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HYPERBOLIC AND UNIFORM DOMAINS IN BANACH SPACES

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Abstract. A domain G in a Banach space is said to be δ-hyperbolic if it is a Gromov δ-hyperbolic space in the quasihyperbolic metric. Then G has the Gromov boundary $\partial^* G$ and the norm boundary ∂G. We show that the following properties are quantitatively equivalent: (1) G is C-uniform. (2) G is δ-hyperbolic and there is a natural bijective map $G \cup \partial^* G \to G \cup \partial G$, which is η-quasimöbius rel $\partial^* G$. (3) G is δ-hyperbolic and there is a natural η-quasimöbius homeomorphism $\partial^* G \to \partial G$. In a euclidean space, this improves a result of Bonk–Heinonen–Koskela, whose estimates depend on dimension and on a base point.

1. Introduction

Let E be a real Banach space and let $G \subsetneq E$ be a domain (open connected nonempty set). We say that G is *δ-hyperbolic*, $\delta \ge 0$, if G is a δ-hyperbolic metric space in the sense of M. Gromov when equipped with the quasihyperbolic metric k_G, defined by the element of length $|dx|/d(x, \partial G)$.

In a recent paper [BHK], M. Bonk, J. Heinonen and P. Koskela study, among other things, relations between hyperbolic and uniform domains in R^n. They show that the following properties are equivalent for a bounded domain $G \subset \mathsf{R}^n$:

(1) G is C-uniform,

(2) G is δ-hyperbolic and there is a natural η-quasisymmetric homeomorphism between the Gromov boundary $\partial^* G$ and the euclidean boundary ∂G of G,

(3) G is δ-hyperbolic and c-linearly locally connected.

The terminology is recalled in Section 2. The metric $d_{p,\varepsilon}$ of $\partial^* G$ in (2) depends on a base point $p \in G$ and on a real parameter ε, $0 < \varepsilon \le 1 \wedge (1/5\delta)$.

The purpose of this paper is to study whether this result can be extended from R^n to arbitrary Banach spaces. We show that in this general case, (1) and (2) are still equivalent and that (1) implies (3). By an example we show that (3) does not imply (1).

We also improve the result of [BHK] in two other directions:

(a) Instead of bounded domains we consider arbitrary domains. ([BHK] considers arbitrary domains in the spherical metric.) The quasisymmetry must then be replaced by quasimöbius. This has the further advantage that the result is independent of the base point.

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(b) In (2), we get a natural bijective map between the Gromov closure $G^* = G \cup \partial G$ and the norm closure $\overline{G}$, and this map is η-quasimöbius rel $\partial^* G$.

The main results are given in 2.12, 2.29 and 3.27. A summary with a precise formulation of the quantitativeness of the results is given in 3.1. In particular, we obtain a dimension-free version of the result (1) $\Leftrightarrow$ (2) of [BHK].

The proof for (1) $\Rightarrow$ (2) is a variation of the proof of the corresponding result in [BHK]. On the other hand, the proof for (2) $\Rightarrow$ (1) is entirely different. In fact, the proof for this in [BHK] is definitely n-dimensional, as it makes use of the modulus of a path family.

The metric space (G, k_G) need not be geodesic, and it is not locally compact if $\dim E = \infty$. Therefore we cannot directly make use of the theory of Gromov hyperbolic spaces given in standard textbooks like [GdH] and [BH]. An exposition of the basic theory of hyperbolic spaces in the more general setting is given in the author's article [Vä8].

It turns out that the lack of geodesics and local compactness is somewhat inconvenient but not a real difficulty. The hard problem in general Banach spaces compared with the euclidean space is the lack of Lebesgue measure and integration. Balls have no volumes. If the proof of an n-dimensional result makes use of volume or space integration, the proof of its Banach version (if true) requires a new method.

On the other hand, line integrals are available, and the quasihyperbolic metric of a domain can be defined as in R^n.

We prove that (1) $\Rightarrow$ (2) in Section 2 and the converse in Section 3. In Section 4 we give a counterexample proving that (3) $\not\Rightarrow$ (1). In Section 5 we give a lower bound for the hyperbolicity constant of a domain.

2. Uniform domains are hyperbolic with a boundary condition

In this section we show that if G is a uniform domain, then G is hyperbolic and there is a natural bijection $G^* \to \overline{G}$, which is quasimöbius rel $\partial^* G$. The result is given in 2.12 and 2.29.

2.1. *Notation and terminology.* Throughout the paper we assume that E is a real Banach space with $\dim E \geq 2$. The norm of a point $x \in E$ is written as $|x|$.

The basic notation is fairly standard and the same as in [Vä8]. We let $\mathsf{R}, \mathsf{Z}, \mathsf{N}$ denote the sets of real numbers, integers and positive integers, respectively. Balls and spheres in a metric space X are written as

$$B(a,r) = \{x : |x-a| < r\},\ \overline{B}(a,r) = \{x : |x-a| \leq r\},$$
$$S(a,r) = \{x : |x-a| = r\}.$$

In a vector space, we may drop the center a if $a = 0$. More generally, if $\emptyset \neq A \subset X$, we set

$$\overline{B}(A,r) = \{x \in X : d(x,A) \leq r\}.$$

The distance between nonempty sets A, B is $d(A,B)$, and the diameter of a set A is $d(A)$. If there is a symbol denoting the metric, say k, we use the notation $k(A,B)$ and $k(A)$ for the distance and the diameter, and we may use the notation $\overline{B}_k(a,r)$ etc. for balls and r-neighborhoods. In a vector space, $[a,b]$ is the closed line segment between points a and b, and $[a,b) = [a,b] \setminus \{b\}$.

We write $\alpha\colon x \curvearrowright y$ if α is an arc joining x and y. If needed, this notation also gives an orientation for α from x to y. For an arc α, we let $\alpha[u,v]$ denote the closed subarc of α between points $u,v \in \alpha$, and for half open subarcs we write $\alpha[u,v)$. Occasionally, we consider a singleton $\{x\}$ as an arc $\alpha\colon x \curvearrowright x$. For real numbers s, t we set $s \wedge t = \min\{s,t\}$, $s \vee t = \max\{s,t\}$. By an expression like ab/cde we mean $(ab)/(cde)$. To simplify expressions we often omit parentheses writing $fx = f(x)$ etc.

In hyperbolic spaces we use the terminology and notation given in [Vä8]. However, the general distance $|x-y|$ will usually be replaced by the quasihyperbolic distance $k(x,y)$; see 2.7.

2.2. *Quasihyperbolic metric.* Let $G \subset E$ be a domain. Without further notice, we always assume that $G \neq E$. For $x \in G$ we write $d(x) = d(x, \partial G)$. The *quasihyperbolic length* of a rectifiable arc $\alpha \subset G$ is defined by

$$l_k(\alpha) = \int_\alpha \frac{|dx|}{d(x)},$$

and the *quasihyperbolic distance* between points $x,y \in G$ is the number

$$k(x,y) = k_G(x,y) = \inf\{l_k(\alpha) \mid \alpha\colon x \curvearrowright y,\ \alpha \subset G\}. \tag{2.3}$$

Then $l_k(\alpha)$ is the length of α in the metric k; see [BHK, A.7]. Hence the space (G,k) is *intrinsic*, but it need not be geodesic; see [Vä6, 3.5]. We shall also consider the norm distance $|x-y|$ and the distance

$$j(x,y) = j_G(x,y) = \log\left(1 + \frac{|x-y|}{d(x) \wedge d(y)}\right).$$

We recall that always

$$\log\frac{d(x)}{d(y)} \le j(x,y) \le k(x,y);$$

see [Vä6, 3.7(1)]. These inequalities will be frequently used without special reference.

2.6. *Uniform domains.* We recall two approaches to uniform domains: through quasihyperbolic metric and through uniform arcs.

Let $C \geq 1$. A domain G is *quasihyperbolically* C*-uniform* or briefly QH C-uniform if $k(x,y) \leq Cj(x,y)$ for all $x, y \in G$.

An arc $\alpha\colon x \curvearrowright y$ in G is C*-uniform* in G if

(1) $l(\alpha[x,z]) \wedge l(\alpha[z,y]) \leq Cd(z)$ for all $z \in \alpha$ (cigar condition),

(2) $l(\alpha) \leq C|x-y|$ (turning condition).

The domain G is C*-uniform* if each pair of points in G can be joined by a C-uniform arc in G.

The properties C-uniform and QH C-uniform are quantitatively equivalent; see [Vä6, 10.17].

2.5. *Quasiconvexity and quasigeodesics.* A metric space is c*-quasiconvex* if each pair of points x, y can be joined by an arc with length at most $c|x-y|$. An arc α in a domain G is a c*-quasigeodesic* in G if it is c-quasiconvex in the quasihyperbolic metric, that is, $l_k(\alpha[x,y]) \leq ck(x,y)$ for all $x, y \in \alpha$. Quasigeodesics have been called neargeodesics in my earlier papers.

The following results are from [Vä6, 9.4 and 10.9].

2.6. Lemma. (1) *Let $c > 1$ and let x, y be points in a domain G. Then there is a c-quasigeodesic $\alpha\colon x \curvearrowright y$.*

(2) *A c-quasigeodesic in a C-uniform domain is a C_1-uniform arc with $C_1 = C_1(C,c)$.*

2.7. *Hyperbolic domains.* Let $\delta \geq 0$. We say that a domain $G \subset E$ is δ*-hyperbolic* if (G,k) is a δ-hyperbolic metric space. This means that

$$(x\,|\,z)_p \geq (x\,|\,y)_p \wedge (y\,|\,z)_p - \delta \tag{2.8}$$

for all $x, y, z, p \in G$, where $(x\,|\,y)_p$ is the Gromov product, defined by

$$2(x\,|\,y)_p = k(p,x) + k(p,y) - k(x,y).$$

The constant δ cannot be arbitrarily close to 0; see 5.3.

We shall apply the theory of hyperbolic spaces given in [Vä8] to the space (G,k). In doing so, one must replace the distance $|x-y|$ and the length $l(\alpha)$ by $k(x,y)$ and $l_k(\alpha)$; in the present paper $|x-y|$ and $l(\alpha)$ denote distance and length in the norm metric. An arc $\alpha\colon x \curvearrowright y$ in G is h*-short*, $h \geq 0$, if

$$l_k(\alpha) \leq k(x,y) + h.$$

Every subarc of an h-short arc is also h-short.

I recall the *standard estimate* [Vä8, 2.33] for an h-short arc $\alpha\colon x \curvearrowright y$ in a δ-hyperbolic domain G:

$$k(p,\alpha) - 2\delta - h \le (x\,|\,y)_p \le k(p,\alpha) + h/2 \tag{2.9}$$

for all $p \in G$.

If an arc $\alpha \subset G$ is an h-short 2-quasigeodesic, we briefly say that α is an *h-arc*.

2.10. Lemma. *Let $h > 0$ and let x and y be distinct points in a domain G. Then there is an h-arc $\alpha\colon x \curvearrowright y$.*

Proof. Choose a c-quasigeodesic $\alpha\colon x \curvearrowright y$ with $c = 2 \wedge \bigl(1 + h/k(x,y)\bigr)$. Then $l_k(\alpha) \le ck(x,y) \le k(x,y) + h$. $\square$

2.11. *Examples.* We show in 2.12 that every uniform domain is hyperbolic. Since hyperbolicity is preserved by quasi-isometries [Vä8, 3.18], it follows that every quasiconformal image of a uniform domain $G \subset \mathsf{R}^n$ is hyperbolic. In particular, every simply connected proper subdomain of the plane is δ-hyperbolic with a universal constant δ.

Some examples of nonhyperbolic domains are:

(1) $G = \mathsf{R}^2 \setminus \{ne_1 : n \in \mathsf{Z}\}$.

(2) $G = \{x \in \mathsf{R}^3 : 0 < x_3 < 1\}$.

(3) More generally, $G = E_1 \times B_2 \subset E = E_1 \times E_2$, where E_1 is a Banach space of dimension at least two and B_2 is a ball in E_2.

The following result is a free version of the first part of Theorem 3.6 of [BHK]. The proof is essentially the same.

2.12. Theorem. *A C-uniform domain $G \subset E$ is δ-hyperbolic with $\delta = \delta(C)$.*

Proof. Let $\mathscr{A}$ be the family of all 2-quasigeodesics in G. By 2.10 this family satisfies the conditions in [Vä8, 2.26].

