mathrm{loc}}(\Omega)$ *for some $\tilde{\delta} > 0$, and hence it is a classical weak solution of* (1.1).

Recently, a similar theorem on very weak solutions for parabolic equations (in case $X = \nabla$) was proved by J. Lewis and J. Kinnunen [18], [19].

The idea of Iwaniec and Sbordone was to use the Hodge decomposition in construction of a test function. Later J. Lewis [17] showed another proof, where a construction of a test function was based on a Hardy–Littlewood maximal function. We follow the idea of Lewis. We also follow the way of Iwaniec, Sbordone and Lewis to show the higher integrability of the gradient by application of the Gehring Lemma. We use it in a version formulated by Giaquinta [10, Chapter V, Proposition 1.1], introducing changes that are necessary to adapt it to arbitrary metric spaces with a doubling measure. To the best of the author's knowledge, this lemma is not available in the mathematical literature in such generality. We need a metric version of the theorem (see Theorem 3.3 in Section 3) because of the change of a metric in $\mathbf{R}^n$. This is a result of working with a differential operator X instead of a classical gradient. The idea of the proof is analogous to that in the euclidean case. We cannot, however, use tools which are strictly connected with the euclidean geometry: decomposition into dyadic cubes, the classical Calderon–Zygmund Theorem etc. In general metric spaces one then has to use different arguments, see e.g. Lemma 3.1 which replaces the classical Calderon–Zygmund decomposition.

The Gehring Lemma is widely used in the theory of quasi-regular mappings and nonlinear p.d.e.'s (see [14], [11], [20]). For the proof in the euclidean case see for instance [1], [10], [23].

In Section 2 we present basic information on Carnot–Carathéodory spaces. Section 3 contains the proof of the metric version of Gehring's Lemma. Section 4 contains a precise statement of the assumptions on the operator and the proof of Theorem 1.2. As an application of the theorem we have the following compactness theorem in Section 5:

Theorem 1.3. *Let F be a compact subset of Ω and δ be a constant defined by Theorem* 1.2. *Let $\{u_i\}_{i\in\mathbf{N}}$ be a family of very weak solutions of* (1.1) *such that $u_i \in W^{1,r}_X(\Omega)$ for some $p-\delta<r<p$. If the family is bounded in $W^{1,r}_X(\Omega)$, then it is compact in $W^{1,p}_X(F)$.*

2. Carnot–Carathéodory spaces

Let $X_1,\dots,X_k$ be a family of vector fields in $\mathbf{R}^n$ with real, C^∞ coefficients. The family satisfies the Hörmander condition if there exists an integer m such that a family of commutators of the vector fields up to the length m, i.e. the family of vector fields

$$X_1,\dots,X_k,[X_{i_1},X_{i_2}],\dots,\bigl[X_{i_1},\bigl[X_{i_2},[\dots,X_{i_m}]\bigr]\dots\bigr],\quad i_j=1,2,\dots,k,$$

spans the tangent space $T_x\mathbf{R}^n$ at every point $x\in\mathbf{R}^n$.

For $u\in\operatorname{Lip}(\mathbf{R}^n)$ we define X_ju by

$$X_ju(x)=\langle X_j(x),\nabla u(x)\rangle,\quad j=1,2,\dots,k,$$

and set $Xu=(X_1u,\dots,X_ku)$. Its length is given by

$$|Xu(x)|=\biggl(\sum_{j=1}^k|X_ju(x)|^2\biggr)^{1/2},$$

where X_j^* is a formal adjoint to X_j in L^2, i.e.

$$\int_{\mathbf{R}^n}(X_j^*u)v\,\mathrm{d}x=-\int_{\mathbf{R}^n}uX_jv\,\mathrm{d}x\quad\text{for functions } u,v\in C_0^\infty(\mathbf{R}^n).$$

Given $\mathbf{R}^n$ with the family of vector fields, we define a distance function ϱ. We say that an absolutely continuous curve $\gamma\colon[a,b]\to\mathbf{R}^n$ is *admissible*, if there exist functions $c_j\colon[a,b]\to\mathbf{R}$, $j=1,\dots,k$, such that

$$\dot\gamma(t)=\sum_{j=1}^kc_j(t)X_j\bigl(\gamma(t)\bigr)\quad\text{and}\quad\sum_{j=1}^kc_j(t)^2\le1.$$

Functions c_j do not need to be unique, because vector fields X_j do not need to be linearly independent. The distance $\varrho(x,y)$ between points x and y is defined as the infimum of those $T>0$ for which there exists an admissible curve $\gamma\colon[0,T]\to\mathbf{R}^n$ such that $\gamma(0)=x$ and $\gamma(T)=y$. If such a curve does not exist, we set $\varrho(x,y)=\infty$. The function ϱ is called the Carnot–Carathéodory distance. In general it does not need to be a metric. When the family $X_1,\dots,X_k$ satisfies the Hörmander condition, then ϱ is a metric and we say that $(\mathbf{R}^n,\varrho)$ is a Carnot–Carathéodory space. For more information about such spaces and their geometry see for instance [26], [22], [12].

Here and subsequently all the distances will be with respect to the metric ϱ. In particular all the balls B are balls with respect to the C.-C. metric. If $\sigma>0$ and $B=B(x,r)$ then σB will denote a ball centered in x of radius $\sigma\cdot r$. By $\operatorname{diam}\Omega$ we will denote the diameter of the set Ω.

The metric ϱ is locally Hölder continuous with respect to the euclidean metric. Thus the space $(\mathbf{R}^n,\varrho)$ is homeomorphic with the euclidean space $\mathbf{R}^n$, and every set which is bounded in euclidean metric is also bounded in the metric ϱ. The reverse implication is not true. However, if $X_1,\dots,X_k$ have globally Lipschitz coefficients, then Garofalo and Nhieu [8] have shown that every bounded set with respect to ϱ is also bounded in euclidean metric.

We will consider the Lebesgue measure in the Carnot–Carathéodory space. As we change the metric, the measure of $B(x,r)$ is no longer equal to the familiar $\omega_n r^n$. However, the important fact is that the Lebesgue measure in the Carnot–Carathéodory space satisfies the so-called doubling condition (although only locally—see [22]):

Theorem 2.1. *Let Ω be an open, bounded subset of $\mathbf{R}^n$. There exists a constant $C_d\geq 1$ such that*

$$|B(x_0,2r)|\leq C_d|B(x_0,r)| \tag{2.3}$$

provided $x_0\in\Omega$ and $r<5\operatorname{diam}\Omega$.

The best constant C_d is known as the doubling constant and we call a measure satisfying the above condition a doubling measure. Iterating (2.3) we obtain a lower bound on $\mu\bigl(B(x,r)\bigr)$.

Lemma 2.2. *Let μ be a Borel measure in a metric space Y, finite on bounded sets. Assume that μ satisfies the doubling condition on an open, bounded set $\Omega\subset Y$. Then for every ball $B=B(x,r)$ such that $x\in\Omega$ and $r<\operatorname{diam}\Omega$ the following inequality holds*:

$$\mu(B)\geq\frac{\mu(\Omega)r^s}{(2\operatorname{diam}\Omega)^s}$$

where $s=\log_2 C_d$.

We say that Q is of homogeneous dimension relative to Ω, if there exists a constant $C>0$ such that for every ball B_0 with a center in Ω and with a radius $r_0<\operatorname{diam}\Omega$ we have

$$\frac{\mu(B)}{\mu(B_0)}\geq C\left(\frac{r}{r_0}\right)^Q$$

where $B=B(x,r)$ is any ball such that $x\in B_0$ and $r\leq r_0$. If $\Omega\subset\mathbf{R}^n$ is open and bounded and a family of vector fields on Ω satisfies the Hörmander condition, then the Carnot–Carathéodory space (Ω,ϱ) with a Lebesgue measure has the homogeneous dimension $Q=s=\log_2 C_d$.

Given a first-order differential operator $X=(X_1,\dots,X_k)$, we define the Sobolev space $W_X^{1,p}$ in the following way:

$$W_X^{1,p}(\Omega)=\{u\in L^p(\Omega):X_ju\in L^p(\Omega),\ j=1,2,\dots,k\},$$

where X_ju is distributional derivative. The $W_X^{1,p}$ norm is defined by

$$\|u\|_{1,p}=\|u\|_p+\|Xu\|_p.$$

Smooth functions are dense in $W_X^{1,p}(\Omega)$ ([6], [7]). The existence of smooth cut-off functions in C.-C. spaces was shown in [4] and [8]. We have Sobolev and Poincaré type inequalities ([8], [12], [16]):

Theorem 2.3. *Let Q be a homogeneous dimension relative to Ω. There exist constants $C_1,C_2>0$, such that for every metric ball $B=B(x,r)$, where $x\in\Omega$ and $r\leq\operatorname{diam}\Omega$, the following inequalities hold*:

$$\left(\fint_B|u-u_B|^{s^*}\,\mathrm{d}x\right)^{1/s^*}\leq C_1r\left(\fint_B|Xu|^s\,\mathrm{d}x\right)^{1/s}\quad \text{for } 1\leq s<Q,$$

where $s^*=Qs/(Q-s)$ and

$$\fint_B|u-u_B|^s\,\mathrm{d}x\leq C_2r^s\fint_B|Xu|^s\,\mathrm{d}x\quad\text{for } 1\leq s<\infty.$$

We will consider the following maximal functions:

$$M_\Omega f(x):=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{\Omega\cap B(x,r)}|f|\,dy$$

and

$$M_{\Omega,R}f(x):=\sup_{R\geq r>0}\frac{1}{|B(x,r)|}\int_{\Omega\cap B(x,r)}|f|\,dy.$$

Our setting requires the use of the theory of maximal functions and Muckenhoupt weights in metric spaces equipped with a doubling measure. We refer to [12], [23] and [25] for more details.

Theorem 2.4 (Hardy–Littlewood). *Assume Y is a metric space and μ is a doubling measure on an open set $\Omega \subset Y$. Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$|\{x \in \Omega : M_\Omega u(x) > t\}| \leq \frac{C}{t} \int_\Omega |u| \, d\mu$$

for $t > 0$, where the constant C depends only on the doubling constant (C_d) and

$$\|M_\Omega u\|_{L^p(\Omega,\mu)} \leq C \|u\|_{L^p(\Omega,\mu)}$$

for $1 < p \leq \infty$, where $C = C(C_d, p)$.