Let $h > 0$. By [Vä8, 2.34] it suffices to show that (G,k) is $(\delta, h, \mathscr{A})$-Rips with $\delta = \delta(C)$. Let Δ be an h-short triangle in G with sides α, β, γ in $\mathscr{A}$ and with opposite vertices a, b, c. Let $x \in \gamma$. It suffices to show that

$$k(x, \alpha \cup \beta) \le \delta(C). \tag{2.13}$$

Set $s = l(\gamma[a,x])$. We may assume that $s \le l(\gamma[x,b])$. By Lemma 2.6(2), the arcs α, β, γ are $C_1(C)$-uniform in G. Hence

$$2s \le l(\gamma) \le C_1|a-b| \le C_1(|a-c| + |c-b|) \le C_1\bigl(l(\beta) + l(\alpha)\bigr). \tag{2.14}$$

We next show that there is a point $y \in \alpha \cup \beta$ such that

$$|x-y| \le 3s, \quad d(y) \ge s/2C_1^2. \tag{2.15}$$

Case 1. $l(\beta) \le s/C_1$. Now $l(\alpha) \ge s/C_1$ by (2.14). Hence there is $y \in \alpha$ with $l(\alpha[c,y]) = s/2C_1 \le l(\alpha[y,b])$. Then

$$|x-y| \le s + l(\beta) + l(\alpha[c,y]) \le s + s/C_1 + s/2C_1 \le 3s.$$

Since α is C_1-uniform, we have $d(y) \ge l(\alpha[c,y])/C_1 = s/2C_1^2$.

Case 2. $l(\beta) \ge s/C_1$. Now we can choose $y \in \beta$ with $l(\beta[a,y]) = s/2C_1 \le l(\beta[y,c])$. Then

$$|x-y| \le s + s/2C_1 \le 2s.$$

The second inequality of (2.15) follows as in Case 1.

Since γ is C_1-uniform, we have $d(x) \ge s/C_1$. As G is QH C_2-uniform with $C_2 = C_2(C)$, these estimates yield

$$k(x,y) \le C_2 \log\left(1 + \frac{|x-y|}{d(x) \wedge d(y)}\right) \le C_2 \log\left(1 + \frac{3s}{s/2C_1^2}\right) = C_2 \log(1 + 6C_1^2),$$

which implies (2.13). □

2.16. Remark. A modification of the proof above shows that, more generally, all inner uniform domains are hyperbolic. A domain G is *inner C-uniform* if each pair $x, y \in G$ can be joined by an arc α satisfying the cigar condition 2.4(1) and the inner turning condition

(2i) $l(\alpha) \le C l(x,y) = C \inf\{l(\gamma) \mid \gamma\colon x \curvearrowright y,\ \gamma \subset G\}$.

A domain G is a *C-John domain* if each pair $x, y \in G$ can be joined by an arc α satisfying the cigar condition 2.4(1). Obviously

C-uniform $\Rightarrow$ inner C-uniform $\Rightarrow$ C-John.

The planar domain $\mathsf{R}^2 \setminus \{(x,0) : x \ge 0\}$ is inner uniform but not uniform, and $\mathsf{R}^2 \setminus \{(n,0) : n \in \mathsf{N}\}$ is a John domain but not inner uniform and not hyperbolic.

In the rest of this section, we show that if G is a uniform domain, we can extend the identity map of G to a bijection between the Gromov closure and the norm closure of G and that this map is quasimöbius rel boundary.

2.17. *The one-point extension.* Let X be a metric space and let ∞ be an element not in X. The *one-point extension* of X is the set

$$\dot{X} = X \cup \{\infty\}.$$

The topology of $\dot{X}$ consists of all open sets in X and of all sets U containing ∞ such that $\dot{X} \setminus U = X \setminus U$ is closed and bounded in X. Then $\dot{X}$ is a Hausdorff space, and the subspace $X \subset \dot{X}$ has its original topology. In fact, $\dot{X}$ is metrizable but we do not need this fact. The space $\dot{X}$ is compact if and only if X is proper.

The one-point extension of a metric space was already considered by F. Hausdorff [Ha, p. 285] in 1914, but it is fairly seldom mentioned in the literature, except for the case where $\dot{X}$ is compact, in which case it agrees with the familiar one-point compactification.

2.18. *Terminology and notation.* We recall some theory from [Vä8, Section 5]. Let $G \subset E$ be a δ-hyperbolic domain. We fix a base point $p \in G$ and write

$$(x \,|\, y) = (x \,|\, y)_p$$

for $x, y \in G$. A sequence $\bar{x} = (x_i)$ in G is a *Gromov sequence* if $(x_i \,|\, x_j) \to \infty$ as $i, j \to \infty$. Two Gromov sequences $\bar{x}$ and $\bar{y}$ are *equivalent*, $\bar{x} \sim \bar{y}$, if $(x_i \,|\, y_i) \to \infty$. We let $\hat{x}$ denote the equivalence class containing $\bar{x}$. The set of equivalence classes is the *Gromov boundary* $\partial^* G$ of G, and $G^* = G \cup \partial^* G$ is the *Gromov closure* of G. The product $(x \,|\, y)$ is extended to G^* as in [Vä8, (5.8)]. Then the basic inequality (2.8) holds for all $x, y, z \in G^*$ and $p \in G$. Moreover, $(x \,|\, y) = \infty$ if and only if $x = y \in \partial G$.

Let

$$0 < \varepsilon \le 1 \wedge (1/5\delta) \tag{2.19}$$

and write $\varrho_{p,\varepsilon}(x, y) = \varrho_\varepsilon(x, y) = e^{-\varepsilon (x \,|\, y)}$ for $x, y \in G^*$ with the convention $e^{-\infty} = 0$. To the function $\varrho_{p,\varepsilon}$ we associate a function $d_{p,\varepsilon} = d_\varepsilon$ with

$$d_\varepsilon \le \varrho_\varepsilon \le 2 d_\varepsilon \tag{2.20}$$

as in [Vä8, (5.15)]. The function d_ε is a metametric of G^*, that is, it satisfies the axioms of a metric except that $d_\varepsilon(x, x)$ may be positive. In fact, $d_\varepsilon(x, y) = 0$ if and only if $x = y \in \partial^* G$. Hence d_ε defines a metric in $\partial^* G$.

The metametric d_ε defines a topology $\mathscr{T}^*$ in G^*; see [Vä8, 4.2]. In this topology, the points of G are isolated, and a sequence $\bar{x}$ in G converges to a point $a \in \partial^* G$ if and only if $\bar{x}$ is a Gromov sequence and $\bar{x} \in a$.

The norm boundary ∂G and the norm closure $\overline{G} = G \cup \partial G$ of G are taken in the extended space $\dot{E}$. Thus $\infty \in \partial G$ if and only if G is unbounded. We say that a sequence $\bar{x}$ in E *converges in norm* to a point $b \in \dot{E}$ if it converges to b in the topology of $\dot{E}$, that is, $|x_i - b| \to 0$ if $b \ne \infty$ and $|x_i| \to \infty$ if $b = \infty$.

2.21. *Natural maps.* Suppose that $G \subset E$ is a hyperbolic domain. Since $\mathscr{T}^* \,|\, G$ is discrete, the identity map id: $G \to G$ is continuous from the topology $\mathscr{T}^*$ to the norm topology. If it has a continuous extension φ: $G^* \to \overline{G}$, we say that φ is a *natural map*.

2.22. Lemma. *The natural map $\varphi\colon G^* \to \overline{G}$ exists if and only if every Gromov sequence $\bar{x}$ in G has a limit $b(\bar{x})$ in norm. Moreover, $\varphi(a) = b(\bar{x}) \in \partial G$ for each $a \in \partial^* G$ and $\bar{x} \in a$.*

Proof. If $\bar{x}$ is a Gromov sequence in G, then $\bar{x}$ converges to the equivalence class $\hat{x}$ in the topology $\mathscr{T}^*$ by [Vä8, 5.21]. If φ exists, then $x_i = \varphi(x_i) \to \varphi(\hat{x})$ in $\dot{E}$.

Conversely, assume that each Gromov sequence $\bar{x}$ has a limit $b(\bar{x})$ in $\dot{E}$. Then $k(x_i, p) = (x_i \,|\, x_i) \to \infty$, whence $b(\bar{x}) \in \partial G$. If $\bar{x}$ and $\bar{y}$ are equivalent Gromov sequences, then $\bar{z} = (x_1, y_1, x_2, y_2, \ldots)$ is a Gromov sequence, and thus $b(\bar{x}) = b(\bar{z}) = b(\bar{y})$. Setting $\varphi(\hat{x}) = b(\bar{x})$ we therefore obtain a well-defined continuous extension $\varphi\colon G^* \to \overline{G}$ of id. □

2.23. Remark. If φ exists, it defines a continuous map ψ from the metric space $\partial^* G$ into the set $\partial G \subset \dot{E}$. Also ψ is called a natural map.

2.24. *Some constants.* In the study of a C-uniform domain, we let $C_1, C_2, \ldots$ denote constants $C_j \geq C$ depending only on C. By the results mentioned in 2.4 and 2.6, we can fix constants C_1 and C_2 such that if G is C-uniform, then

(1) every 2-quasigeodesic in G is a C_1-uniform arc,

(2) G is QH C_2-uniform, that is,

$$k(x,y) \leq C_2 j(x,y) = C_2 \log\left(1 + \frac{|x-y|}{d(x) \wedge d(y)}\right)$$

for all $x, y \in G$.

One can choose $C_2 = 4C^2$ by [BHK, (2.16)].

Furthermore, we fix a constant $\delta_1 = \delta_1(C)$ such that every C-uniform domain is δ_1-hyperbolic; see 2.12.

The notation C_1, C_2, δ_1 will be fixed for the rest of this section.

2.25. Lemma. *Suppose that G is a C-uniform domain and that $a \in \partial G$. Then $(x \,|\, y) \to \infty$ as $x, y \to a$ in G in the topology of $\dot{E}$.*

Proof. Assume first that $a \neq \infty$. Fix $h = 1$, let $r > 0$, and let $x, y \in B(a, r) \cap G$. Choose an h-arc $\alpha\colon x \curvearrowright y$ and let $z \in \alpha$. By (2.9) it suffices to find an estimate $k(p, z) \geq M(r) \to \infty$ as $r \to 0$. We have

$$d(z) \leq |z-a| \leq |z-x| + |x-a| \leq C_1|x-y| + r \leq (2C_1+1)r,$$

which implies that

$$k(p,z) \geq \log \frac{d(p)}{d(z)} \geq \log \frac{d(p)}{(2C_1+1)r} = M(r).$$

Next assume that $a=\infty$. By translation we may assume that $p=0$. Let $R>0$ and let $x,y\in G\setminus B(R)$. Choose again an h-arc $\alpha\colon x\curvearrowright y$ and let $z\in\alpha$. Now we need an estimate $k(p,z)\ge M(R)\to\infty$ as $R\to\infty$.

Fix a point $b\in\partial G\setminus\{\infty\}$ and let $R>C_1|b|$. We have

$$R-|z|\le|z-x|\wedge|z-y|\le C_1d(z)\le C_1|z-b|\le C_1|z|+C_1|b|,$$

whence $|z|\ge(R-C_1|b|)/(1+C_1)$. Since $k(p,z)\ge\log\bigl(1+|z|/d(p)\bigr)$, this gives the desired estimate. □

2.26. Proposition. *If G is a C-uniform domain, then the natural map $\varphi\colon G^*\to\overline{G}$ exists and is bijective. Moreover, a sequence $\bar x$ in G converges to $a\in\partial G$ in norm if and only if $\bar x$ is a Gromov sequence with $\varphi(\hat x)=a$.*

Proof. By translation we may assume that $p=0$. Let $\bar x$ be a Gromov sequence in G. We show that $\bar x$ converges in norm. By 2.22, this will imply that φ exists.

Case 1. $\bar x$ is bounded. We show that $\bar x$ is Cauchy in the norm metric. Assume that $r_{ij}=|x_i-x_j|>0$, let $h>0$, and choose an h-arc $\alpha\colon x_i\curvearrowright x_j$. Let $z\in\alpha$ be the point with $l(\alpha[x_i,z])=r_{ij}/2$. Then $d(z)\ge r_{ij}/2C_1$. Choose $M>0$ such that $|x_i|\le M$ for all i. Since $|z-x_i|\le r_{ij}/2\le M$, we have

$$|z|\le|z-x_i|+|x_i|\le 2M.$$

By (2.9) we obtain

$$(x_i\,|\,x_j)-h/2\le k(p,z)\le C_2\log\left(1+\frac{|z|}{d(p)\wedge d(z)}\right)\le C_2\log\left(1+\frac{2M}{d(p)\wedge(r_{ij}/2C_1)}\right).$$

Since $(x_i\,|\,x_j)\to\infty$ as $i,j\to\infty$, this implies that $r_{ij}\to 0$, whence $\bar x$ is Cauchy.