We will use the above theorem on a bounded and open set Ω in Carnot–Carathéodory space and also on balls σB, such that $B \subset \Omega$ and $\sigma > 1$. Such balls are contained in $\Omega' = \{x : \varrho(x, \partial\Omega) < \sigma \operatorname{diam} \Omega\}$ which is open and bounded. Therefore the doubling constant may change and so may the constants in the Hardy–Littlewood Theorem, but this does not affect the final result.

Theorem 2.5. *Assume Ω is an open and bounded subset of $\mathbf{R}^n$ with Carnot–Carathéodory metric and $u \in L^1_{\mathrm{loc}}(\Omega)$, $s \geq 1$. Then for almost all $x, y \in \Omega$ we have*

$$|u(x) - u(y)| \leq C \varrho(x,y) \big[(M_{\Omega,2\varrho} |Xu|^s(x))^{1/s} + (M_{\Omega,2\varrho} |Xu|^s(y))^{1/s} \big],$$

and for any metric ball $B \subset \Omega$ with radius r and for almost every $x \in B$ we have

$$|u(x) - u_B| \leq C r \big(M_\Omega |Xu|^s(x) \big)^{1/s}.$$

We say that a nonnegative, locally integrable function w belongs to the space A_p for $p > 1$, if

$$\sup_{B \subset \mathbf{R}^n} \left(\fint_B w \, \mathrm{d}x \right) \left(\fint_B w^{1/(1-p)} \, \mathrm{d}x \right)^{p-1} < \infty.$$

A function w belongs to the space A_1 if there exists a constant $c \geq 1$ such that for every ball $B \subset \mathbf{R}^n$

$$\fint_B w \, \mathrm{d}x \leq c \operatorname*{ess\,inf}_B w.$$

Functions in A_p are called Muckenhoupt weights.

Theorem 2.6 (Muckenhoupt Theorem). *Assume $v \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ is nonnegative and $1 < p < \infty$. Then $v \in A_p$ if and only if there exists a constant $C > 0$ such that*

$$\int_{\mathbf{R}^n} |Mf|^p v \, \mathrm{d}x \leq C \int_{\mathbf{R}^n} |f|^p v \, \mathrm{d}x \quad \textit{for all } f \in L^p(\mathbf{R}^n, v);$$

i.e., M is a bounded operator from $L^p(\mathbf{R}^n, v)$ into $L^p(\mathbf{R}^n, v)$.

A metric version of this theorem, with some additional assumptions (in fact—unnecessary and easy to remove(1)) can be found in [25].

(1) The author would like to thank J. Kinnunen for pointing this out.

3. Gehring's Lemma for metric spaces

In this section we assume (Y,ϱ,μ) to be an arbitrary metric space with a doubling (Borel regular) measure μ, i.e. there exists a constant C_d, such that

$$\mu\big(B(x,2r)\big) \le C_d\mu\big(B(x,r)\big).$$

The doubling condition implies the inequality

$$\frac{\mu(B)}{\mu(B_0)} \ge \frac{1}{4^Q}\left(\frac{r}{r_0}\right)^Q, \tag{3.4}$$

where $Q=\log_2 C_d$ and B_0 has the radius r_0, and $B=B(x,r)$ is any ball such that $x\in B_0$ and $r\le r_0$.

Fix $\sigma>1$. Given a ball $B_0=B_0(x_0,R)\subset Y$ define a decomposition of a ball σB_0 into sets C^k, $k=0,1,2,\dots$, defined by

$$\begin{aligned} &C^0 = B_0,\\ &C^k = \left\{x\in\sigma B_0 : \frac{(\sigma-1)R}{2^{k-1}} \ge \operatorname{dist}\big(x,\partial(\sigma B_0)\big) > \frac{(\sigma-1)R}{2^k}\right\} \quad \text{for } k\ge 1. \end{aligned}$$

The following lemma is a version of the Calderon–Zygmund decomposition for metric spaces:

Lemma 3.1. *Assume a function $u\in L^1(\sigma B_0,\mu)$ is nonnegative. Let α be such that*

$$\fint_{\sigma B_0} u(x)\,\mathrm{d}\mu < \alpha.$$

Then, for every $k=0,1,2,\dots$, there exists a countable family of pairwise disjoint balls $\mathscr{F}^k=\{B_j^k\}$ centered in C^k such that

$$u(x)\le \alpha 2^{kQ} \quad \textit{for almost all } x\in C^k\setminus\bigcup_j 5B_j^k \tag{3.5}$$

and

$$\alpha 2^{kQ} < \fint_{5B_j^k} u(x)\,\mathrm{d}\mu \le \alpha 2^{kQ}K, \tag{3.6}$$

where the constant

$$K=\max\left\{C_d, 8^Q\left(\frac{\sigma^2}{\sigma-1}\right)^Q\right\}.$$

Proof. Define $\mathscr{G}_0^k := \emptyset$ and $S_0^k := C^k$. Define a family of balls

$$\mathscr{B}_1^k = \left\{ B(x,r)x \in S_0^k;\, r = \frac{(\sigma-1)R}{5\cdot 2^{k+1}\sigma} \right\}.$$

Let $\widetilde{\mathscr{B}}_1^k$ be a subfamily of $\mathscr{B}_1^k$ defined by

$$\widetilde{\mathscr{B}}_1^k = \left\{ B \in \mathscr{B}_1^k : \alpha 2^{k\beta} < \fint_{5B} u(x)\,\mathrm{d}\mu \right\}.$$

The Vitali covering lemma implies that we can choose from $\widetilde{\mathscr{B}}_1^k$ a countable subfamily $\mathscr{F}_1^k$ of pairwise disjoint balls such that

$$\bigcup_{B\in\mathscr{F}_1^k} 5B \supset \bigcup_{B\in\widetilde{\mathscr{B}}_1^k} B.$$

Then we put

$$\mathscr{G}_1^k = \mathscr{F}_1^k, \quad S_1^k = C^k \setminus \bigcup_{B\in\mathscr{G}_1^k} 5B.$$

Iteration of this procedure gives in the ith step

$$\mathscr{B}_i^k = \left\{ B(x,r) : x \in S_{i-1}^k;\, r = \frac{(\sigma-1)R}{5\cdot 2^{k+i}\sigma} \right\},$$
$$\widetilde{\mathscr{B}}_i^k = \left\{ B \in \mathscr{B}_i^k : \alpha 2^{kQ} < \fint_{5B} u(x)\,\mathrm{d}\mu \right\},$$

$\mathscr{F}_i^k$ being a countable subfamily of pairwise disjoint balls such that

$$\bigcup_{B\in\mathscr{F}_i^k} 5B \supset \bigcup_{B\in\widetilde{\mathscr{B}}_i^k} B.$$

We also have

$$\mathscr{G}_i^k = \mathscr{G}_{i-1}^k \cup \mathscr{F}_i^k, \quad \text{and} \quad S_i^k = C^k \setminus \bigcup_{B\in\mathscr{G}_i^k} 5B.$$

Define $\mathscr{F}^k = \{B_j^k\} = \bigcup_i \mathscr{G}_i^k = \bigcup_i \mathscr{F}_i^k$. For every ball belonging to that family we have

$$\alpha 2^{kQ} < \fint_{5B_j^k} u(x)\,\mathrm{d}\mu,$$

which gives us the lower estimation of (3.6). We proceed to show the upper estimation of (3.6).

Assume $B \in \mathscr{F}_1^k$ has radius r. Thus we have

$$r = \frac{(\sigma - 1)R}{5 \cdot 2^{k+1}\sigma}$$

and applying (3.4) we obtain

$$\fint_{5B} u(x)\,\mathrm{d}\mu \le \frac{\mu(B_0)}{\mu(5B)} \fint_{B_0} u(x)\,\mathrm{d}\mu < \alpha 2^{kQ} 8^Q \left(\frac{\sigma^2}{\sigma - 1}\right)^Q.$$

If $B = B(x, r) \in \mathscr{F}_i^k$ for $i > 1$, then

$$x \in S_{i-1}^k \subset S_{i-2}^k \quad \text{and} \quad r = \frac{(\sigma - 1)R}{5 \cdot 2^{k+i}\sigma}.$$

For the ball $2B = B(x, 2R)$ we have $x \in S_{i-2}^k$ and

$$2r = \frac{(\sigma - 1)R}{5 \cdot 2^{k+i-1}\sigma};$$

thus $2B \in \mathscr{B}_{i-1}^k$. By construction, $2B$ does not belong to $\widetilde{\mathscr{B}}_{i-1}^k$, because

$$x \in S_{i-1}^k = C^k \setminus \bigcup_{B \in \mathscr{G}_{i-1}^k} 5B \subset C^k \setminus \bigcup_{B \in \mathscr{F}_{i-1}^k} 5B \subset C^k \setminus \bigcup_{B \in \widetilde{\mathscr{B}}_{i-1}^k} B.$$

Therefore

$$\fint_{5 \cdot 2B} u(x)\,\mathrm{d}\mu \le \alpha 2^{kQ}.$$

The doubling condition leads to

$$\fint_{5B} u(x)\,\mathrm{d}\mu < \alpha 2^{kQ} C_d.$$

To obtain (3.5) assume $x \in C^k \setminus \bigcup_{B \in \mathscr{F}^k} 5B$ and let $\{B_i\}$ be a sequence of balls centered in x with

$$r_i = \frac{(\sigma - 1)R}{5 \cdot 2^{k+i}\sigma}.$$

For every $i = 1, 2, \ldots$, we have $x \in S_i^k$. Therefore B_i does not belong to $\widetilde{\mathscr{B}}_i^k$. Thus

$$\fint_{5B_i} u(x)\,\mathrm{d}\mu \le \alpha 2^{kQ} \quad \text{for } i = 1, 2, \ldots.$$

The Lebesgue Theorem implies

$$u(x) \le \alpha 2^{kQ}$$

for almost all $x \in C^k \setminus \bigcup_j 5B_j^k$. The proof is complete. □

The lemma below is standard (see e.g. [9]).

Lemma 3.2. *Fix a ball $B \subset Y$. Assume that functions F, G are nonnegative and belong to the space $L^q(B,\mu)$ for some $q > 1$. If there exists a constant $a > 1$ such that for every $t \geq 1$ we have*

$$\int_{E(G,t)} G^q \, \mathrm{d}\mu \leq a\left[t^{q-1} \int_{E(G,t)} G \, \mathrm{d}\mu + \int_{E(F,t)} F^q \, \mathrm{d}\mu\right],$$

where $E(G,t) = \{x \in B : G(x) > t\}$ and $E(F,t) = \{x \in B : F(x) > t\}$. Then the following inequality holds:

$$\int_B G^p \, \mathrm{d}\mu \leq \mu_p\left(\int_B G^q \, \mathrm{d}\mu + a \int_B F^p \, \mathrm{d}\mu\right)$$

for $p \in [q, q+\varepsilon)$, where $\varepsilon = (q-1)/(a-1)$ and $\mu_p = (p-1)/\bigl(p-1-a(p-q)\bigr)$.

The following theorem is a version of the Gehring Lemma for metric spaces with a doubling measure (see e.g. [9], [10]).

Theorem 3.3. *Let $q \in [q_0, 2Q]$, where $q_0 > 1$ is fixed. Assume the functions f, g to be nonnegative and such that $g \in L^q_{\mathrm{loc}}(Y,\mu)$, $f \in L^{r_0}_{\mathrm{loc}}(Y,\mu)$ for some $r_0 > q$. Assume that there exist constants $b > 1$ and θ such that for every ball $B \subset \sigma B \subset Y$ the following inequality holds*

$$⨍_B g^q \, \mathrm{d}\mu \leq b\left[\left(⨍_{\sigma B} g \, \mathrm{d}\mu\right)^q + ⨍_{\sigma B} f^q \, \mathrm{d}\mu\right] + \theta ⨍_{\sigma B} g^q \, \mathrm{d}\mu.$$

Then there exist nonnegative constants θ_0 and ε_0, $\theta_0 = \theta_0(q_0, Q, C_d, \sigma)$ and $\varepsilon_0 = \varepsilon_0(b, q_0, Q, C_d, \sigma)$ such that if $0 < \theta < \theta_0$ then $g \in L^p_{\mathrm{loc}}(Y,\mu)$ for $p \in [q, q+\varepsilon_0)$ and moreover

$$\left(⨍_B g^p \, \mathrm{d}\mu\right)^{1/p} \leq C\left[\left(⨍_{\sigma B} g^q \, \mathrm{d}\mu\right)^{1/q} + \left(⨍_{\sigma B} f^p \, \mathrm{d}\mu\right)^{1/p}\right]$$

for $C = C(b, q_0, Q, C_d, \sigma)$.

Remark. For the definitions of the constants θ_0, ε_0 see (3.19), (3.20) and (3.22).

Proof. Fix a ball $B \subset \sigma B \subset \Omega$. Let u be given by

$$u(x) = \frac{g^q(x)}{⨍_{\sigma B} g^q \, dx}.$$

Take $s > t \geq 1$ (their precise values shall be fixed later). Let $\alpha = s^q > 1$. By Lemma 3.1 we obtain a decomposition of σB into sets C^k, $k = 1, 2, \ldots$, and for every k a family of pairwise disjoint balls $\{B_j^k\}_{j=1,2\ldots} \subset C^k$ such that

$$u(x) \leq \alpha 2^{kQ} \quad \text{for a.e. } x \in C^k \setminus \bigcup_j 5B_j^k$$

and

$$\alpha 2^{kQ} < \fint_{5B_j^k} u(x)\, dx \leq \alpha 2^{kQ} K,$$

where

$$K = \max\left\{C_d, 8^Q \left(\frac{\sigma^2}{\sigma - 1}\right)^Q\right\}.$$

Assume $x \in C^k$. Define functions

$$F(x) := \frac{f(x)}{2^{kQ/q}\left(\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{1/q}} \tag{3.7}$$

and

$$G(x) := \frac{g(x)}{2^{kQ/q}\left(\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{1/q}}. \tag{3.8}$$

By the assumptions of the theorem

$$\fint_{5B_j^k} g^q \,\mathrm{d}\mu \leq b\left[\left(\fint_{\sigma 5B_j^k} g \,\mathrm{d}\mu\right)^q + \fint_{\sigma 5B_j^k} f^q \,\mathrm{d}\mu\right] + \theta \fint_{\sigma 5B_j^k} g^q \,\mathrm{d}\mu.$$

Consider a ball $5\sigma B_j^k$. It is centered in C^k with radius $r \leq (\sigma - 1)R/2^{k+1}\sigma$; hence $5\sigma B_j^k \subset \bigcup_{i=(k-1)^+}^{k+1} C^i$. Therefore for any $x \in 5\sigma B_j^k$ we have

$$f(x) \leq F(x)\left(2^{(k+1)q} \fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{1/q}$$

and

$$g(x) \leq G(x)\left(2^{(k+1)q} \fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{1/q}.$$

It follows that

$$\fint_{5B_j^k} G^q \,\mathrm{d}\mu \leq b \cdot 2^q \left[\left(\fint_{\sigma 5B_j^k} G \,\mathrm{d}\mu\right)^q + \fint_{\sigma 5B_j^k} F^q \,\mathrm{d}\mu\right] + \theta \cdot 2^q \fint_{\sigma 5B_j^k} G^q \,\mathrm{d}\mu. \tag{3.9}$$

By the definition of G, for every ball B_j^k we have

$$s^q < \fint_{5B_j^k} G^q \,\mathrm{d}\mu \le s^q K. \tag{3.10}$$

Let us now set

$$s := 2^{Q/q}\, b^{1/(q-1)}\, \frac{2q}{q-1} t > t. \tag{3.11}$$

Combining (3.9) with (3.10) and applying (3.11) we obtain, after simple computations,

$$\begin{aligned} \frac{2q}{q-1} t\mu(\sigma 5B_j^k) \le \int_{\sigma 5B_j^k} G \,\mathrm{d}\mu + \big(\mu(\sigma 5B_j^k)\big)^{1-(1/q)} \Big(\int_{\sigma 5B_j^k} F^q \,\mathrm{d}\mu\Big)^{1/q} \\ + \Big(\frac{\theta}{b}\Big)^{1/q} \big(\mu(\sigma 5B_j^k)\big)^{1-(1/q)} \Big(\int_{\sigma 5B_j^k} G^q \,\mathrm{d}\mu\Big)^{1/q}. \end{aligned} \tag{3.12}$$

Let $E(G,s) = \{x \in \sigma B : G(x) > s\}$. Since

$$G(x) = \Big(\frac{u(x)}{2^{kQ}}\Big)^{1/q}$$

for almost all $x \in B \setminus \bigcup_j 5B_j^k$, we have $G \le s$. Thus

$$\mu\big(E(G,s)\big) = \mu\Big(E(G,s) \cap \Big(\bigcup_{j,k} 5B_j^k\Big)\Big)$$

and

$$\int_{E(G,s)} G^q \,\mathrm{d}\mu = \int_{E(G,s)\cap\{\cup_{j,k} 5B_j^k\}} G^q \,\mathrm{d}\mu \le \sum_{j,k} \int_{5B_j^k} G^q \,\mathrm{d}\mu.$$

Combining this with (3.10) and (3.11) and applying the doubling condition we obtain

$$\int_{E(G,s)} G^q \,\mathrm{d}\mu \le s^q K \sum_{j,k} \mu(5B_j^k) \le b\, 2^Q\, K\, C_d^3\, t^q \Big(\frac{2q}{q-1}\Big)^q \sum_{j,k} \mu(B_j^k). \tag{3.13}$$

By the definitions of $E(F,t)$ and $E(G,t)$ we have

$$\int_{\sigma 5B_j^k} F \,\mathrm{d}\mu \le \int_{\sigma 5B_j^k \cap E(F,t)} F \,\mathrm{d}\mu + t\mu(\sigma 5B_j^k)$$

and

$$\int_{\sigma 5B_j^k} G\,\mathrm{d}\mu \le \int_{\sigma 5B_j^k\cap E(G,t)} G\,\mathrm{d}\mu + t\mu(\sigma 5B_j^k).$$

Applying Young's inequality we obtain

$$\begin{aligned}\big(t^{q-1}\mu(B')^{1-(1/q)}\big)\bigg(\int_{B'} F^q\,\mathrm{d}\mu\bigg)^{1/q} &\le \frac{q-1}{q}\big(t^{q-1}\mu(B')^{(q-1)/q}\big)^{q/(q-1)} + \frac{1}{q}\int_{B'} F^q\,\mathrm{d}\mu\\ &\le \frac{q-1}{q}t^q\mu(B') + \frac{1}{q}\int_{B'} F^q\,\mathrm{d}\mu\\ &\le \int_{B'\cap E(F,t)} F^q\,\mathrm{d}\mu + t^q\mu(B').\end{aligned}$$

Hence

$$\big(\mu(\sigma 5B_j^k)\big)^{1-(1/q)}\bigg(\int_{\sigma 5B_j^k} F^q\,\mathrm{d}\mu\bigg)^{1/q} \le t^{1-q}\int_{\sigma 5B_j^k\cap E(F,t)} F^q\,\mathrm{d}\mu + t\mu(\sigma 5B_j^k).$$

In the same manner we check that

$$\bigg(\frac{\theta}{b}\bigg)^{1/q}\big(\mu(\sigma 5B_j^k)\big)^{1-(1/q)}\bigg(\int_{\sigma 5B_j^k} G^q\,\mathrm{d}\mu\bigg)^{1/q} \le \frac{\theta t^{1-q}}{b}\int_{\sigma 5B_j^k\cap E(G,t)} G^q\,\mathrm{d}\mu + t\mu(\sigma 5B_j^k).$$