Case 2. $\bar x$ is unbounded. We show that $|x_i|\to\infty$. If this is not true, then there is $R>0$ such that $|x_i|\le R$ for infinitely many i. Let $\alpha\colon x_i\curvearrowright x_j$ be an h-arc as in Case 1, where $i<j$, $|x_i|\le R$ and $|x_j|\ge 3R$. Pick a point $z\in\alpha$ with $|z|=2R$. Then $d(z)\ge R/C_1$ by uniformity, whence

$$k(p,\alpha)\le k(p,z)\le C_2\log\left(1+\frac{2R}{d(p)\wedge(R/C_1)}\right)=K.$$

By (2.9) this yields $(x_i\,|\,x_j)\le K+h/2$. As $\bar x$ is a Gromov sequence, this gives a contradiction.

We have proved that the natural map $\varphi\colon G^*\to\overline{G}$ exists.

To show that φ is injective, let $\bar x$ and $\bar y$ be Gromov sequences with $\varphi(\hat x)=\varphi(\hat y)=a$. Then $(x_i\,|\,y_i)\to\infty$ by 2.25, and hence $\hat x=\hat y$.

Finally, let $a\in\partial G$ and choose a sequence $\bar x$ in G converging to a in norm. By 2.25 we have $(x_i\,|\,x_j)\to\infty$ as $i,j\to\infty$. Thus $\bar x$ is a Gromov sequence and $\varphi(\hat x)=a$, whence φ is surjective.

The last statement of the lemma follows from the proof. □

2.27. *Notation.* In view of 2.26, we simplify the notation by writing $\varphi x = x$ for $x \in \partial^* G$ if G is a uniform domain.

2.28. *Quasimöbius maps in the extended space.* Let X be a metric space and let $Q = (x, y, z, w)$ be a quadruple in $\dot{X} = X \cup \{\infty\}$ with $x \neq z$, $y \neq w$. For finite points, the cross ratio of Q is defined by

$$\operatorname{cr} Q = \frac{|x - y|\,|z - w|}{|x - z|\,|y - w|}$$

as in [Vä8, 4.5]. If one of the points is ∞, the cross ratio $\operatorname{cr} Q$ is defined by deleting the distances containing ∞. For example, $\operatorname{cr}(x, y, z, \infty) = |x - y|/|x - z|$.

The definition of quasimöbius maps and its relative version (see [Vä8, 4.8]) can now be extended in an obvious manner to the case where the spaces may contain the point ∞. One could consider extended metametric spaces but they are not needed, because the metametric space $(G^*, d_{p,\varepsilon})$ is bounded.

We are ready to formulate the second main result of this section; cf. [BHK, 3.6].

2.29. Theorem. *Suppose that G is a C-uniform domain. Then the natural bijective map $\varphi\colon G^* \to \overline{G}$, which exists by 2.26, is η-quasimöbius rel $\partial^* G$ with respect to the metametric $d_{p,\varepsilon}$ of G^* and the norm metric of $\overline{G}$ where $0 < \varepsilon \le \varepsilon_0(C)$. The function η depends only on C and ε but not on p.*

We first give a version of [BHK, 3.14].

2.30. Lemma. *Suppose that G is a C-uniform domain. Let $a, b, z \in G$ with $|a - b| = s > 0$ and $|z - a| \ge 2s$. Let $h > 0$ and let $\alpha\colon z \curvearrowright a$ and $\beta\colon a \curvearrowright b$ be h-arcs. Define a point $y \in \alpha$ by $l(\alpha[y, a]) = s$. Then*

(a) $k(y, \beta) \le C_3$,

(b) $k(z, \beta) - C_3 \le k(z, y) \le k(z, \beta) + C_4 + h$.

The constants C_3 and C_4 depend only on C.

Proof. The first inequality of (b) follows from (a). It remains to prove (a) and the second inequality of (b).

Let $x \in \beta$ be the point bisecting the length of β. Recall the constants C_1, C_2 from 2.24. Since α and β are C_1-uniform, we have $d(x) \wedge d(y) \ge s/2C_1$. Since $|x - y| \le |x - a| + |a - y| \le C_1 s + s$, we get

$$k(x, y) \le C_2 \log\bigl(1 + 2C_1(C_1 + 1)\bigr) = C_3,$$

which proves (a).

To prove (b) assume that $u \in \beta$. We must find an estimate

$$k(z, y) \le k(z, u) + C_4 + h. \tag{2.31}$$

Case 1. $l(\beta[a, u]) \wedge l(\beta[u, b]) \ge s/2$. Now $d(u) \ge s/2C_1$, $|u - y| \le C_1 s + s$, and we obtain an estimate $k(u, y) \le C_3$ as above. This implies (2.31).

Case 2. $l(\beta[a,u]) \wedge l(\beta[u,b]) \le s/2$. Choose an h-arc $\gamma\colon z \curvearrowright u$. Then

$$l(\gamma) \ge |u-z| \ge |a-z| - s - |b-u| \wedge |a-u| \ge 2s - s - s/2 = s/2.$$

Choose a point $u' \in \gamma$ with $l(\gamma[u,u']) = s/4$. Then $d(u') \ge s/4C_1$. Since

$$|u'-y| \le |u'-u| + |u-a| + |a-y| \le s/4 + C_1 s + s \le 3C_1 s,$$

we get

$$k(u',y) \le C_2 \log(1+12C_1^2) = C_4.$$

As γ is h-short, this implies that

$$k(z,u) \ge k(z,u') - h \ge k(z,y) - k(u',y) - h \ge k(z,y) - C_4 - h,$$

which is (2.31). □

2.32. Definition. A homeomorphism $f\colon G \to G'$ between domains G and G' is M-*quasihyperbolic* if f is M-bilipschitz in the quasihyperbolic metrics of G and G'.

2.33. *Inversion.* For a Banach space E, the *inversion* of E (in the unit sphere) is the map $u\colon \dot{E} \to \dot{E}$, defined by $u(x) = x/|x|^2$ for $0 \ne x \ne \infty$, $u(0) = \infty$, $u(\infty) = 0$. Then u is a homeomorphism with $u^{-1} = u$. We recall the following properties of u:

(1) u is η-quasimöbius with $\eta(t) = 81t$, [Vä6, 6.22].

(2) For each domain $G \subset E \setminus \{0\}$, the map $u_1\colon G \to uG$ defined by u is M-quasihyperbolic with a universal constant M. This was proved with $M = 36$ in [Vä6, 5.14] and with $M = 12$ in [Vä4, 2.9].

(3) u maps each C-uniform domain $G \subset E \setminus \{0\}$ onto a C'-uniform domain with $C' = C'(C) \ge C$. This follows from (1) and from [Vä6, 10.22].

It follows from (3) and 2.12 that u maps each C-uniform domain $G \subset E\setminus\{0\}$ onto a δ_2-hyperbolic domain, $\delta_2 = \delta_2(C) = \delta_1(C') \ge \delta_1(C)$. By (2) and by [Vä8, 5.38], u extends to a homeomorphism $\bar{u}_1\colon G^* \to (uG)^*$, which is θ-quasimöbius in the metametrics d_ε with arbitrary base points for all $\varepsilon \le 1 \wedge (1/5\delta_2)$ with $\theta = \theta_C$.

2.34. *Proof of Theorem* 2.29. Recall from 2.26 that the natural map $\varphi\colon G^* \to \overline{G}$ is bijective and that $\varphi\partial^* G = \partial G$. It suffices to show that the map $\varphi^{-1}\colon \overline{G} \to G^*$ is θ-quasimöbius rel ∂G with $\theta = \theta_{C,\varepsilon}$. As before, we write $\varphi x = x$ for $x \in G^*$, and then $G^* = \overline{G}$ as a set. Let $Q = (a,b,c,w)$ be a quadruple of distinct points in $\overline{G}$ with $a, w \in \partial G$. We want to find an estimate

$$\operatorname{cr}(Q, d_{p,\varepsilon}) \le \theta\bigl(\operatorname{cr}(Q, \operatorname{norm})\bigr), \tag{2.35}$$

where $\theta(s) \to 0$ as $s \to 0$.

By 2.33 we may use translations and the inversion to normalize the situation so that G is unbounded and $w = \infty$. From 2.33(3) it follows that we can replace C by a larger constant, still denoted by C, so that G is C-uniform after the normalization. The domain G is δ_1-hyperbolic and we set $\varepsilon_0(C) = 1 \wedge (1/5\delta_1)$. We show that (2.35) holds for $0 < \varepsilon \le \varepsilon_0$.

Choose sequences $(a_i), (b_i), (c_i), (w_i)$ in G converging in norm to a, b, c, w, respectively. If $b \in G$ [or $c \in G$], we choose $b_i = b$ [or $c_i = c$] for all i. By 2.26, these sequences converge also in the metametric $d_{p,\varepsilon}$ (to the same limits). We may assume that for each i, the points a_i, b_i, c_i, w_i are distinct and that $|a_i - b_i| \vee |a_i - c_i| \le |w_i - a_i|/2$.

Writing

$$Q_i = (a_i, b_i, c_i, w_i), \quad d_i = d_{w_i,\varepsilon}, \quad \varrho_i = \varrho_{w_i,\varepsilon}$$

we have $d_i(w_i, x) = 1$ for all $x \in \overline{G}$, and $\mathrm{cr}(Q_i, d_{p,\varepsilon}) \to \mathrm{cr}(Q, d_{p,\varepsilon})$. Since $\mathrm{cr}(Q_i, d_{p,\varepsilon}) \le 16\, \mathrm{cr}(Q_i, d_i)$ by [Vä8, 5.28], we get by (2.20)

$$\mathrm{cr}(Q_i, d_{p,\varepsilon}) \le 16 \frac{d_i(a_i, b_i)}{d_i(a_i, c_i)} \le 32 \frac{\varrho_i(a_i, b_i)}{\varrho_i(a_i, c_i)} = 32 e^{-\varepsilon\mu_i} \le 32 e^{|\mu_i|}, \tag{2.36}$$

where $\mu_i = (a_i \,|\, b_i)_{w_i} - (a_i \,|\, c_i)_{w_i}$. Setting $t_i = |a_i - b_i|/|a_i - c_i|$, $t = |a-b|/|a-c|$ we have $t = \mathrm{cr}(Q, \mathrm{norm})$, and $t_i \to t$ as $i \to \infty$. We want to get a lower bound for μ_i.

Fix $h = 1$ and choose h-arcs $\alpha_i\colon w_i \curvearrowright a_i$, $\beta_i\colon a_i \curvearrowright b_i$, $\gamma_i\colon a_i \curvearrowright c_i$. Let $y(b_i) \in \alpha_i$ be the point chosen as in 2.30 where now $a = a_i$, $b = b_i$, $z = w_i$, $\alpha = \alpha_i$, $\beta = \beta_i$, and let $y(c_i) \in \alpha_i$ be the corresponding point for c_i. Since α_i is C_1-uniform, we have

$$d\big(y(b_i)\big) \ge |a_i - b_i|/C_1, \quad d\big(y(c_i)\big) \ge |a_i - c_i|/C_1. \tag{2.37}$$

We consider two cases.

Case 1. $t \ge 1$. We may assume that $t_i \ge 1/2$ for all i. By (2.37) we have

$$d\big(y(b_i)\big) \wedge d\big(y(c_i)\big) \ge |a_i - c_i|/2C_1$$

for all i. Since

$$|y(b_i) - y(c_i)| \le |y(b_i) - a_i| + |a_i - y(c_i)| \le |a_i - b_i| + |a_i - c_i| \le 3|a_i - b_i|,$$

we get

$$k\big(y(b_i), y(c_i)\big) \le C_2 \log(1 + 6C_1 t_i).$$

By (2.9) and 2.30 we have

$$\begin{aligned} |\mu_i| &\le |k(w_i, \beta_i) - k(w_i, \gamma_i)| + 2\delta_1 + 2h \\ &\le \big|k\big(w_i, y(b_i)\big) - k\big(w_i, y(c_i)\big)\big| + C_3 + C_4 + 2\delta_1 + 3h. \end{aligned}$$

Since α_i is h-short and since $h = 1$, it follows that

$$|\mu_i| \le k\big(y(b_i), y(c_i)\big) + C_5 \le C_2 \log(1 + 6C_1 t_i) + C_5.$$

As $i \to \infty$, this and (2.36) give (2.35) with $\theta(t) = 32 \exp[C_2 \log(1 + 6C_1 t) + C_5]$.