Substituting the last two inequalities into (3.12) yields

$$\begin{aligned}\frac{2q}{q-1}\mu(\sigma 5B_j^k) \le t^{-1}&\int_{\sigma 5B_j^k\cap E(G,t)} G\,\mathrm{d}\mu + t^{-q}\int_{\sigma 5B_j^k\cap E(F,t)} F^q\,\mathrm{d}\mu\\ &+ \frac{\theta}{b}t^{-q}\int_{\sigma 5B_j^k\cap E(G,t)} G^q\,\mathrm{d}\mu + 2\mu(\sigma 5B_j^k),\end{aligned}$$

and therefore

$$\begin{aligned}\mu(\sigma 5B_j^k) \le \frac{q-1}{2t^q}\bigg[&t^{q-1}\int_{\sigma 5B_j^k\cap E(G,t)} G\,\mathrm{d}\mu\\ &+ \int_{\sigma 5B_j^k\cap E(F,t)} F^q\,\mathrm{d}\mu + \frac{\theta}{b}\int_{\sigma 5B_j^k\cap E(G,t)} G^q\,\mathrm{d}\mu\bigg].\end{aligned}\tag{3.14}$$

Let $D^k = \bigcup_j \sigma 5B_j^k$. There exists a countable subfamily of pairwise disjoint balls $\big(\sigma 5B_{j(h)}^k\big)_{j(1),j(2),\dots}$ such that $D^k \subset \bigcup_h \sigma 25B_{j(h)}^k$. Hence

$$\mu(D^k) \le C_d^3 \sum_h \mu(\sigma 5B_{j(h)}^k).$$

From (3.14) it follows that

$$\mu(D^k) \le \frac{C_d^3(q-1)}{2t^q} \sum_h \Big[t^{q-1} \int_{\sigma 5 B^k_{j(h)} \cap E(G,t)} G \,\mathrm{d}\mu + \int_{\sigma 5 B^k_{j(h)} \cap E(F,t)} F^q \,\mathrm{d}\mu + \frac{\theta}{b} \int_{\sigma 5 B^k_{j(h)} \cap E(G,t)} G^q \,\mathrm{d}\mu \Big].$$

The balls $\sigma 5 B^k_{j(h)}$ are pairwise disjoint and contained in $\bigcup_{i=(k-1)^+}^{k+1} C^i$. Thus

$$\mu(D^k) \le \frac{C_d^3(q-1)}{2t^q} \sum_{i=(k-1)^+}^{k+1} \Big[t^{q-1} \int_{C^i \cap E(G,t)} G \,\mathrm{d}\mu + \int_{C^i \cap E(F,t)} F^q \,\mathrm{d}\mu + \frac{\theta}{b} \int_{C^i \cap E(G,t)} G^q \,\mathrm{d}\mu \Big].$$

By summing over $k = 1, 2, \dots$ we obtain (note that each C^k can appear at most 3 times)

$$\sum_k \mu(D^k) \le \frac{3 \cdot C_d^3(q-1)}{2t^q} \sum_k \Big[t^{q-1} \int_{C^k \cap E(G,t)} G \,\mathrm{d}\mu + \int_{C^k \cap E(F,t)} F^q \,\mathrm{d}\mu + \frac{\theta}{b} \int_{C^k \cap E(G,t)} G^q \,\mathrm{d}\mu \Big].$$

Therefore we have

$$\sum_k \mu(D^k) \le \frac{3 \cdot C_d^3(q-1)}{2t^q} \Big[t^{q-1} \int_{E(G,t)} G \,\mathrm{d}\mu + \int_{E(F,t)} F^q \,\mathrm{d}\mu + \frac{\theta}{b} \int_{E(G,t)} G^q \,\mathrm{d}\mu \Big]. \tag{3.15}$$

By the definition of D_k we also have

$$\sum_{j,k} \mu(B^k_j) \le \sum_k \mu(D^k). \tag{3.16}$$

Combining (3.13) with (3.15) and (3.16) we obtain

$$\int_{E(G,s)} G^q \,\mathrm{d}\mu \le \frac{3\, K\, 2^Q\, C_d^6\, (2q)^q}{2(q-1)^{q-1}} \Big[t^{q-1} b \int_{E(G,t)} G \,\mathrm{d}\mu + b \int_{E(F,t)} F^q \,\mathrm{d}\mu + \theta \int_{E(G,t)} G^q \,\mathrm{d}\mu \Big]. \tag{3.17}$$

We also have

$$
\begin{aligned}
\int_{E(G,t)\setminus E(G,s)} G^q \,\mathrm{d}\mu &\le s^{q-1} \int_{E(G,t)} G \,\mathrm{d}\mu \\
&\le 2^{Q(q-1)/q} \left(\frac{2q}{q-1}\right)^{q-1} t^{q-1} b \int_{E(G,t)} G \,\mathrm{d}\mu.
\end{aligned}
\tag{3.18}
$$

Adding both sides of (3.17) and (3.18) we conclude that

$$
\begin{aligned}
\int_{E(G,t)} G^q \,\mathrm{d}\mu \le (a_1 + a_2) \cdot t^{q-1} b \int_{E(G,t)} G \,\mathrm{d}\mu \\
+ a_1 \left[b \int_{E(F,t)} F^q \,\mathrm{d}\mu + \theta \int_{E(G,t)} G^q \,\mathrm{d}\mu \right],
\end{aligned}
$$

where the constants

$$
a_1 = \frac{3 \cdot 2^Q \, K \, C_d^6 \, (2q)^q}{2(q-1)^{q-1}}, \quad a_2 = \frac{2^{Q(1-(1/q))} \, (2q)^q}{2q(q-1)^{q-1}}.
$$

Assume $q \in [q_0, 2Q]$. Then $a_1,\ a_2 < a_0$, where

$$
a_0 = 2 \, K \, C_d^6 \, 32^Q \, Q^{2Q} \quad \text{for } K = \max\left\{ C_d; 8^Q \left(\frac{\sigma^2}{\sigma - 1}\right)^Q \right\}.
\tag{3.19}
$$

Define

$$
\theta_0 := \frac{1}{2a_0}.
\tag{3.20}
$$

Then for $\theta < \theta_0$ we have $a_1 \theta < \frac{1}{2}$ and therefore

$$
\int_{E(G,t)} G^q \,\mathrm{d}\mu \le 4a_0 b \left[t^{q-1} \int_{E(G,t)} G \,\mathrm{d}\mu + \int_{E(F,t)} F^q \,\mathrm{d}\mu \right].
$$

Since $t \ge 1$ was arbitrary and the constants $a_1,\ a_2$ do not depend on t, by Lemma 3.2 we obtain

$$
\int_B G^p \,\mathrm{d}\mu \le \mu_p \left(\int_B G^q \,\mathrm{d}\mu + 4a_0 b \int_B F^p \,\mathrm{d}\mu \right),
\tag{3.21}
$$

where

$$
\mu_p = \frac{p-1}{p-1-4a_0 b(p-q)}
$$

and $p \in [q, q+\varepsilon_0)$ for

$$\varepsilon_0 = \frac{q_0 - 1}{4a_0 b}. \tag{3.22}$$

By inequality (3.21) and definitions of F and G we get

$$\sum_k \frac{\displaystyle\int_{C^k} g^p \,\mathrm{d}\mu}{2^{kQp/q}\left(\displaystyle\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{p/q}} \le \mu_p \sum_k \frac{\displaystyle\int_{C^k} g^q \,\mathrm{d}\mu}{2^{kQ}\left(\displaystyle\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)} + 4\mu_p a_0 b \sum_k \frac{\displaystyle\int_{C^k} f^p \,\mathrm{d}\mu}{2^{kQp/q}\left(\displaystyle\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{p/q}}.$$

Since $C^0 = B$ we obtain after simple computations

$$\int_B g^p \,\mathrm{d}\mu \le \mu_p 2^Q \left(\fint_{\sigma B} g^q \,\mathrm{d}\mu\right)^{p/q} + 4\mu_p a_0 b 2^Q \fint_{\sigma B} f^p \,\mathrm{d}\mu.$$

Taking $C = (4\mu_p a_0 b 2^Q)^{1/p}$ completes the proof. □

4. Proof of the main theorem

Throughout this section we assume that $\Omega \subset \mathbf{R}^n$ is open and bounded and that vector fields $X_1, \dots, X_k$, defined on a neighborhood of Ω, have smooth (C^∞), globally Lipschitz coefficients and satisfy the Hörmander condition.

The functions $A = (A_1, \dots, A_k)$: $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \to \mathbf{R}^k$ and B: $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^k \to \mathbf{R}$ are both Carathéodory functions, i.e. they are measurable in x and continuous in v, ξ. Moreover, there exist constants $\alpha, \beta > 0$ such that

$$\begin{aligned} |A(x,v,\xi)| &\le \alpha(|v|^{p-1} + |\xi|^{p-1}), \\ |B(x,v,\xi)| &\le \alpha(|v|^{p-1} + |\xi|^{p-1}), \\ \langle A(x,v,\xi)|\xi\rangle &\ge \beta|\xi|^p \end{aligned} \tag{4.23}$$

for some $p \ge 2$.

We consider the following equation in Ω:

$$X^* A(x,u,Xu) + B(x,u,Xu) = 0. \tag{1.1}$$

Theorem (1.2). *There exists $\delta > 0$, such that if a function u is a very weak solution of (1.1), i.e. $u \in W^{1,p-\delta}_{X,\mathrm{loc}}(\Omega)$ and it satisfies the equation*

$$\int_\Omega \langle A(x,u,Xu)|X\phi(x)\rangle \,\mathrm{d}x + \int_\Omega B(x,u,Xu)\phi(x)\,\mathrm{d}x = 0$$

for every function $\phi \in C_0^\infty(\Omega)$, then $u \in W^{1,p+\delta}_{X,\mathrm{loc}}(\Omega)$, and hence it is a weak solution of (1.1).