Case 2. $t < 1$. We may assume that $t_i < 1$ for all i. Applying (2.9) and 2.30 as in Case 1 we get

$$\mu_i \ge k\big(w_i, y(b_i)\big) - k\big(w_i, y(c_i)\big) - C_3 - C_4 - 2\delta_1 - 3h.$$

Since $t_i < 1$, we have $l\big(\alpha_i[y(b_i), a_i]\big) < l\big(\alpha_i[y(c_i), a_i]\big)$ by the definition of y in 2.30. Hence the point $y(c_i)$ lies between w_i and $y(b_i)$ on α_i. As α_i is 1-short, it follows that $\mu_i \ge k\big(y(b_i), y(c_i)\big) - C_6$. Writing $r_i = |a_i - a|$ we have

$$d\big(y(b_i)\big) \le |y(b_i) - a| \le |y(b_i) - a_i| + r_i \le |a_i - b_i| + r_i.$$

As in Case 1 we have $d\big(y(c_i)\big) \ge |a_i - c_i|/C_1$. Hence

$$k\big(y(c_i), y(b_i)\big) \ge \log \frac{d\big(y(c_i)\big)}{d\big(y(b_i)\big)} \ge -\log(C_1 t_i + C_1 r_i/|a_i - c_i|).$$

Here $r_i/|a_i - c_i| \to 0$ as $i \to \infty$. Combining the estimates with (2.36) and letting $i \to \infty$ yields (2.35) with $\theta(t) = 32 e^{C_6} (C_1 t)^{\varepsilon} \to 0$ as $t \to 0$. □

2.38. *Bounded domains.* A θ-quasimöbius map between bounded spaces is η-quasisymmetric, and a relative version of this is also true; see [Vä6, 6.29]. However, we cannot choose η depending only on θ. In order to get a quantitative result we must normalize the map in a suitable way; see [Vä6, 6.29 and 6.31]. These ideas and 2.29 give Theorem 2.39 below. An alternative proof is obtained by modifying the proof of 2.29 as in [BHK, 3.6]. I omit the details.

2.39. Theorem. *Suppose that G is a bounded C-uniform domain with a base point p such that $d(x) \le c d(p)$ for all $x \in G$. Then the natural bijective map $\varphi\colon G^* \to \overline{G}$, which exists by 2.26, is η-quasisymmetric rel $\partial^* G$ with respect to the metametric $d_{p,\varepsilon}$ of G^* and the norm metric of $\overline{G}$, where $0 < \varepsilon \le \varepsilon_0(C)$. The function η depends only on (C, c, ε).* □

2.40. *An application to the free quasiworld.* In the free quasiworld (see [Vä6]) we study (with different terminology) domains $G \subset E$ and $G' \subset E'$ in Banach spaces and homeomorphisms $f\colon G \to G'$ that are (λ, μ)-quasi-isometries in the quasihyperbolic metric, as well as some subclasses of these maps. If G and G' are C-uniform, then f extends to a homeomorphism $\bar f\colon \overline{G} \to \overline{G}'$, and $\bar f$ is η-quasimöbius rel ∂G with η depending only on (C, λ, μ); see [Vä6, 11.8].

We remark that a new proof for this result is obtained by combining Theorems [Vä8, 5.38] and 2.29. In fact, the extension $\bar{f}$ is obtained, somewhat more generally, for maps f that need not be homeomorphisms, only weakly surjective in the sense of [Vä8, 5.32]. The induced map $\partial G \to \partial G'$ will nevertheless be a quasimöbius homeomorphism.

In [Vä5, Section 5] we considered domains of the type $G = E_1 \times B_2$ in a product space $E_1 \times E_2$, where $B_2 \subset E_2$ is a ball. If $\dim E_1 \geq 2$, these domains are not hyperbolic, but each homeomorphic (λ, μ)-quasi-isometry between such domains still extends to a homeomorphism $\bar{f}$ between the closures, and $\bar{f}$ is η-quasisymmetric rel ∂G with η depending on (λ, μ). I do not know whether this result can be obtained with the aid of Gromov hyperbolicity.

Furthermore, since hyperbolicity is preserved by quasi-isometries [Vä8, 3.18], it follows from 2.12 that there is no quasihyperbolic quasi-isometry from a uniform domain onto a nonhyperbolic domain, for example, from a ball onto a domain between two parallel hyperplanes in a space of dimension at least three. In particular, these domains are not (freely) quasiconformally equivalent. Again, [Vä5, Section 5] contains further results on domains $G = E_1 \times B_2$ that do not directly follow from the results of this paper.

3. Hyperbolic domains with a boundary condition are uniform

3.1. *Introduction to Section* 3. We proved in Section 2 that a C-uniform domain $G \subset E$ has the following properties:

(a) G is δ-hyperbolic.

(b) The natural map $\varphi\colon G^* \to \overline{G}$ exists and φ is bijective and η-quasimöbius rel $\partial^* G$ from the metametric $d_{p,\varepsilon}$ into the norm metric.

In this section we show that also the converse is true in the stronger form where (b) is replaced by

(b′) The natural map $\varphi\colon G^* \to \overline{G}$ exists and its restriction $\psi\colon \partial^* G \to \partial G$ is bijective and η-quasimöbius.

These results are quantitative in the following sense:

If G is C-uniform, then (a) holds with $\delta = \delta(C)$ and (b) holds for all $p \in G$, for all $0 < \varepsilon \leq \varepsilon_0(C)$ and for η depending only on C and ε.

If (b′) holds, then G is C-uniform with $C = C(\delta, \eta, \varepsilon)$.

Observe that the result is independent of the base point p.

The case where $E = \mathsf{R}^n$ and G is bounded (or more generally, G is a domain in S^n) was proved in [BHK, 1.11 and 7.11] with C depending on $\delta, \eta, \varepsilon, n$. However, in [BHK] the quasimöbius condition was replaced by quasisymmetry, and the base point p had to be chosen so that $d(p)$ is maximal. The proof in [BHK] made use of the modulus of a path family, and it cannot be extended to general Banach spaces. Our proof makes use of h-short arcs and (μ, h)-biroads in G as well as of the rough starlikeness of G. Several times we shall make use of the idea of [BHK] to divide the points of a domain into annulus points and arc points.

As mentioned before, we assume throughout the paper that $\dim E \geq 2$. The case $\dim E = 1$ is trivial: A domain is an open interval (bounded or unbounded), and each domain $G \neq E$ is 1-uniform and 0-hyperbolic.

The main result is given in 3.27. We start with various auxiliary results.

3.2. Lemma. *If α: $x \curvearrowright y$ is a rectifiable arc in a domain G, then $l(\alpha) \leq (e^{l_k(\alpha)} - 1)d(x)$.*

Proof. Set $L = l_k(\alpha)$ and let φ: $[0, L] \to \alpha$ be the parametrization of α by quasihyperbolic length with $\varphi(0) = x$. Then $d\bigl(\varphi(t)\bigr) \leq e^t d(x)$, whence

$$l(\alpha) = \int_0^L d\bigl(\varphi(t)\bigr)\, dt \leq (e^L - 1)d(x). \square$$

We recall that a *length map* between arcs is a map preserving the length of each subarc. For arcs in the space (G, k), such a map is called a *quasihyperbolic length map.*

We next give an estimate for the change of ordinary length in a quasihyperbolic length map.

3.3. Lemma. *Let α and β be arcs in a domain G with $l_k(\alpha) \leq l_k(\beta)$, and let f: $\alpha \to \beta$ be a quasihyperbolic length map. If $k(fx, x) \leq c$ for all $x \in \alpha$, then $e^{-c} l(\alpha) \leq l(f\alpha) \leq e^c l(\alpha)$.*

Proof. We may assume that $f\alpha = \beta$. By symmetry, it suffices to prove the second inequality. Since

$$\log \frac{d(fx)}{d(x)} \leq k(fx, x) \leq c,$$

we have $d(fx) \leq e^c d(x)$ for all $x \in \alpha$. Set $L = l_k(\alpha) = l_k(\beta)$, and let φ: $[0, L] \to \alpha$ be the parametrization of α by quasihyperbolic length. Then

$$l(\beta) = \int_0^L d\bigl(f\varphi(t)\bigr)\, dt \leq e^c \int_0^L d\bigl(\varphi(t)\bigr)\, dt = e^c l(\alpha). \square$$

The following basic result on the geometry of normed spaces is from [Sc, Theorem 4J, p. 18]:

3.4. Theorem. *Let E be a normed space with $\dim E \geq 2$. Then every sphere in E is 2-quasiconvex.*

We next study the behavior of an h-short arc in a domain containing a large ring.

3.5. Lemma. *Suppose that* $\alpha\colon x_1 \curvearrowright x_2$ *is an* h*-short arc with* $h \le 1/10$ *in a domain* G *and that* $a \in \partial G$.

(1) *If* $B(a,16t) \setminus \overline{B}(a,t/16) \subset G$ *and if* $|x_i - a| \ge 8t$, $i = 1,2$, *then* $|z-a| > t$ *for all* $z \in \alpha$.

(2) *If* $B(a,7s) \setminus \overline{B}(a,s) \subset G$ *and if* $|x_i - a| \le 4s$, $i = 1,2$, *then* $|z-a| < 64s$ *for all* $z \in \alpha$.

Proof. (1) Assume that there is a point $z \in \alpha$ with $|z-a| \le t$. Choose points $u_1, u_2 \in \alpha \cap S(a,8t)$ such that $z \in \alpha[u_1,u_2]$. By 3.4, there is an arc $\beta\colon u_1 \curvearrowright u_2$ in $S(a,8t)$ with $l(\beta) \le 32t$. For all $x \in S(a,8t)$ we have $d(x) \ge (8 - 1/16)t$. Hence

$$k(u_1,u_2) \le l_k(\beta) \le 32/(8-1/16) = 4.031\ldots.$$

On the other hand, $|u_i - z| \ge 7t$, $d(z) \le |z-a| \le t$, and we obtain the contradiction

$$k(u_1,u_2) \ge k(u_1,z) + k(z,u_2) - h \ge 2\log(1+7) - 1/10 = 4.058\ldots.$$

(2) Assume that $z \in \alpha$ and that $|z-a| \ge 64s$. Choose $u_1, u_2 \in S(a,4s)$ with $z \in \alpha[u_1,u_2]$. Now 3.4 gives $k(u_1,u_2) \le 16/3$. Since $|u_i - z| \ge 60s$ and $d(u_i) \le |u_i - a| = 4s$, we get the contradiction $k(u_1,u_2) \ge 2\log(1+15) - 1/10 = 5.44\ldots$. □

3.6. *Endcuts and crosscuts.* Let $G \subset E$ be a domain. An arc $\gamma\colon a \curvearrowright b$ in $\overline{G}$ is an *endcut* of G if $\gamma \cap \partial G = \{b\}$, and γ is a *crosscut* of G if $\gamma \cap \partial G = \{a,b\}$. Such arcs are called c*-quasigeodesics* if $l_k(\gamma[u,v]) \le ck(u,v)$ for all $u,v \in \gamma \cap G$. The definition of a C-uniform arc (see 2.4) makes also sense for endcuts and crosscuts with finite endpoints.

If $\dim E < \infty$, then each point $x_0 \in G$ can be joined to a nearest point $b \in \partial G$ by the segmental geodesic endcut $[x_0,b]$. In the general case, a nearest boundary point need not exist. However, there always exists a quasigeodesic endcut consisting of a sequence of line segments. This was proved in [Vä3, 3.10]. I give a more precise formulation of this result in 3.9. In the proof, it is possible to make use of the following result of [BHK]:

3.7. Lemma. *Suppose that* G *is a domain and that* $\alpha\colon a \curvearrowright b$ *is a rectifiable arc in* $\overline{G}$ *such that*

(1) $\alpha \setminus \{a,b\} \subset G$,
(2) $l(\alpha[a,x]) \wedge l(\alpha[x,b]) \le Cd(x)$ *for all* $x \in \alpha \setminus \{a,b\}$,
(3) α *is* c*-quasiconvex in norm.*

Then α *is a* c_1*-quasigeodesic with* $c_1 = 4cC$.

Proof. Let $u, v \in \alpha \cap G$ and set $\alpha_1 = \alpha[u, v]$. Then also α_1 satisfies the conditions of the lemma. By [BHK, (2.15) and (2.12)] we get

$$l_k(\alpha_1) \le 4C \log\left(1 + \frac{l(\gamma)}{d(u) \wedge d(v)}\right) \le 4C \log\left(1 + c\frac{|u - v|}{d(u) \wedge d(v)}\right) \le 4Cck(u, v). \square$$

3.8. Remark. If the cigar condition (2) is replaced by the carrot condition

$(2')$ $l(\alpha[x, b]) \le Cd(x)$,

the proof of [BHK, (2.15)] shows that 3.7 holds with $c_1 = 2cC$.

3.9. Proposition. *Suppose that x_0 is a point of a domain G, that $0 < s \le 1/4$, and that $a \in \partial G$ with $|a - x_0| < (1 + s/2)d(x_0)$. Let x_1 be the unique point in $[x_0, a] \cap S\bigl(x_0, d(x_0)\bigr)$. Then there is an endcut γ: $x_0 \curvearrowright b \in \partial G$ such that:*

(1) $l(\gamma) < (1 + s)d(x_0)$.
(2) $\gamma \subset B\bigl(x_0, (1 + s)d(x_0)\bigr)$.
(3) $|a - b| < \frac{31}{30} sd(x_0) < 2sd(x_0)$.
(4) $[x_0, x_1] \subset \gamma$ *and* $\gamma \setminus [x_0, x_1] \subset B\bigl(x_1, sd(x_0)\bigr)$.
(5) $l(\gamma[x, b]) \le C_0 d(x)$ *for all* $x \in \gamma \setminus \{b\}$ *with a universal* $C_0 \le 47/15 < 4$.
(6) γ *is* c_0*-quasiconvex in norm with a universal* $c_0 \le 4$.
(7) γ *is a* λ_0*-quasigeodesic with a universal* $\lambda_0 \le 26$.

Proof. I recall the construction of γ in [Vä3, 3.10]. This arc is the union of a finite or infinite sequence of line segments $\gamma_i = [x_{i-1}, x_i)$ and b. The points $x_1, x_2, \dots$ are found inductively as follows. Set $s_i = 2^{-i}s$, $y_1 = a$. Assume that the points $x_0, \dots, x_i$ have been chosen. If $x_i \in \partial G$, the process stops. If $x_i \in G$, we choose a point $y_{i+1} \in \partial G$ with $|y_{i+1} - x_i| < (1 + s_{i+1})d(x_i)$ and let x_{i+1} be the unique point in $[x_i, y_{i+1}] \cap S\bigl(x_i, d(x_i)\bigr)$.

Setting $d_i = d(x_i)$ we have

$$l(\gamma_{i+1}) = |x_i - x_{i+1}| = d_i \le |x_i - y_i| < s_i d_{i-1} \tag{3.10}$$

for $i \ge 1$. Hence

$$d_{i+k} \le s_{i+1}s_{i+2}\cdots s_{i+k}d_i \le s_{i+1}^k d_i$$

for $i \ge 0$, $k \ge 1$. From (3.10) it also follows that the sequence (x_i) converges to a point $b \in \partial G$.

We proved in [Vä3, 3.10] that γ is an arc, but the conditions (1)–(7) make sense if we interpret γ as a path in an obvious way, and then (6) implies that γ is an arc. Writing $r_i = l(\gamma[x_i, b])$ we get

$$r_i = \sum_{j \ge i} d_j \le d_i/(1 - s_{i+1}) \tag{3.11}$$

for $i \geq 0$. In particular, $l(\gamma) = r_0 \leq d_0/(1-s_1) < (1+s)d_0$, which implies (1) and (2). Furthermore,

$$r_1 \leq d_1/(1-s_2) \leq s_1 d_0/(1-1/16) = 8sd_0/15 < sd_0,$$
$$|a-b| \leq |a-x_1| + r_1 < s_1 d_0 + 8sd_0/15 = 31sd_0/30,$$

and we obtain (4) and (3).

(5): Assume that $x \in \gamma_i$. Then $d(x) \geq |x-x_i|$ and

$$l(\gamma[x,b]) = |x-x_i| + r_i \leq |x-x_i| + d_i/(1-s_{i+1}) \leq |x-x_i| + 16d_i/15,$$

because $s_{i+1} \leq s_2 \leq 1/16$. If $|x-x_i| \geq d_i/2$, then the right-hand side is at most $(1+32/15)d(x)$, and (5) follows. If $|x-x_i| \leq d_i/2$, then $d(x) \geq d_i/2$, and we again obtain (5).

(6): Assume that $x \in \gamma_i$, $y \in \gamma_j$ with $i < j$, and set $\varrho = l(\gamma[x,y])/|x-y|$. We must show that $\varrho \leq 4$. If $j = i+1$, we make use of some results in the appendix. Applying Lemma A.3(3) with $(x,y,z) \mapsto (x_{i-1}, x_i, y_{i+1})$ we see that the deviation (see A.2) between the rays from x_i through x_{i-1} and x_{i+1} is at least 1, and hence $\varrho \leq 3$ by A.4.

If $j \geq i+2$, then

$$l(\gamma[x,y]) \leq l(\gamma[x,b]) = |x-x_i| + d_i + r_{i+1}.$$

By (3.10) we have $d_{i+1} \leq s_{i+1} d_i \leq d_i/16$. As $s_{i+2} \leq s_3 \leq 1/32$, (3.11) implies that

$$r_{i+1} \leq d_{i+1}/(1-s_{i+2}) \leq 2d_i/31 < d_i/15.$$

Since $|y-x_{i+1}| \leq r_{i+1}$ and since $\gamma_i \cup \gamma_{i+1}$ is 3-quasiconvex, we obtain

$$|x-y| \geq |x-x_{i+1}| - r_{i+1} \geq (|x-x_i| + d_i)/3 - d_i/15.$$

Consequently,

$$\varrho \leq 3\frac{|x-x_i| + 16d_i/15}{|x-x_i| + 4d_i/5}.$$

This expression is maximal for $x = x_i$, whence $\varrho \leq 4$.

(7): This follows from (5), (6) and 3.8 with $\lambda_0 = 2c_0C_0 < 26$. In fact, (7) holds with $\lambda_0 = 17$; see [Vä3, 3.10]. ▫

We prove in 3.13 that quasigeodesic endcuts produce Gromov sequences. The proof makes use of the following corollary of the stability theorem [Vä8, 3.11]:

3.12. Stability lemma. *Let G be a δ-hyperbolic domain and let γ and γ' be arcs in G with common endpoints. If γ is a c-quasigeodesic and if γ' is h-short, then the quasihyperbolic Hausdorff distance $k_H(\gamma,\gamma')$ is bounded by a constant $M(\delta,c,h)$.*

3.13. Lemma. *Suppose that $\gamma\colon x_0 \curvearrowright b \in \partial G$ is a quasigeodesic endcut of a hyperbolic domain G and that $\bar{x}$ is a sequence on $\gamma \cap G$ converging to b in norm. Then $\bar{x}$ is a Gromov sequence. If $\bar{y}$ is another such sequence, then $\bar{x} \sim \bar{y}$.*

Proof. Assume that G is δ-hyperbolic and that γ is c-quasigeodesic. It suffices to show that $(x_i \,|\, y_j) \to \infty$ as $i,j \to \infty$. Fix $h>0$ and choose h-short arcs $\alpha_{ij}\colon x_i \curvearrowright y_j$. Then $k_H(\alpha_{ij}, \gamma[x_i,y_j]) \le M(\delta,c,h)$ by 3.12. By (2.9) we obtain

$$(x_i \,|\, y_j) \ge k(p,\alpha_{ij}) - 2\delta - h \ge k(p,\gamma[x_i,y_j]) - M - 2\delta - h.$$

Since γ is an endcut, this implies that $(x_i \,|\, y_j) \to \infty$ as $i,j\to\infty$. □

3.14. Remarks. 1. Lemma 3.13 is, in fact, a special case of [Vä8, 6.31].

2. It follows from 3.13 that a quasigeodesic endcut $\gamma\colon x_0 \curvearrowright b$ determines an element of $\partial^* G$, which will also be written as b.

3. Similarly, a quasigeodesic crosscut $\gamma\colon a \curvearrowright b$ determines two elements of $\partial^* G$, which will be written as a and b.

3.15. Lemma. *Suppose that $\bar{x}$ is a sequence in a domain G and that $\bar{x}$ converges in norm to an isolated point $b_0 \in \partial G$. Then $\bar{x}$ is a Gromov sequence in the quasihyperbolic metric.*

Proof. We prove the case $b_0 = \infty$. The case of a finite b_0 is rather similar, and it is not needed in this paper. Choose a number $R>0$ such that $\partial G \setminus \{\infty\} \subset B(R)$. Let $a,b \in G$ with $2R \le |a| \le |b|$ and set $z = |b|a/|a|$. By 3.4 there is an arc $\beta\colon z \curvearrowright b$ in $S(|b|)$ with $l(\beta) \le 4|b|$. Then $\alpha = [a,z] \cup \beta$ is an arc joining a and b. Since $\big|d(x) - |x|\big| \le R$ for $x \in \alpha$, we obtain

$$k(a,b) \le l_k(\alpha) \le \log\frac{|b|-R}{|a|-R} + \frac{4|b|}{|b|-R} \le \log\frac{|b|}{|a|/2} + 8 \le \log\frac{|b|}{|a|} + 9,$$
$$k(a,b) \ge \log\frac{d(b)}{d(a)} \ge \log\frac{|b|/2}{2|a|} \ge \log\frac{|b|}{|a|} - 2.$$

Choose the base point p with $|p| = 2R$. By the definition of the Gromov product we have

$$2(a\,|\,b) \ge \log\frac{|a|}{2R} + \log\frac{|b|}{2R} - \log\frac{|b|}{|a|} - 13 = 2\log\frac{|a|}{2R} - 13,$$

and the lemma follows. □

3.16. Lemma. *Let G be a hyperbolic domain and let U be an open set in $\dot{E}$ meeting ∂G. Then there is a Gromov sequence $\bar{x}$ in G such that $\bar{x}$ converges in norm to a point $b \in U \cap \partial G$.*

Proof. If $U \cap \partial G = \{\infty\}$, the lemma follows from 3.15. If $U \cap \partial G \neq \{\infty\}$, there is a quasigeodesic endcut $\gamma\colon x_0 \curvearrowright b \in U \cap \partial G$ by 3.9, and the lemma follows from 3.13. □

3.17. *Annulus and arc points, anchors.* We adopt the terminology of [BHK, p. 65] with a slight change, because a nearest boundary point need not exist.

Let $0 < \lambda \leq 1/2$. A point x in a domain G is said to be a λ-*annulus point* of G if there is a point $a \in \partial G$ such that for $t = |x - a|$, the annulus $B(a, t/\lambda) \setminus \overline{B}(a, \lambda t)$ is contained in G. Observe that $(1 - \lambda)t \leq d(x) \leq t$.

If x is not a λ-annulus point of G, it is a λ-*arc point* of G.

Let $x_0 \in G$ and let $c \geq 1$. A crosscut $\tau\colon a \curvearrowright b$ of G is a c-*anchor* of x_0 in G if $x_0 \in \tau$ and if

(1) τ is a c-quasigeodesic,
(2) τ is c-quasiconvex in norm,
(3) $l(\tau[a, x]) \leq cd(x)$ for all $x \in \tau[x_0, a)$ and $l(\tau[b, x]) \leq cd(x)$ for all $x \in \tau[x_0, b)$.

Observe that these conditions imply that $a \neq \infty \neq b$. Moreover, a c-anchor $\tau\colon a \curvearrowright b$ has the following properties:

(4) τ is c-uniform in G,
(5) $l(\tau) \leq 2cd(x_0)$,
(6) $d(x_0) \leq l(\tau)/2 \leq c|a - b|/2$.

By an important result of [BHK, 7.2], every λ-arc point of a domain $G \subset \mathsf{R}^n$ has a c-anchor in G with $c = c(\lambda)$. We next extend this result to all Banach spaces, and it will be used in several proofs of this article.

3.18. Anchor lemma. *If x_0 is a λ-arc point of a domain G, then x_0 has a c-anchor in G with $c = c(\lambda)$.*

Proof. I follow the idea of [BHK, 7.2], but the proof is more complicated, because segmental endcuts are replaced by arcs given by 3.9 and, moreover, the geometry of normed spaces is not so simple as the euclidean geometry.

We choose a small positive number s. To be safe, we take $s = \lambda/100 \leq 1/200$. Set $d = d(x_0)$ and choose a point $a \in \partial G$ with

$$|a - x_0| = t < (1 + s/2)d.$$

We may assume that $a = 0$. Let $\gamma\colon x_0 \curvearrowright b$ be an endcut of G given by 3.9 for these a and s. Since x_0 is a λ-arc point, there is a point $y \in \partial G$ with $\lambda t < |y| < t/\lambda$. Let L be the ray from 0 through y, and let z be the unique point in $L \cap S(t)$. Let T be a 2-dimensional plane containing $0, x_0, z$ (unique unless $z = \pm x_0$), and let α be the shorter arc between x_0 and z in $S(t) \cap T$. If $z = x_0$, the arc α degenerates to $\{x_0\}$.