Assume the function $u \in W^{1,p-\delta}_{\mathrm{loc}}(\Omega)$ is a very weak solution of the equation (1.1). We can assume also that $\delta < \frac{1}{2}$. Let $B \subset \Omega$ be a ball with a radius r. Define

$$s := \frac{(p-\delta)Q}{Q+1} < p - \delta.$$

Let ϕ be a smooth cut-off function, i.e. $\phi \in C_0^\infty(2B)$ such that $0 \le \phi \le 1$, $\phi = 1$ on B and $|X\phi| \le c/r$. Define

$$\tilde{u} = (u - u_{2B})\phi$$

and

$$E_\lambda = \{(M_\Omega |X\tilde{u}|^s)^{1/s} \le \lambda\} \quad \text{for } \lambda > 0.$$

Then the function $\tilde{u}$ is a Lipschitz function on E_λ with the Lipschitz constant $c\lambda$ (see Theorem 2.5). By the Kirszbraun theorem we can prolong $\tilde{u}$ to the Lipschitz function v_λ defined on the whole $\mathbf{R}^n$ with the same Lipschitz constant (see e.g. [5]). Moreover, there exists λ_0 such that for every $\lambda \ge \lambda_0$ the function v_λ has a compact support. Indeed, if $x \in \mathbf{R}^n \setminus 3B$, then

$$(M_\Omega |X\tilde{u}(x)|^s)^{1/s} = \sup_{B' \ni x,\, B' \cap 2B \neq \emptyset} \left(\fint_{B'} |X\tilde{u}|^s \, dx \right)^{1/s} \le \left(C_d \fint_{2B} |X\tilde{u}|^s \, dx \right)^{1/s}$$

because $|B'| \ge |B|$. Define $\lambda_0 := \big(C_d \fint_{2B} |X\tilde{u}|^s \, dx\big)^{1/s}$. Then we have

$$(M_\Omega |X\tilde{u}(x)|^s)^{1/s} < \lambda \quad \text{for } \lambda \ge \lambda_0, \tag{4.24}$$

and that implies $v_\lambda(x) = \tilde{u}(x) = 0$. We will take the function v_λ as a test function in equation (1.2).

Lemma 4.1. *Let $\tilde{u}$ be defined as above. Then the function $(M_\Omega |X\tilde{u}|^s)^{-\delta/s}$ belongs to the space A_r, where $r = p/s$.*

Proof. Fix a ball $B \subset \mathbf{R}^n$. Define $w(x) = (M_\Omega |X\tilde{u}(x)|^s)^{-\delta/s}$. Then we have

$$\fint_B w \, \mathrm{d}x \le \left(\inf_B M_\Omega |X\tilde{u}|^s \right)^{-\delta/s}$$

and

$$\left(\fint_B w^{1/(1-r)} \, \mathrm{d}x \right)^{r-1} = \left(\fint_B (M_\Omega |X\tilde{u}|^s)^{\delta/(p-s)} \, \mathrm{d}x \right)^{r-1}.$$

Since $\delta < p - s$ it follows that $(M_\Omega |X\tilde{u}|^s)^{\delta/(p-s)} \in A_1$. Hence

$$\left(\fint_B w^{1/(1-r)} \, \mathrm{d}x \right)^{r-1} \le \left(c \inf_B (M_\Omega |X\tilde{u}|^s)^{\delta/(p-s)} \right)^{(p-s)/s} = c \left(\inf_B M_\Omega |X\tilde{u}|^s \right)^{\delta/s}.$$

It follows immediately that

$$\left(\fint_B w \, \mathrm{d}x \right) \left(\fint_B w^{1/(1-r)} \, \mathrm{d}x \right)^{r-1} \le C,$$

and the proof is complete. □

Lemma 4.2. *Let $B \subset \Omega$ be a metric ball with radius r, and let $0 < \sigma \leq 5$. The following inequality holds*:

$$\int_{\sigma B} |u|^{p-1}(M_{\sigma B}|Xu|^s)^{(1-\delta)/s} \leq c_1 \int_{\sigma B} |u|^{p-\delta}\,\mathrm{d}x + c_2|\sigma B|\biggl(\fint_{\sigma B} |Xu|^{(p-\delta)Q/(Q+1)}\,\mathrm{d}x\biggr)^{(Q+1)/Q},$$

where the constants $c_1 = c_1(p)$ and $c_2 = c_2(p,r)$.

Proof. By Hölder's inequality we have

$$\fint_{\sigma B} |u|^{p-1}(M_{\sigma B}|Xu|^s)^{(1-\delta)/s} \leq \biggl(\fint_{\sigma B} |u|^{(p-1)s_1} dx\biggr)^{1/s_1} \times \biggl(\fint_{\sigma B} (M_{\sigma B}|Xu|^s)^{(1-\delta)s_2/s}\,dx\biggr)^{1/s_2},$$

where

$$s_1 = \frac{(p-\delta)Q}{(p-1)Q-(1-\delta)}, \quad s_2 = \frac{(p-\delta)Q}{(1-\delta)(Q+1)}.$$

To the right-hand side of the above inequality we apply first the Hardy–Littlewood Theorem (for the maximal function $M_{\sigma B} f$; all the balls σB, where $B \subset \Omega$, are contained in some open and bounded set). Then by Young inequality with the exponents $(p-\delta)/(p-1)$ and $(p-\delta)/(1-\delta)$ we obtain

$$\begin{aligned}\fint_{\sigma B} |u|^{p-1}(M_{\sigma B}|Xu|^s)^{(1-\delta)/s}\,dx &\leq c\biggl(\fint_{\sigma B} |u|^{(p-1)s_1}\,dx\biggr)^{1/s_1} \times \biggl(\fint_{\sigma B} |Xu|^{(1-\delta)s_2}\,dx\biggr)^{1/s_2}\\ &\leq c\biggl(\fint_{\sigma B} |u|^{(p-1)s_1}\,dx\biggr)^{(p-\delta)/(s_1(p-1))} + c\biggl(\fint_{\sigma B} |Xu|^{(p-\delta)Q/(Q+1)}\,dx\biggr)^{(Q+1)/Q}.\end{aligned} \tag{4.25}$$

For the first integral on the right-hand side we have

$$\biggl(\fint_{\sigma B} |u|^{(p-1)s_1}\,dx\biggr)^{1/s_1(p-1)} \leq \biggl(\fint_{\sigma B} |u-u_{\sigma B}|^{(p-1)s_1}\,dx\biggr)^{1/s_1(p-1)} + |u_{\sigma B}|.$$

Applying Hölder's inequality and then Sobolev's inequality we obtain

$$\begin{aligned} &c\left(\fint_{\sigma B}|u|^{(p-1)s_1}\,dx\right)^{(p-\delta)/s_1(p-1)} \\ (4.26)\quad &\le cr^{p-\delta}2^p\left(\fint_{\sigma B}|Xu|^{(p-\delta)Q/(Q+1)(p-1)}\,dx\right)^{(p-1)(Q+1)/Q}+2^p\fint_{\sigma B}|u|^{p-\delta}\,dx \end{aligned}$$

Then (4.25), (4.26) and Hölder's inequality (as $p\ge 2$) imply part (i) of the lemma. □

Corollary 4.3. *We have from Poincaré's inequality that*

$$\begin{aligned} \int_{\sigma B}|u|^{p-1}(M_{\sigma B}|X\tilde u|^s)^{(1-\delta)/s} &\le c_1\int_{\sigma B}|u|^{p-\delta}\,dx \\ &\quad + c_2|\sigma B|\left(\fint_{\sigma B}|Xu|^{(p-\delta)Q/(Q+1)}\,dx\right)^{(Q+1)/Q}. \end{aligned}$$

Proof of Theorem 1.2. We first show that $|Xu|\in L^{p+\tilde\delta}_{\mathrm{loc}}$ for some $\tilde\delta>0$. Let $\lambda\ge\lambda_0$. Take v_λ as a test function in (1.2):

$$\int_{3B}A(x,u,Xu)\cdot Xv_\lambda\,dx+\int_{3B}B(x,u,Xu)\cdot v_\lambda\,dx=0.$$

We will show that the assumptions of Theorem 3.3 are satisfied.

By definitions of E_λ, v_λ and by the growth conditions on A and B we have

$$\begin{aligned} &\int_{2B\cap E_\lambda}A(x,u,Xu)\cdot X\tilde u\,dx+\int_{2B\cap E_\lambda}B(x,u,Xu)\cdot\tilde u\,\mathrm{d}x \\ &\qquad\le\int_{3B\setminus E_\lambda}|A(x,u,Xu)|\cdot|Xv_\lambda|\,\mathrm{d}x+\int_{3B\setminus E_\lambda}|B(x,u,Xu)|\cdot|v_\lambda|\,dx \\ &\qquad\le c\int_{3B\setminus E_\lambda}\lambda|Xu|^{p-1}\,dx+c\int_{3B\setminus E_\lambda}\lambda|u|^{p-1}\,dx. \end{aligned}$$

The last inequality holds because vector fields X_j are Lipschitz continuous and there exists a constant c such that $|Xv_\lambda|\le c\lambda$ and $|v_\lambda|\le cr\lambda$, where r is the radius of B.

Multiplying both sides of the last inequality by $\lambda^{-(1+\delta)}$ and integrating over $(\lambda_0,+\infty)$ we obtain

$$\begin{aligned} L&=\int_{\lambda_0}^{\infty}\int_{2B\cap E_\lambda}\lambda^{-(1+\delta)}\big(A(x,u,Xu)\cdot X\tilde u+B(x,u,Xu)\tilde u\big)\,dx\,d\lambda \\ &\le c\int_{\lambda_0}^{\infty}\int_{3B\setminus E_\lambda}\lambda^{-\delta}(|u|^{p-1}+|Xu|^{p-1})\,dx\,d\lambda=P. \end{aligned}\tag{4.27}$$

Estimation of P. Changing the order of integration and using (4.24) we obtain

$$P \le \frac{c}{1-\delta}\int_{3B\setminus E_{\lambda_0}} (M_\Omega|X\tilde{u}|^s)^{1-\delta/s}(|u|^{p-1}+|Xu|^{p-1})\,dx$$
$$\le c\int_{3B}(M_\Omega|X\tilde{u}|^s)^{1-\delta/s}|u|^{p-1}\,dx + c\int_{3B}(M_\Omega|X\tilde{u}|^s)^{1-\delta/s}|Xu|^{p-1}\,dx.$$

To estimate the first component of the right-hand side we apply Lemma 4.2. To estimate the second component we apply the Hölder inequality and then the Hardy–Littlewood theorem. It follows that

(4.28)
$$P \le c\int_{3B}|u|^{p-\delta}\,dx + c\Big(\int_{3B}|Xu|^{(p-\delta)Q/(Q+1)}\,dx\Big)^{(Q+1)/Q} + c\int_{3B}|Xu|^{p-\delta}\,dx$$

Estimation of L. By changing the order of integration we obtain

$$L = \frac{1}{\delta}\int_{2B\setminus E_{\lambda_0}} \big(A(x,u,Xu)\cdot X\tilde{u} + B(x,u,Xu)\tilde{u}\big)(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx$$
$$+ \frac{1}{\delta}\int_{2B\cap E_{\lambda_0}} \big(A(x,u,Xu)\cdot X\tilde{u} + B(x,u,Xu)\tilde{u}\big)\lambda_0^{-\delta}\,dx.$$

Since $2B\setminus E_{\lambda_0} = 2B\setminus(2B\cap E_{\lambda_0})$, the growth conditions on A and B imply

$$\begin{aligned} L &\ge \frac{1}{\delta}\int_{2B}\big(A(x,u,Xu)\cdot X\tilde{u}\big)(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx \\ &\quad - \frac{2\alpha}{\delta}\int_{2B\cap E_{\lambda_0}}\big(|u|^{p-1}+|Xu|^{p-1}\big)|X\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx \\ &\quad - \frac{3\alpha}{\delta}\int_{2B}\big(|u|^{p-1}+|Xu|^{p-1}\big)|\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx \\ &= \frac{1}{\delta}(I_1 - 2\alpha I_2 - 3\alpha I_3), \end{aligned} \tag{4.29}$$

where

$$I_1 = \int_{2B}\big(A(x,u,Xu)\cdot X\tilde{u}\big)(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx,$$
$$I_2 = \int_{2B\cap E_{\lambda_0}}\big(|u|^{p-1}+|Xu|^{p-1}\big)|X\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx,$$
$$I_3 = \int_{2B}\big(|u|^{p-1}+|Xu|^{p-1}\big)|\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx.$$

Estimation of I_1. Define sets

$$D_1 = \big\{x\in 2B\setminus B : (M_\Omega|X\tilde{u}|^s)^{1/s} \le \delta(M_{2B}|Xu|^s)^{1/s}\big\}$$

and

$$D_2 = \{x \in 2B \backslash B : (M_\Omega |X\tilde{u}|^s)^{1/s} > \delta (M_{2B} |Xu|^s)^{1/s}\}.$$

Hence

$$\begin{aligned}
I_1 &\geq \int_{B \cup D_2} A(x,u,Xu) \cdot Xu (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad + \int_{D_2} A(x,u,Xu)(u - u_{2B}) X\phi (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad - \alpha \int_{D_1} (|u|^{p-1} + |Xu|^{p-1}) |X\tilde{u}| (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\geq \beta \int_B |Xu|^p (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad - \frac{c\alpha}{r} \int_{D_2} (|u|^{p-1} + |Xu|^{p-1}) |u - u_{2B}| (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad - \alpha \int_{D_1} (|u|^{p-1} + |Xu|^{p-1}) |X\tilde{u}| (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx.
\end{aligned}$$

Lemma 4.1 yields

$$\begin{aligned}
I_1 &\geq c\beta \int_B (M_B |Xu|^s)^{p/s} (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad - \frac{c\alpha}{r} \int_{D_2} (|u|^{p-1} + |Xu|^{p-1}) |u - u_{2B}| (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx \\
&\quad - \alpha \int_{D_1} (|u|^{p-1} + |Xu|^{p-1}) |X\tilde{u}| (M_\Omega |X\tilde{u}|^s)^{-\delta/s} \, dx =: I_{1,1} - I_{1,2} - I_{1,3}.
\end{aligned}$$

We will estimate each integral $I_{1,k}$, for $k = 1,2,3$.

If $x \in \frac{1}{2}B$ then we have

$$\begin{aligned}
(M_\Omega |X\tilde{u}|^s)^{1/s}(x) &\leq \sup_{B' \ni x,\, B' \subset B} \left(\fint_{B'} |X\tilde{u}|^s \right)^{1/s} + \sup_{B' \ni x,\, B' \cap \partial B \neq \emptyset} \left(\fint_{B'} |X\tilde{u}|^s \right)^{1/s} \\
&\leq (M_B |Xu|^s)^{1/s} + c \left(\fint_{2B} |Xu|^s \, dx \right)^{1/s} \\
&\quad + \frac{c}{r} \left(\fint_{2B} |u - u_{2B}|^s \, dx \right)^{1/s} \\
&\leq (M_B |Xu|^s)^{1/s} + c \left(\fint_{2B} |Xu|^s \, dx \right)^{1/s}.
\end{aligned}$$

The second inequality comes from the doubling condition and the last one from Poincaré's inequality. Let $G \subset \frac{1}{2}B$ be such that if $x \in G$ then

$$(M_B|Xu|^s)^{1/s} \geq c\Big(\fint_{2B} |Xu|^s\,dx\Big)^{1/s}.$$

Then we have

$$\begin{aligned} I_{1,1} &\geq c\beta \int_G (M_B|Xu|^s)^{p/s}(M_B|Xu|^s)^{-\delta/s}\,dx \\ &= c\int_{B/2} (M_B|Xu|^s)^{(p-\delta)/s}\,dx - c\Big(\fint_{2B} |Xu|^s\,dx\Big)^{(p-\delta)/s} \int_{B/2\setminus G} dx. \end{aligned}$$

Hence

$$I_{1,1} \geq c\int_{B/2} |Xu|^{p-\delta}\,dx - c|B|\Big(\fint_{2B} |Xu|^s\,dx\Big)^{(p-\delta)/s}. \tag{4.30}$$

By the definition of D_2, Theorem 2.5 and the properties of maximal function we have

$$\begin{aligned} I_{1,2} &\leq \frac{c\alpha\delta^{-\delta}}{r}\int_{2B} (|u|^{p-1}+|Xu|^{p-1})|u-u_{2B}|(M_{2B}|Xu|^s)^{-\delta/s}\,dx \\ &\leq c\alpha\delta^{-\delta}\Big[\int_{2B} |u|^{p-1}(M_{2B}|Xu|^s)^{(1-\delta)/s}\,dx \\ &\quad + \frac{1}{r}\int_{2B} |u-u_{2B}|(M_{2B}|Xu|^s)^{(p-1-\delta)/s}\,dx\Big]. \end{aligned}$$

The first component of the right-hand side is estimated, by Lemma 4.2,

$$c\int_{2B} |u|^{p-\delta}\,dx + c\Big(\fint_{2B} |Xu|^{(p-\delta)Q/(Q+1)}\,dx\Big)^{(Q+1)/Q}.$$

To the second component of the right-hand side we apply Hölder's inequality with exponents

$$\frac{(p-\delta)Q}{Q+1} \quad \text{and} \quad \frac{p-\delta}{p-1-\delta}\frac{Q}{Q+1}.$$

Next, by Poincaré's inequality and the Hardy–Littlewood Theorem, we have

$$\begin{aligned} &\frac{1}{r}\int_{2B} |u-u_{2B}|(M_{2B}|Xu|^s)^{(p-1-\delta)/s}\,dx \\ &\quad \leq \frac{|2B|}{r}\Big(\fint_{2B} |u-u_{2B}|^{(p-\delta)Q/(Q+1)}\,dx\Big)^{(Q+1)/(p-\delta)Q} \\ &\quad\quad \times \Big(\fint_{2B} (M_{2B}|Xu|^s)^{(p-\delta)Q/s(Q+1)}\,dx\Big)^{(p-1-\delta)(Q+1)/(p-\delta)Q} \\ &\quad \leq |2B|\Big(\fint_{2B} |Xu|^{(p-\delta)Q/(Q+1)}\Big)^{(Q+1)/Q}. \end{aligned}$$

Thus

$$I_{1,2} \le c\alpha\delta^{-\delta}\left[\int_{2B}|u|^{p-\delta}\,dx + |2B|\left(\fint_{2B}|Xu|^{(p-\delta)Q/(Q+1)}\right)^{(Q+1)/Q}\right]. \tag{4.31}$$

For the integral $I_{1,3}$ we have

$$I_{1,3} \le \alpha\int_{D_1}(|u|^{p-1}+|Xu|^{p-1})(M_\Omega|X\tilde{u}|^s)^{(1-\delta)/s}\,dx,$$

and, using the definition of D_1,

$$\begin{aligned}
I_{1,3} &\le \alpha\delta^{1-\delta}\int_{2B}(|u|^{p-1}+|Xu|^{p-1})(M_{2B}|Xu|^s)^{(1-\delta)/s}\,dx\\
&\le \alpha\delta^{1-\delta}\int_{2B}|u|^{p-1}(M_{2B}|Xu|^s)^{(1-\delta)/s}\,dx\\
&\quad + \alpha\delta^{1-\delta}\int_{2B}(M_{2B}|Xu|^s)^{(p-\delta)/s}\,dx.
\end{aligned}$$

To the first component of the right-hand side we apply Lemma 4.2. Because of the coefficient $\delta\cdot\delta^{-\delta}$ it will be consumed in the inequality (4.31). The second component, by the Hardy–Littlewood Theorem, is estimated by

$$c\alpha\delta^{1-\delta}\int_{2B}|Xu|^{p-\delta}\,dx.$$

Combining (4.30) and (4.31) with the estimation of $I_{1,3}$, we obtain finally

$$\begin{aligned}
I_1 &\ge c\int_{B/2}|Xu|^{p-\delta}\,dx - c\int_{2B}|u|^{p-\delta}\,dx - c\delta\int_{2B}|Xu|^{p-\delta}\,dx\\
&\quad - c|2B|\left(\fint_{2B}|Xu|^{(p-\delta)Q/(Q+1)}\right)^{(Q+1)/Q}.
\end{aligned} \tag{4.32}$$

Estimation of I_2. We have

$$I_2 \le \int_{2B}|u|^{p-1}(M_\Omega|X\tilde{u}|^s)^{1-\delta/s}\,dx + \int_{2B\cap E_{\lambda_0}}|Xu|^{p-1}|X\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx. \tag{4.33}$$