Let β: $x_0 \curvearrowright y$ be the arc $\beta = \alpha \cup [z, y]$ and set $r = \lambda t/3 \le t/6$. Since $y \in \partial G$ and since $r < d$, there is a first point x' of β, $x' \ne x_0$, such that $d(x') = r$. Choose a point $a' \in \partial G$ with $|a' - x'| < (1 + s/2)r$ and then a point $x_0' \in \beta[x_0, x']$ such that $|a' - x_0'| = d(a', \beta[x_0, x'])$. Setting $d' = d(x_0')$ we have

$$r \le d' \le |a' - x_0'| \le |a' - x'| < (1 + s/2)r < (1 + s/2)^2 d/6 < d.$$

Let γ': $x_0' \curvearrowright b'$ be an endcut of G given by 3.9 with the substitution $(x_0, a, s) \mapsto (x_0', a', s)$. Setting $\beta_0 = \beta[x_0, x_0']$ we show that

$$\tau = \gamma \cup \beta_0 \cup \gamma' \colon b \curvearrowright b'$$

is the desired anchor. It is not yet quite clear that τ is an arc, but the conditions (1)–(3) of 3.17 make sense in an obvious way. For example, for points $u \in \beta_0$ and $v \in \gamma'$, condition (2) means that $l(\beta_0[u, x_0']) + l(\gamma'[x_0', v]) \le c|u - v|$. We show that τ has properties (1)–(3), and then (2) implies that τ is an arc.

We first verify (3). The case $x \in \gamma \cup \gamma'$ follows from 3.9(5). If $x \in \beta_0$, then $d(x) \ge r \ge \lambda d/3$. Furthermore, $|z - y| \le t/\lambda - t$ and $l(\alpha) \le 4t$ by 3.4. Thus $l(\beta_0) \le 3t + t/\lambda$. Since $l(\gamma') \le (1+s)d' \le (1+s)d$ by 3.9(1) and since $t < (1+s)d$, we have $l(\beta_0 \cup \gamma') \le (1 + s)(4 + 1/\lambda)d$, and (3) follows.

Property (2) will be proved in A.6 in the appendix, and (1) follows from (2) and (3) by 3.7. □

3.19. *Roads and biroads.* Recall from [Vä8, Section 6] that for $\mu \ge 0$, $h \ge 0$, a (μ, h)-*road* in a δ-hyperbolic domain G is a sequence $\bar\alpha$ of h-short arcs α_i: $y_i \curvearrowright u_i$ such that $l_k(\alpha_i) \nearrow \infty$ and such that for $i \le j$, the quasihyperbolic length map g_{ij}: $\alpha_i \to \alpha_j$ with $g_{ij} y_i = y_j$ satisfies $k(g_{ij}x, x) \le \mu$ for all $x \in \alpha_i$. The sequence (u_i) is then a Gromov sequence defining an element $\hat u \in \partial^* G$, and we write $\bar\alpha$: $\bar y \curvearrowright \hat u$. Each pair $y \in G$, $b \in \partial^* G$ can be joined by a (μ, h)-road with $\mu = 4\delta + 2h$.

Furthermore, each pair $a, b \in \partial^* G$ can be joined by a (μ, h)-*biroad* $\bar\alpha$: $a \curvearrowright b$ for each $h > 0$ and $\mu = 12\delta + 10h$. This means that $\bar\alpha$ is a sequence of arcs α_i: $u_i \curvearrowright v_i$ in G and that for $i \le j$ there are quasihyperbolic length maps g_{ij}: $\alpha_i \to \alpha_j$ with the following properties:

(1) Each α_i is h-short.
(2) $u_i \to a$, $v_i \to b$.
(3) $g_{ii} = \mathrm{id}$, $g_{ik} = g_{ij} \circ g_{jk}$ for $i \le j \le k$.
(4) $k(g_{ij}x, x) \le \mu$ for all $i \le j$ and $x \in \alpha_i$.

The *locus* $|\bar\alpha|$ of a road or a biroad $\bar\alpha$ is the union of all arcs α_i.

We recall the *extended standard estimate* [Vä8, 6.20] for a (μ,h)-biroad $\bar\alpha$: $a \curvearrowright b$ in a δ-hyperbolic domain:

$$k(p,|\bar\alpha|) - 4\delta - h \le (a\,|\,b)_p \le k(p,|\bar\alpha|) + \mu + h/2. \tag{3.20}$$

We next show that hyperbolic domains are roughly starlike. The proof is a variation of [BHK, 7.8]. Recall from [Vä8, 6.33] that a hyperbolic domain G is said to be (K,μ)-*roughly starlike* with respect to a point $y \in G$ if for each $x \in G$ and for each $h > 0$ there is a (μ,h)-road $\bar\alpha$: $y \curvearrowright b \in \partial^* G$ with $k(x,|\bar\alpha|) \le K$.

Furthermore, G is (K,μ)-roughly starlike with respect to a *boundary point* $a \in \partial^* G$ if for each $x \in G$ and for each $h > 0$ there is a (μ,h)-biroad $\bar\alpha$: $a \curvearrowright b \in \partial^* G$ with $k(x,|\bar\alpha|) \le K$.

3.12. Lemma. *Let $0 < r < s$, let G be a hyperbolic domain containing the annulus $B(s) \setminus \bar B(r)$, and let $\bar x$ and $\bar y$ be equivalent Gromov sequences in G. If $|x_i| \le r$ for all i, then $|y_i| < s$ for large i. If $|y_i| \ge s$ for all i, then $|x_i| > r$ for large i.*

Proof. Assume that $|x_i| \le r$, $|y_i| \ge s$. Let $h = 1$ and choose h-short arcs γ_i: $x_i \curvearrowright y_i$. Let p be a base point of G with $|p| = t = (r+s)/2$. Since γ_i meets $S(t)$ and since $k\big(S(t)\big) \le 4t/(r-s)$ by 3.4, the standard estimate 2.9 gives

$$(x_i\,|\,y_i) \le k(p,\gamma_i) + h/2 \le 4t/(r-s) + 1/2.$$

As $\bar x \sim \bar y$, we have $(x_i\,|\,y_i) \to \infty$, and the lemma follows. □

3.22. Theorem. *A δ-hyperbolic domain G is (K,μ_1)-roughly starlike with respect to each point of G and (K,μ_2)-roughly starlike with respect to each point of $\partial^* G$, where $K = K(\delta)$, $\mu_1 = 4\delta + 1$, $\mu_2 = 12\delta + 1$.*

Proof. We prove the second part, which is needed later. The proof for the first part is rather similar.

Let $a \in \partial^* G$, $x_0 \in G$, $0 < h \le 1/10$. It suffices to find a (μ_2,h)-biroad $\bar\sigma$ from a with $k(x_0,|\bar\sigma|) \le K(\delta)$.

Case 1. x_0 is a $\frac{1}{3}$-arc point. Let τ: $b \curvearrowright c$ be a c-anchor of x_0 given by 3.18; now $c = c(1/3)$ is a universal constant. By Remark 3.14.3, the points b and c can be considered as elements of the Gromov boundary $\partial^* G$. Let $\bar\alpha$: $b \curvearrowright c$ be a (μ_2,h)-biroad given by [Vä8, 6.13]. By the extended stability theorem [Vä8, 6.32] there is a point $x_1 \in |\bar\alpha|$ with $k(x_0,x_1) \le K_1(\delta)$. If $a \in \{b,c\}$, we may put $\bar\sigma = \bar\alpha$. If $a \notin \{b,c\}$, we choose (μ_2,h)-biroads $\bar\beta$: $a \curvearrowright c$ and $\bar\gamma$: $a \curvearrowright b$. By the extended Rips condition [Vä8, 6.24] there is $x_2 \in |\bar\beta| \cup |\bar\gamma|$ with $k(x_1,x_2) \le K_2(\delta)$. Then $k(x_0,x_2) \le K_1 + K_2$, whence either $\bar\beta$ or $\bar\gamma$ is the desired biroad.

Case 2. x_0 is a $\frac{1}{3}$-annulus point. There is a point $b \in \partial G$ such that $B(b, 3t) \setminus \overline{B}(b, t/3) \subset G$ for $t = |x_0 - b|$. Choose a Gromov sequence $\bar{u} \in a$. Since $k(x_0, u_i) \to \infty$, we may assume that either $|u_i - b| < t/2$ for all i or $|u_i - b| > 2t$ for all i. Assume that $|u_i - b| < t/2$ for all i. By 3.16 we find a Gromov sequence $\bar{v}$ such that $|v_i - b| > 2t$ for all i. Choose a (μ_2, h)-biroad $\bar{\sigma}$: $a \curvearrowright \hat{v}$. From 3.21 it follows that σ_i meets $S(b, t)$ for large i. Since $k\big(S(b, t)\big) \le 8$ by 3.4, we have $k(x_0, |\bar{\sigma}|) \le 8$.

The case $|u_i - b| > 2t$ is treated similarly, choosing a Gromov sequence $\bar{v}$ with $|v_i - b| < t/2$ for all i. □

In the next three lemmas we estimate the Gromov product $(a \,|\, b) = (a \,|\, b)_p$ for an anchor τ: $a \curvearrowright b$ of a point in a hyperbolic domain with a base point p. Since τ is a quasigeodesic, the points a, b can be considered as elements of $\partial^* G$; see Remark 3.14.3. Then $(a \,|\, b)$ is defined.

3.23. Lemma. *Suppose that τ: $a \curvearrowright b$ is a c-anchor of a point x_0 in a δ-hyperbolic domain G. Let α: $p \curvearrowright u$ be an h-short arc containing a point y_0 with $k(x_0, y_0) \le \mu$. Then $(a \,|\, b) \le (a \,|\, u) + K(\delta, c, \mu, h)$.*

Proof. Choose sequences (x_i) on $\tau[x_0, a)$ and (y_i) on $\tau[x_0, b)$ such that $x_i \to a$, $y_i \to b$. Fix i and set $t = (x_i \,|\, y_0)$. As $t \le k(p, y_0)$, we can choose a point $z \in \alpha[p, y_0]$ with $k(p, z) = t$. By [Vä8, 2.8(6)] we have $(z \,|\, y_0) \wedge (z \,|\, u) \ge t - h/2$. Hence

$$(x_i \,|\, u) \ge (x_i \,|\, y_0) \wedge (y_0 \,|\, z) \wedge (z \,|\, u) - 2\delta \ge t - h/2 - 2\delta. \tag{3.24}$$

Choose h-short arcs β: $x_i \curvearrowright y_0$ and γ: $x_i \curvearrowright y_i$. By [Vä8, (2.23) and 2.24] we find a point $z_1 \in \beta$ with $k(z_1, z) \le 4\delta + 4h$. Since τ is a c-quasigeodesic, it follows from the stability lemma 3.12 that $k(x_0, \gamma) \le M(\delta, c, h)$, whence $k(y_0, \gamma) \le M + \mu$. By the second ribbon lemma [Vä8, 2.18] this yields $k(z_1, \gamma) \le K_1(\delta, c, \mu, h)$, and we obtain $k(z, \gamma) \le 4\delta + 4h + K_1$. Since

$$(x_i \,|\, y_i) \le k(p, \gamma) + h/2 \le t + k(z, \gamma) + h/2$$

by 2.9, this and (3.24) imply that $(x_i \,|\, y_i) \le (x_i \,|\, u) + K_1 + 6\delta + 5h$. As $i \to \infty$, this gives the lemma. □

3.25. Lemma. *Let τ: $a \curvearrowright b$ be a c-anchor of a point x_0 of a δ-hyperbolic domain G. Then $(a \,|\, b) \le k(p, x_0) + K(\delta, c)$.*

Proof. Choose sequences (x_i) and (y_i) on τ as in the proof of 3.23. Set $h = 1$ and let γ_i: $x_i \curvearrowright y_i$ be an h-short arc. It follows from (2.9) and stability 3.12 that

$$(x_i \,|\, y_i) \le k(p, \gamma_i) + h/2 \le k(p, x_0) + K_1(\delta, c).$$

As $i \to \infty$, this implies the lemma. □

In the next lemma we show that under certain additional conditions, the inequality of 3.25 can be reversed.