Estimation of the first component follows from Lemma 4.2. We will work with the second one. Fix a constant $\gamma>0$. Assume that $y\in 2B\cap E_{\lambda_0}$. If $|Xu(y)|\ge\lambda_0/\gamma$, we have

$$\begin{aligned}
\int_{2B\cap E_{\lambda_0}}|Xu|^{p-1}|X\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx &\le \lambda_0^{1-\delta}\int_{2B}|Xu|^{p-1}\,dx\\
&\le \gamma^{1-\delta}\int_{2B}|Xu|^{p-\delta}\,dx.
\end{aligned}$$

If $|Xu(y)| \le \lambda_0/\gamma$ then, since

$$\lambda_0 := c\Big(\fint_{2B} |X\tilde{u}|^s \, dx\Big)^{1/s} \quad \text{and} \quad \lambda_0 \le \inf_{2B}\big(M_\Omega |X\tilde{u}|^s\big)^{1/s},$$

we obtain

$$\begin{aligned}\int_{2B\cap E_{\lambda_0}} |Xu|^{p-1}|X\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\, dx &\le \lambda_0^{p-1}\gamma^{1-p}\lambda_0^{-\delta}\int_{2B}|X\tilde{u}|\,dx\\ &\le c\gamma^{1-p}|2B|\Big(\fint_{2B}|X\tilde{u}|^s dx\Big)^{p-1-\delta/s}\fint_{2B}|X\tilde{u}|\,dx\\ &\le c\gamma^{1-p}|2B|\Big(\fint_{2B}|Xu|^s\,dx\Big)^{p-\delta/s},\end{aligned}$$

where the last inequality follows from Poincaré's inequality. Thus the second component of the right-hand side in (4.33) is estimated by

$$\gamma^{1-\delta}\int_{2B}|Xu|^{p-\delta}\,dx + c\gamma^{1-p}|2B|\Big(\fint_{2B}|Xu|^s\,dx\Big)^{p-\delta/s}.$$

Since $s = (p-\delta)Q/(Q+1)$, the integral I_2 satisfies

$$\begin{aligned}I_2 \le c\int_{2B}|u|^{p-\delta}\,dx + (c+\gamma^{1-p})|2B|\Big(\fint_{2B}|Xu|^{(p-\delta)Q/(Q+1)}\Big)^{(Q+1)/Q}&\\ +\gamma^{1-\delta}\int_{2B}|Xu|^{p-\delta}\,dx.&\end{aligned}\tag{4.34}$$

Estimation of I_3. For the integral I_3 we have

$$I_3 \le \int_{2B}|u|^{p-1}|\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx + \int_{2B}|Xu|^{p-1}|\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx.$$

By Corollary 4.3 the first component on the right-hand side gives the parts which were considered earlier in the integral $I_{1,2}$. We might estimate the second component

$$\begin{aligned}\int_{2B}|Xu|^{p-1}|\tilde{u}|(M_\Omega|X\tilde{u}|^s)^{-\delta/s}\,dx &\le \int_{2B}|Xu|^{p-1}|\tilde{u}|\,|X\tilde{u}|^{-\delta}\,dx\\ &\le \int_{2B}|Xu|^{p-1-\delta}|u-u_{2B}|\,dx\\ &\quad + cr^\delta\fint_{2B}|Xu|^{p-1}|u-u_{2B}|^{1-\delta}dx,\end{aligned}\tag{4.35}$$

because $|X\tilde{u}| \le |Xu| + (c/r)|u - u_{2B}|$. To the last two integrals we apply Hölder's inequality choosing for the first integral

$$s_1 = \frac{Q(p-\delta)}{(Q+1)(p-1-\delta)}, \quad s_2 = \frac{Q(p-\delta)}{(Q+1)-(p-\delta)}$$

and for the second integral

$$t_1 = \frac{(p-\delta)Q}{(p-1)(Q+1)}, \quad t_2 = \frac{(p-\delta)Q}{Q(1-\delta)-(p-1)}.$$

Then, by the Sobolev inequality, the right-hand side of (4.35) does not exceed

$$c|2B|\biggl(\fint_{2B} |Xu|^{Q(p-\delta)/(Q+1)}\,dx\biggr)^{(Q+1)/Q}.$$

Thus the integral I_3 can be estimated by the same expression as $I_{1,2}$, i.e.

$$I_3 \le c\int_{2B} |u|^{p-\delta}\,dx + c|2B|\biggl(\fint_{2B} |Xu|^{(p-\delta)Q/(Q+1)}\biggr)^{(Q+1)/Q}. \tag{4.36}$$

Combining (4.28), (4.29) with (4.32), (4.34) and (4.36) we obtain

$$\begin{aligned}
&\frac{1}{\delta}\biggl[c\int_{B/2} |Xu|^{p-\delta}\,dx - c\int_{2B} |u|^{p-\delta}\,dx - (c\delta+\gamma^{1-\delta})\int_{2B} |Xu|^{p-\delta}\,dx \\
&\qquad - (c+\gamma^{1-p})|2B|\biggl(\fint_{2B} |Xu|^{(p-\delta)Q/(Q+1)}\biggr)^{(Q+1)/Q}\biggr] \\
&\quad \le c\int_{3B} |Xu|^{p-\delta}\,dx.
\end{aligned}$$

Therefore u satisfies a reverse Hölder inequality:

$$\begin{aligned}
\fint_{B/2} |Xu|^{p-\delta}\,dx &\le c\gamma^{1-p}\biggl[\biggl(\fint_{3B} |Xu|^{(p-\delta)Q/(Q+1)}\biggr)^{(Q+1)/Q} + \fint_{3B} |u|^{p-\delta}\,dx\biggr] \\
&\quad + c(\gamma^{1-\delta}+\delta)\fint_{3B} |Xu|^{p-\delta}\,dx.
\end{aligned} \tag{4.37}$$

We can apply Theorem 3.3 with

$$g = |Xu|^{(p-\delta)Q/(Q+1)}, \quad f = |u|^{(p-\delta)Q/(Q+1)},$$

and

$$\theta = c(\gamma^{1-\delta}+\delta), \quad b = c\gamma^{1-p}, \quad q = \frac{Q+1}{Q}.$$

By Sobolev's inequality the function f is in L^{r_0}, where $r_0 = (Q+1)/(Q-p+\delta) > q$. The constants γ and δ can be chosen sufficiently small such that $\theta < \theta_0$. Then we obtain

Lemma 4.4. *If the function u is a very weak solution of* (1.1) *and* $|Xu| \in L^{p-\delta}$, *where* $\delta \in (-\delta_0, \delta_0)$ *for some* $\delta_0 > 0$, *then there exists* $\varepsilon_0 > 0$ *such that* $|Xu| \in L^{p-\delta+\varepsilon_0}$.

By iteration of the procedure we prove that $|Xu| \in L^{p+\tilde{\delta}}$, for some $\tilde{\delta} > 0$, say $\tilde{\delta} = \frac{1}{2}\varepsilon_0$, where ε_0 is given by Theorem 3.3. It is important to control the constant b and thus the constant ε_0. They both depend on α, β, C_d, constants in Poincaré's and Sobolev's inequalities and in the Hardy–Littlewood Theorem. The last one depends formally on the exponent $p-\delta$, but by the Riesz–Thorin Theorem (Riesz Convexity Theorem, see e.g. [24]), if $p \in [2, p_0]$, then the Hardy–Littlewood constant may be chosen independently of p.

Now we show that $u \in L^{p+\tilde{\delta}}_{\mathrm{loc}}$. Assume that Ω' is a compact subset of Ω. Let $\phi \in C_0^\infty(\Omega)$ be a cut-off function, such that $\phi = 1$ onto Ω'. Let $v = u \cdot \phi$. Then $v \in W^{1,p-\delta}(\mathbf{R}^n)$ and $|Xv| \in L^{p+\tilde{\delta}}(\mathbf{R}^n)$. By Poincaré's inequality

$$\left(\fint_B |v - v_B|^{p+\tilde{\delta}}\,\mathrm{d}x\right)^{1/p+\tilde{\delta}} \le cr\left(\fint_B |Xv|^{p+\tilde{\delta}}\,\mathrm{d}x\right)^{1/p+\tilde{\delta}}.$$

Define

$$E_\lambda = \bigl\{x : \bigl(M_\Omega |Xv|^{p+\tilde{\delta}}(x)\bigr)^{1/p+\tilde{\delta}} \le \lambda\bigr\}.$$

Then

$$|\mathbf{R}^n \setminus E_\lambda| \le c\lambda^{-p} \int_{\mathbf{R}^n} |Xv|^{p+\tilde{\delta}}\,\mathrm{d}x.$$

For every $\lambda > 0$ there exists a Lipschitz function v_λ on $\mathbf{R}^n$ such that $v = v_\lambda$ for a.e. $x \in E_\lambda$. Then

$$\int_{\mathbf{R}^n} |Xv_\lambda|^{p+\tilde{\delta}}\,\mathrm{d}x \le \int_{E_\lambda} |Xv|^{p+\tilde{\delta}}\,\mathrm{d}x + c\lambda^{p+\tilde{\delta}}|\mathbf{R}^n \setminus E_\lambda| \le \int_{\mathbf{R}^n} |Xv|^{p+\tilde{\delta}}\,\mathrm{d}x,$$

and this implies that $|Xv_\lambda|$ are uniformly bounded. It follows from the same argument as previously that there exists λ_0 such that for $\lambda > \lambda_0$ we have $\operatorname{supp} v_\lambda = B \supset \Omega'$. Thus the set $\{v_\lambda : \lambda > \lambda_0\}$ is bounded in $W_0^{1,p+\tilde{\delta}}(B)$. Since $v_\lambda \to v$ for $\lambda \to \infty$ we have $v \in L^{p+\tilde{\delta}}(B)$ and $u \in W^{1,p+\tilde{\delta}}(B)$. □

5. Compactness

In this section we additionally assume that for all $v, w \in \mathbf{R}$ the following inequalities hold:

$$|A(x, v, \xi) - A(x, w, \zeta)| \le \alpha|\xi - \zeta|(|\xi| + |\zeta|)^{p-2}$$

and

$$\langle A(x, v, \xi) - A(x, w, \zeta)|\xi - \zeta\rangle \ge \beta|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}.$$

Theorem 5.3 below is a classical compactness theorem for weak solutions. It follows from Caccioppoli type estimates with the natural exponent p. Together with Theorem 1.2 they imply Theorem 5.4; i.e., the compactness result where it is enough to assume that very weak solutions are bounded in $W_X^{1,r}$ for some $r < p$. In this way, for the nonlinear elliptic case, Theorem 5.4 was proved by Iwaniec and Sbordone [15]. The alternative approach to prove the result is to use the Caccioppoli type inequalities below the natural exponent.

Theorem 5.1. *Let* $\{u_i\}_{i\in\mathbf{N}}$, $u_i \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$, *be a family of weak local solutions of the equation* (1.1). *Then for* j, $k \in \mathbf{N}$ *we have*

$$\|\phi X(u_j - u_k)\|_p^p \le c\|(u_j - u_k)X\phi\|_p^2 \cdot \left(\|\phi X u_j\|_p^{p-2} + \|\phi X u_k\|_p^{p-2}\right) + c\sup_{i\in\mathbf{N}} \|u_i\|^{p-1}_{W_X^{1,p}} \cdot \|\phi(u_j - u_k)\|_p$$

for every function $\phi \in C_0^\infty(\Omega)$, *where the constant* c *depends only on* p, α, β.

Proof. Assume $\phi \in C_0^\infty(\Omega)$ and let $\eta = \phi^p(u_j - u_k)$ be a test function in the equation (1.1). By conditions on A and B we obtain

$$\beta \int_\Omega |\phi|^p |Xu_j - Xu_k|^2 (|Xu_j| + |Xu_k|)^{p-2}\,\mathrm{d}x \le I_1 + I_2, \tag{5.38}$$

where

$$I_1 := p\alpha \int_\Omega |X\phi|\,|\phi|^{p-1}|u_j - u_k|\,|Xu_j - Xu_k|(|Xu_j| + |Xu_k|)^{p-2}\,\mathrm{d}x$$

and

$$I_2 := \alpha \int_\Omega |\phi|^p |u_j - u_k|\left(|u_j|^{p-1} + |u_k|^{p-1} + |Xu_j|^{p-1} + |Xu_k|^{p-1}\right)\mathrm{d}x.$$

Applying to I_1 Young's inequality with exponents $2,2$ we can estimate the integral by sum of expressions:

$$p\alpha\theta^2 \int_\Omega |\phi|^p |Xu_j - Xu_k|^2 (|Xu_j| + |Xu_k|)^{p-2}\,\mathrm{d}x \tag{5.39}$$

and

$$\frac{p\alpha c}{\theta^2}\|(u_j - u_k)X\phi\|_p^2\left(\|\phi Xu_j\|_p^{p-2} + \|\phi Xu_k\|_p^{p-2}\right), \tag{5.40}$$

where the constant c depends only on p (to obtain the second component we used Hölder's inequality with exponents $\frac{1}{2}p$ and $p/(p-2)$). We put the expression (5.39) on the left-hand side of (5.38). The constant $\theta > 0$ is such that $\beta - p\alpha\theta^2 > 0$.

To estimate the integral I_2 we apply Hölder's inequality with exponents $p, p/(p-1)$. We obtain the estimation of I_2 by

$$\alpha\|\phi(u_j-u_k)\|_p \cdot \sup_{i\in\mathbf{N}} \|\phi u_i\|^{p-1}_{W^{1,p}_{\mathrm{loc}}}. \tag{5.41}$$

Combining (5.38)–(5.41) we obtain an inequality

$$\begin{aligned}\|\phi X(u_j-u_k)\|_p^p \le{}& \frac{p\alpha c}{\theta^2(\beta-p\alpha\theta^2)}\|(u_j-u_k)X\phi\|_p^2 \cdot \left(\|\phi X u_j\|_p^{p-2} + \|\phi X u_k\|_p^{p-2}\right)\\ &+\frac{\alpha}{\beta-p\alpha\theta^2}\sup_{i\in\mathbf{N}}\|u_i\|^{p-1}_{W^{1,p}_X}\cdot\|\phi(u_j-u_k)\|_p,\end{aligned}$$

and the proof is complete. □

Theorem 5.2. *If a function $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ is a weak local solution of the equation* (1.1), *then it satisfies a Caccioppoli type inequality*

$$\|\phi X u\|_p \le c\|uX\phi\|_p + c\|u\phi\|_p$$

for every function $\phi \in C_0^\infty(\Omega)$, where the constant c depends only on constants p, α and β.

Proof. The procedure is similar to the previous one. As a test function we take $\eta = \phi^p u$, where $\phi \in C_0^\infty$. By the conditions on A and B we obtain an inequality

$$\begin{aligned}\beta\|\phi X u\|_p^p \le{}& \left(p\alpha\|uX\phi\|_p + \alpha\|u\phi\|_p\right)\cdot\|u\phi\|_p^{p-1}\\ &+\left(p\alpha\|uX\phi\|_p + \alpha\|u\phi\|_p\right)\cdot\|\phi X u\|_p^{p-1},\end{aligned}$$

which implies the theorem. □

The above theorems imply

Theorem 5.3. *If a family of weak solutions $\{u_i\} \subset W^{1,p}_{X,\mathrm{loc}}(\Omega)$ of the equation* (1.1) *is bounded in $L^p(\Omega)$, then it is compact in $W^{1,p}_X(\Omega)$.*

Combining the theorem on the higher integrability of very weak solutions and the above compactness theorem we obtain the following result:

Theorem 5.4. *Let F be any compact subset of Ω and let δ be the constant from Theorem* 1.2. *Then if a family $\{u_i\}_{i\in\mathbf{N}}$ of very weak solutions of the equation* (1.1), *$u_i \in W^{1,r}_X(\Omega)$, is bounded in $W^{1,r}_X(\Omega)$, where $p-\delta < r < p$, then this family compact in $W^{1,p}_X(F)$.*

References

[1] Bensoussan, A., and J. Frehse: Regularity Results for Nonlinear Elliptic Systems and Applications. - Springer–Verlag, Berlin–Heidelberg, 2002.

[2] Bojarski, B.: Generalized solutions of a system of differential equations of first order and of elliptic type with discontinuous coefficients. - Mat. Sb. N.S. 43:85, 1957, 451–503 (in Russian).

[3] Calderon, A. P.: Inequalities for the maximal function relative to a metric. - Studia Math. 57, 1976, 297–306.

[4] Citti, G., N. Garofalo, and E. Lanconelli: Harnack's inequality for sum of squares plus a potential. - Amer. J. Math. 115, 1993, 699–734.

[5] Federer, H.: Geometric Measure Theory. - Grundlehren Math. Wiss. 153, Springer-Verlag New York Inc., New York, 1969.

[6] Franchi, B., R. Serapioni, and F. Serra Cassano: Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields. - Boll. Un. Mat. Ital. B 11, 1997, 83–117.

[7] Garofalo, N., and D. M. Nhieu: Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces. - Comm. Pure Appl. Math. 49, 1996, 1081–1144.

[8] Garofalo, N., and D. M. Nhieu: Lipchitz continuity, global smooth approximation and extension theorems for Sobolev functions in Carnot–Carathéodory spaces. - J. Analyse Math. 74, 1998, 67–97.

[9] Gehring, F. W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. - Acta Math. 130, 1973, 265–277.

[10] Giaquinta, M.: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. - Ann. Math. Stud. 105, Princeton Univ. Press, Princeton, NJ, 1983.

[11] Giaquinta, M., and G. Modica: Regularity results for some classes of higher order non linear elliptic systems. - J. Reine Angew. Math. 311–312, 1979, 145–169.

[12] Hajłasz, P., and P. Koskela: Sobolev met Poincaré. - Mem. Amer. Math. Soc. 145, 2000.

[13] Iwaniec, T.: p-harmonic tensors and quasilinear mappings. - Ann. of Math. 136, 1992, 589–624.

[14] Iwaniec, T.: The Gehring lemma. - In: Quasiconformal Mappings and Analysis. A Collection of Papers Honoring F. W. Gehring, edited by Duren P. et al. Springer-Verlag, New York, NY, 1998, 181–204.

[15] Iwaniec, T., and C. Sbordone: Weak minima of variational integrals. - J. Reine Angew. Math. 454, 1994, 143–161.

[16] Jerison, D.: The Poincaré inequality for vector fields satisfying Hörmander condition. - Duke Math. J. 53, 1986, 303–343.

[17] Lewis, J. L.: On very weak solutions of certain elliptic systems. - Comm. Partial Differential Equations 18, 1993,1515–1537.

[18] Lewis, J. L., and J. Kinnunen: Higher integrability for parabolic systems of p-Laplacian type. - Duke Math. J. 102, 2002, 253–271.

[19] Lewis, J. L., and J. Kinnunen: Very weak solutions of parabolic systems of p-Laplacian type. - Ark. Mat. 40, 2002, 105–132.

[20] Meyers, N. G., and A. Elcrat: Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions. - Duke Math. J. 42, 1975, 121–136.

[21] Mikkonen, P.: On the Wolff potential and quasilinear elliptic equations involving measures. - Ann. Acad. Sci. Fenn. Math. Diss. 104, 1996.

[22] Nagel, A., E. M. Stein, and S. Wainger: Balls and metrics defined by vector fields I: Basic properties. - Acta Math. 55, 1985, 103–147.

[23] Stein, E. M.: Harmonic Analysis. - Princeton Math. Ser. 43, Monographs in Harmonic Analysis, III. Princeton Univ. Press, Princeton, NJ, 1993.

[24] Stein, E. M., and G. Weiss: Introduction to Fourier Analysis on Euclidean Spaces. - Princeton Math. Ser. 32, Princeton Univ. Press, Princeton, NJ, 1971.

[25] Strömberg, J.-O., and A. Torchinsky: Weighted Hardy Spaces. - Lecture Notes in Math. 1381, Springer–Verlag, Berlin–Heidelberg, 1989.

[26] Varopoulos, N. Th., L. Saloff–Coste, and T. Coulhon: Analysis and Geometry on Groups. - Cambridge Univ. Press, 1992.

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