3.26. Lemma. *Suppose that $\tau\colon a \curvearrowright b$ is a c-anchor of a point x_0 in a δ-hyperbolic domain G. Let $\alpha\colon p \curvearrowright y_0$ be an h-short arc with $k(x_0, y_0) \le \mu$ such that $d(y_0) \le c_1 d(x)$ for all $x \in \alpha$. Then $(a\,|\,b) \ge k(p, x_0) - K(\delta, c, c_1, \mu, h)$.*

Proof. Let $K_0, K_1, \dots$ denote positive constants depending on (δ, c, c_1, μ, h). Let (x_i) be a sequence on $\tau[x_0, a)$ converging to a. Fix $i \in \mathsf{N}$ and choose h-short arcs $\beta\colon x_i \curvearrowright y_0$ and $\gamma\colon x_i \curvearrowright x_0$. As in the proof of 3.23, we choose a point $z \in \alpha$ with $k(p, z) = (y_0\,|\,x_i)$, and then $k(z, \beta) \le 4\delta + 4h$. By the second ribbon lemma [Vä8, 2.18] we have $k_H(\beta, \gamma) \le 8\delta + 5\mu + 5h$. From these estimates and from stability 3.12 it follows that there is a point $z' \in \tau[x_0, x_i]$ with $k(z, z') \le K_1(\delta, c, \mu, h)$. We have

$$\log \frac{d(y_0)}{d(z')} \le \log \frac{c_1 d(z)}{d(z')} \le k(z, z') + \log c_1 \le K_1 + \log c_1,$$

whence $d(x_0) \le e^{\mu} d(y_0) \le K_2 d(z')$. For each $x \in \sigma = \tau[z', x_0]$, this and 3.17(3) yield

$$d(x_0)/K_2 \le d(z') \le |z' - a| \le l(\tau[x, a]) \le c d(x).$$

Moreover, we have $l(\sigma) \le c d(x_0)$ by 3.17(3). Integration along σ gives $k(z', x_0) \le c^2 K_2$, and we obtain

$$k(p, x_0) \le k(p, z) + k(z, z') + k(z', x_0) \le (x_i\,|\,y_0) + K_1 + c^2 K_2 \le (x_i\,|\,x_0) + K_3.$$

As $i \to \infty$, this gives $k(p, x_0) \le (a\,|\,x_0) + K_4$. Similarly $k(p, x_0) \le (b\,|\,x_0) + K_4$. Hence $(a\,|\,b) \ge (a\,|\,x_0) \wedge (b\,|\,x_0) - \delta \ge k(p, x_0) - K_4 - \delta$; see [Vä8, 5.12]. □

We next state the main result of this section. Recall from 2.21–2.23 that for a δ-hyperbolic domain G, the natural map $\varphi\colon G^* \to \overline{G}$ exists if and only if each Gromov sequence $\bar{x}$ in G converges in norm to a limit $b(\bar{x})$; then $\varphi\hat{x} = b(\bar{x})$. The map φ is continuous in the metametric $d_{p,\varepsilon}$ of G^* and the norm metric of $\overline{G}$, where $p \in G$ and $0 < \varepsilon \le 1 \wedge (1/5\delta)$. Moreover φ defines a continuous map $\psi\colon \partial^* G \to \partial G$ between metric spaces.

3.27. Theorem. *Suppose that $G \subset E$ is a δ-hyperbolic domain such that the natural map $\varphi\colon G^* \to \overline{G}$ exists in the sense of 2.12 and defines an η-quasimöbius bijection $\psi\colon \partial^* G \to \partial G$ in the metric $d_{p,\varepsilon}$ and the norm metric for some $p \in G$ and for some $\varepsilon \le 1 \wedge (1/5\delta)$. Then G is C-uniform with $C = C(\delta, \eta, \varepsilon)$.*

3.28. *Notation.* Let $\delta > 0$, $0 < \varepsilon \le 1 \wedge (1/5\delta)$ and let $\eta\colon [0, \infty) \to [0, \infty)$ be a homeomorphism. We let $Q(\delta, \eta, \varepsilon)$ denote the set of all domains G in some Banach space such that

(1) G is unbounded,

(2) G is δ-hyperbolic,

(3) the natural map $\varphi\colon G^* \to \overline{G}$ exists and defines an η-quasimöbius homeomorphism $\psi\colon \partial^* G \to \partial G$ in the metric $d_{p,\varepsilon}$ and in the norm metric for all $p \in G$.

For $G \in Q(\delta, \eta, \varepsilon)$, we simplify the notation by identifying $\partial^* G$ and ∂G by the natural bijection $\psi\colon \partial^* G \to \partial G$ writing $\psi x = x$ for $x \in \partial^* G$.

We shall reduce Theorem 3.27 to the following result:

3.29. Proposition. *If $G \in Q(\delta, \eta, \varepsilon)$, then G is C-uniform with $C = C(\delta, \eta, \varepsilon)$.*

3.30. Lemma. *Proposition* 3.29 *implies Theorem* 3.27.

Proof. Suppose that $G \subset E$ is a domain satisfying the conditions of 3.27 with some $\delta, \eta, \varepsilon, p$. We may assume (by a translation) that $0 \in \partial G$. Let u be the inversion of E; see 2.33. We show that the domain $G' = uG$ lies in $Q(v')$ with some $v' = (\delta', \eta', \varepsilon')$ depending only on $v = (\delta, \eta, \varepsilon)$. By Proposition 3.29, this will imply that G' is $C'(v')$-uniform, whence G is $C(v)$-uniform by 2.33(3).

Since $0 \in \partial G$, the domain G' is unbounded. The inversion u defines homeomorphisms $u_1\colon G \to G'$ and $v_1 = u_1^{-1}\colon G' \to G$, and these maps are 12-quasihyperbolic by 2.33(2). From [Vä8, 3.18] it follows that G' is δ'-hyperbolic with some $\delta'(\delta) \geq \delta$.

It remains to show that G satisfies condition 3.28(3). Set $\varepsilon' = \varepsilon \wedge (1/5\delta')$. Then $\varepsilon' \leq \varepsilon \leq c\varepsilon'$ with $c = c(\delta) = \delta'/\delta$. Let $p' \in G'$ be an arbitrary base point. As v_1 is 12-quasihyperbolic, it follows from [Vä8, 5.38] that v_1 induces a homeomorphism $\psi_1 = \partial v_1\colon \partial^* G' \to \partial^* G$. Moreover, ψ_1 is η_1-quasimöbius in the metrics $d_{p',\varepsilon'}$ and $d_{p,\varepsilon'}$ with η_1 depending only on δ. Consequently, the natural map $\psi'\colon \partial^* G' \to \partial G'$ exists and $\psi' a = u\psi\psi_1 a$ for $a \in \partial^* G'$. By [Vä8, 5.28], the identity map $(\partial^* G, d_{p,\varepsilon'}) \to (\partial^* G, d_{p,\varepsilon})$ is θ-quasimöbius with $\theta(t) = 4^{c+1}(t \vee t^c)$. By 2.33(1), these facts imply that ψ' is η'-quasimöbius with $\eta'(t) = 81\eta\theta\eta_1(t)$. Hence $G' \in Q(\delta', \eta', \varepsilon')$. $\square$

3.31. *Outline of the proof of Proposition* 3.29. Suppose that $G \in Q(\delta, \eta, \varepsilon)$ and let $x_1, x_2 \in G$. If $|x_1 - x_2| \leq d(x_1) \vee d(x_2)$, then the line segment $[x_1, x_2]$ is a 1-uniform arc. Hence we may assume that $|x_1 - x_2| \geq d(x_1) \vee d(x_2)$. We shall show that any h-short arc $\gamma\colon x_1 \curvearrowright x_2$ with $h = 1/10$ is C-uniform, $C = C(\delta, \eta, \varepsilon)$. For this we approximate γ by a (μ, h)-biroad $\bar\alpha\colon a_1 \curvearrowright a_2$ such that $k(x_i, |\bar\alpha|) \leq C_1$. In 3.54 we show that the members of $\bar\alpha$ satisfy conditions close to uniformity. As a crucial step we prove the case $a_2 = \infty$ in the length carrot lemma 3.40, which is preceded by a related result 3.36 on distance carrots.

The proof makes substantial use of the division of a domain into annulus points and arc points and of the anchor lemma 3.18. It will be completed in 3.37.

The following basic estimates follow almost immediately from (2.20):

3.32. Lemma. *Suppose that G is a δ-hyperbolic domain and that $x,y,a,b \in G^*$, $\varepsilon \le 1 \wedge (1/5\delta)$.*

(1) If $d_\varepsilon(x,y) \le c d_\varepsilon(a,b)$, then $(a\,|\,b) \le (x\,|\,y) + K$ with $K = K(c,\varepsilon) = \varepsilon^{-1}\log(2c)$.

(2) If $(a\,|\,b) \le (x\,|\,y) + K$, then $d_\varepsilon(x,y) \le c d_\varepsilon(a,b)$ with $c = c(K) = 2e^K$. □

3.33. Lemma. *Let G be a δ-hyperbolic domain, let $h \le 1/10$, and let $\bar\alpha$ be a (μ,h)-road, $\alpha_i\colon z_i \curvearrowright u_i$, such that $u_i \to b \in \partial G$ in norm. Let $i \in \mathsf{N}$ and let $x \in \alpha_i$ be a λ-annulus point of G with $\lambda^{-1} \ge 16e^\mu$ and let $d(z_i) \ge 9e^{2\mu}d(x)$. Then for $r = d(x)$ we have*

(1) $B\big(b,(\lambda^{-1}-1)r\big) \setminus \overline{B}(b,3\lambda r) \subset G$,
(2) $\alpha_i[x,u_i] \subset B(b,9r)$,
(3) $r \le |x-b| \le (1+4\lambda)r$.

Proof. There is a point $a \in \partial G$ such that G contains the annulus $B(a,t/\lambda) \setminus \overline{B}(a,\lambda t)$ for $t = |x-a|$. Then

$$(1-\lambda)t \le r \le t. \tag{3.34}$$

Let $j \ge i$. There is a point $x_j \in \alpha_j$ with $k(x_j,x) \le \mu$. We have $|x_j - x| \le (e^\mu - 1)r \le (e^\mu-1)t$, whence $|x_j - a| \le e^\mu t$. Furthermore, $k(z_i,z_j) \le \mu$ implies that $d(z_j) \ge e^{-\mu}d(z_i) \ge 9e^\mu r$, whence

$$|z_j - a| \ge d(z_j) \ge 9e^\mu r \ge 8e^\mu t$$

by (3.34).

If $|y-a| \ge 8e^\mu t$ for some $y \in \alpha_j[x_j,u_j]$, then we may apply Lemma 3.5(1) to the arc $\alpha_j[z_j,y]$ with the substitution $t \mapsto e^\mu t$ to conclude that $|x_j - a| > e^\mu t$, a contradiction. Hence $\alpha_j[x_j,u_j] \subset B(a,8e^\mu t)$ for all $j \ge i$. For $j = i$, we choose $x_i = x$ and get the better estimate

$$\alpha_i[x,u_i] \subset B(a,8t). \tag{3.35}$$

Since $u_j \to b$, we obtain $|b-a| \le 8e^\mu t < t/\lambda$. As $b \in \partial G$, this implies that $|b-a| \le \lambda t$. The lemma follows by easy estimates from this, (3.34) and (3.35). □

3.36. Distance carrot lemma. *Let $G \in Q(\delta,\eta,\varepsilon)$, let $h = 1/10$, let $\bar\alpha\colon b_0 \curvearrowright \infty$, $\alpha_i\colon u_i \curvearrowright v_i$ be a (μ,h)-biroad and let $x \in |\bar\alpha|$. Then*

$$|x - b_0| \le C(\delta,\eta,\mu)d(x).$$

Proof. Choose i with $x \in \alpha_i$ and set $\lambda^{-1} = 16e^\mu$.

Case 1. x is a λ-annulus point. There is a point $a \in \partial G$ such that G contains the annulus $B(a, t/\lambda) \setminus \overline{B}(a, \lambda t)$ where $t = |x - a|$. Since $v_i \to \infty$, there is an integer $j_0 \geq i$ such that $|v_j - a| \geq 8e^{\mu}t$ for $j \geq j_0$. Let $j \geq j_0$ and set $x_j = g_{ij}x$; then $k(x_j, x) \leq \mu$.

If $|u_j - a| \geq 8e^{\mu}t$, we may apply Lemma 3.5(1) with the substitution $t \mapsto e^{\mu}t$ and get $|x_j - a| > e^{\mu}t$. On the other hand,

$$|x_j - x| \leq (e^{k(x_j,x)} - 1)d(x) \leq (e^{\mu} - 1)t,$$

which yields $|x_j - a| \leq e^{\mu}t$, a contradiction. Hence $|u_j - a| \leq 8e^{\mu}t$ for all $j \geq j_0$, whence $|b_0 - a| \leq 8e^{\mu}t$, which implies that $|b_0 - a| \leq \lambda t$. Since $d(x) \geq (1 - \lambda)t$, we obtain

$$|x - b_0| \leq |x - a| + |a - b_0| \leq (1 + \lambda)d(x)/(1 - \lambda) < 2d(x).$$

Case 2. x is a λ-arc point. By 3.18 there is a $c(\mu)$-anchor τ: $a \curvearrowright b$ of x. Set $\mu_0 = 12\delta + 1$ and choose a (μ_0, h)-biroad $\bar{\beta}$: $a \curvearrowright b$. By (3.20) and by the extended version [Vä8, 6.32(2)] of the stability lemma we get $(a \,|\, b)_x \leq k(x, |\bar{\beta}|) + \mu + h/2 \leq M(\delta, \mu)$. As $\varepsilon \leq 1$, this yields $d_{x,\varepsilon}(a, b) \geq e^{-M}/2$.

Furthermore, for $j \geq i$ we have $(u_j \,|\, v_j)_x \leq k(x, \alpha_j) + h/2 \leq \mu + 1$, which yields $d_{x,\varepsilon}(u_j, v_j) \geq e^{-\mu-1}/2$ and so $d_{x,\varepsilon}(b_0, \infty) \geq e^{-\mu-1}/2$. Set $Q = (a, b_0, b, \infty)$. As $d_{x,\varepsilon}$ is bounded by 1, these estimates give

$$\operatorname{cr}(Q, d_{x,\varepsilon}) \leq 4e^{M+\mu+1} = K(\delta, \mu).$$

Since the map id: $(\partial G, d_{x,\varepsilon}) \to (\partial G, \text{norm})$ is η-quasimöbius, we obtain

$$|a - b_0|/|a - b| = \operatorname{cr}(Q, \text{norm}) \leq \eta(K).$$

By 3.17(3) we have $|x - a| \leq cd(x)$ and $|a - b| \leq 2cd(x)$, and we get the desired estimate

$$|x - b_0| \leq |x - a| + |a - b_0| \leq \big(1 + 2\eta(K)\big)cd(x). \ \square$$

3.37. Lemma. *Let $1 < c_1 \leq c_2 < c_3$ with $c_1 - 1 \leq c_3 - c_2$. Suppose that G is a domain containing an annulus $B(b, c_3 r) \setminus \overline{B}(b, r)$ and that x and y are points with $c_1 r \leq |x - b| \leq |y - b| \leq c_2 r$. Then $k(x, y) \leq 5c_2/(c_1 - 1)$.*

Proof. We may assume that $b = 0$. Set $z = |y|x/|x|$. By 3.4 there is an arc α: $z \curvearrowright y$ in $S(|y|)$ with $l(\alpha) \leq 2|z - y| \leq 4c_2 r$. The arc $\beta = [x, z] \cup \alpha$ joins x and y with $l(\beta) \leq 5c_2 r$. For every $u \in \beta$ we have $d(u) \geq (|u| - r) \wedge (c_3 r - |u|) \geq c_1 r - r$. Hence $k(x, y) \leq l_k(\beta) \leq 5c_2/(c_1 - 1)$. $\square$

3.38. *Strings.* Let $\bar{\alpha}$: $a \curvearrowright b$, α_i: $u_i \curvearrowright v_i$, be a (μ, h)-biroad in a hyperbolic domain G. We recall from [Vä8, 6.21] that the *string* $\operatorname{str}\bar{\alpha}$ of $\bar{\alpha}$ is obtained by identifying the elements (x, i) and $(g_{ij}x, j)$ in the disjoint union of all α_i. There are natural injective maps

$$\pi_i\colon \alpha_i \to \operatorname{str}\bar{\alpha}, \tag{3.39}$$

defined by $(x, i) \in \pi_i x$, and a metric l_k in $\operatorname{str}\bar{\alpha}$, defined by $l_k(\xi, \zeta) = l_k(\alpha_i[x, z])$ where $\pi_i x = \xi$, $\pi_i z = \zeta$. Furthermore, there is a bijective isometry ω: $\operatorname{str}\bar{\alpha} \to \mathsf{R}$, unique up to an additive constant, and ω defines a linear order in $\operatorname{str}\bar{\alpha}$. The *locus* $|\xi|$ of an element $\xi \in \operatorname{str}\bar{\alpha}$ is the set of all $x \in G$ such that $(x, i) \in \xi$ for some i. Then $k(|\xi|) \le \mu$ and $|\bar{\alpha}| = \bigcup\{|\xi| : \xi \in \operatorname{str}\bar{\alpha}\}$.

3.40. Length carrot lemma. *Let $G \in Q(\delta, \eta, \varepsilon)$ and let $\bar{\alpha}$: $b_0 \curvearrowright \infty$, α_i: $u_i \curvearrowright v_i$ be a (μ, h)-biroad in G with $h = 1/10$. Then*

$$l(\alpha_i[u_i, x]) \le C(\delta, \eta, \varepsilon, \mu) d(x)$$

for all i and for all $x \in \alpha_i$.

Proof. Let $K_1, K_2, \dots$ denote positive constants depending only on $(\delta, \eta, \varepsilon, \mu)$. In particular, let K_1 be the constant $C(\delta, \eta, \mu)$ of 3.36. Set

$$d^*(\xi) = d(|\xi|, \partial G)$$

for $\xi \in \operatorname{str}\bar{\alpha}$. Since $k(|\xi|) \le \mu$, we have $d(x) \le e^\mu d(y)$ for all $x, y \in |\xi|$, whence

$$d^*(\xi) \le d(x) \le e^\mu d^*(\xi) \tag{3.41}$$

for all $x \in |\xi|$. Hence $d^*(\xi) > 0$ for all $\xi \in \operatorname{str}\bar{\alpha}$.

We say that an element $\xi \in \operatorname{str}\bar{\alpha}$ tends to $-\infty$ or to ∞ if the number $\omega(\xi) \in \mathsf{R}$ tends to $-\infty$ or to ∞, respectively. Moreover, we use obvious notation like $[\xi_1, \xi_2]$ and $(-\infty, \xi_0]$ for intervals in $\operatorname{str}\bar{\alpha}$. For a set $A \subset \operatorname{str}\bar{\alpha}$ we write

$$|A| = \bigcup\{|\xi| : \xi \in A\}.$$

Fact 1. $d^*(\xi) \to 0$ as $\xi \to -\infty$ and $d^*(\xi) \to \infty$ as $\xi \to \infty$.

Choose an arbitrary base point $p \in G$ and let $r_0 > 0$. As the natural map φ: $G^* \to \overline{G}$ is continuous at b_0 by Definition 2.21, there is $M_0 > 0$ such that $|x - b_0| < r_0$ for all $x \in G$ with $(x \,|\, b_0) > M_0$. From [Vä8, 6.8] it follows that there is $\xi_0 \in \operatorname{str}\bar{\alpha}$ such that $(x \,|\, b_0) > M_0$ whenever $x \in |\xi|$ for some $\xi \le \xi_0$. Hence $d(|\xi|, b_0) \to 0$ as $\xi \to -\infty$, and the first part of Fact 1 follows.

A similar argument shows that $d(|\xi|, b_0) \to \infty$ as $\xi \to \infty$. Since $d^*(\xi) \ge d(|\xi|, b_0)/K_1$ by 3.36, we obtain the second part of Fact 1.

Fact 2. For each $\xi_0 \in \operatorname{str}\bar\alpha$, the set $|(-\infty, \xi_0]|$ is bounded.

By 3.36 and (3.41), it suffices to show that $d^*(\xi)$ is bounded over $\xi \in (-\infty, \xi_0]$. By Fact 1 there is $\xi_1 \le \xi_0$ such that $d^*(\xi) \le 1$ for $\xi \le \xi_1$. Choose $i \in \mathsf{N}$ such that the natural image $\pi_i \alpha_i$ of α_i covers $[\xi_1, \xi_0]$. Since $d(x)$ is bounded over $x \in \alpha_i$ by compactness, Fact 2 follows.

From Fact 1 it follows that for each $t > 0$, the set $\{\xi \in \operatorname{str}\bar\alpha : d^*(\xi) \le t\}$ is nonempty and bounded from above. Hence it has a supremum in $\operatorname{str}\bar\alpha$. We set

$$\zeta_n = \sup\{\xi \in \operatorname{str}\bar\alpha : d^*(\xi) \le 2^n\}$$

for $n \in \mathsf{Z}$. Then (ζ_n) is an increasing sequence and $\operatorname{str}\bar\alpha$ is the union of the intervals $[\zeta_{n-1}, \zeta_n]$, $n \in \mathsf{Z}$.

We next show that

$$e^{-\mu} 2^n \le d^*(\zeta_n) \le 2^n \tag{3.42}$$

for all $n \in \mathsf{Z}$. Assume that $d^*(\zeta_n) < e^{-\mu}2^n$. Then there is $(z, i) \in \zeta_n$ with $d(z) < e^{-\mu}2^n$. Choose an integer $j \ge i$ such that $g_{ij}z \ne v_j$. Since $k(g_{ij}z, z) \le \mu$, we have $d(g_{ij}z) \le e^{\mu} d(z) < 2^n$. Choose a point $x \in \alpha_j(g_{ij}z, v_j]$ with $d(x) < 2^n$. Then $\pi_j x > \zeta_n$ and therefore $d^*(\pi_j x) > 2^n$, which is impossible because $x \in |\pi_j x|$. Hence the first inequality of (3.42) is true.

Assume that $d^*(\zeta_n) > 2^n$ and let $t > 0$. Since $\pi_i v_i \to \infty$, there is $\xi_0 < \zeta_n$ such that for no $i \in \mathsf{N}$ we have $\xi_0 < \pi_i v_i < \zeta_n$. Choose an element $\xi \in \operatorname{str}\bar\alpha$ with $\xi_0 < \xi < \zeta_n$ such that $d^*(\xi) \le 2^n$ and such that $l_k(\xi, \zeta_n) < t$. If α_i meets $|\xi|$, then α_i also meets $|\zeta_n|$. Consequently, for each $x \in |\xi|$ we have $k(x, |\zeta_n|) \le l_k(\xi, \zeta_n) < t$. Since $k(x, |\zeta_n|) \ge \log\bigl(d^*(\zeta_n)/d(x)\bigr)$, we obtain $d^*(\zeta_n) \le e^t d(x)$ for all $x \in |\xi|$, whence $d^*(\zeta_n) \le e^t d^*(\xi) \le e^t 2^n$. As $t \to 0$, this implies the second inequality of (3.42).

By (3.41) and (3.42) we have

$$e^{-\mu} 2^n \le d(x) \le e^{\mu} 2^n \tag{3.43}$$

for all $n \in \mathsf{Z}$ and $x \in |\zeta_n|$.

We prove in Fact 5 below that $l_k(\zeta_{n-1}, \zeta_n) \le K_2$ for all $n \in \mathsf{Z}$. We show now how this implies the lemma. Let $i \in \mathsf{N}$ and let $x \in \alpha_i$. Choose integers s and n with $s \le n$ such that $\pi_i u_i \in (\zeta_{s-1}, \zeta_s]$ and $\pi_i x \in (\zeta_{n-1}, \zeta_n]$. Next choose $m \ge i$ such that $[\zeta_{s-1}, \zeta_n] \subset \pi_m \alpha_m$. Writing $g = g_{im}$ we have $l(\alpha_i[u_i, x]) \le e^{\mu} l(\alpha_m[gu_i, gx])$ by 3.3. For each $j = s-1, \dots, n$ there is a point $z_j \in \pi_m^{-1}\zeta_j$, and we obtain

$$l(\alpha_m[gu_i, gx]) \le l(\alpha_m[z_{s-1}, z_n]) \le \sum_{j=s}^{n} l(\alpha_m[z_{j-1}, z_j]).$$

Since $l_k(\alpha_m[z_{j-1},z_j]) = l_k(\zeta_{j-1},\zeta_j) \le K_2$ and since $d(z_{j-1}) \le e^\mu 2^{j-1}$ by (3.43), this and Lemma 3.2 yield

$$l(\alpha_m[gu_i,gx]) \le e^{K_2+\mu}\sum_{j=s}^{n} 2^{j-1} \le 2^n e^{K_2+\mu}.$$

As $\pi_i x > \zeta_{n-1}$, we have $d(x) \ge d^*(\pi_i x) > 2^{n-1}$. Hence the lemma holds with $C = 2e^{K_2+2\mu}$.

Set $\lambda = (20e^{2\mu})^{-1}$ and $\lambda_1 = e^{-\mu}\lambda = (20e^{3\mu})^{-1}$. We say that an element $\xi \in \operatorname{str}\bar\alpha$ is a λ-*annulus element* if $|\xi|$ contains a λ-annulus point of G. All other elements of $\operatorname{str}\bar\alpha$ are called λ-*arc elements*. Each point in the locus of a λ-arc element is a λ-arc point.

Fact 3. If $\xi \in \operatorname{str}\bar\alpha$ is a λ_1-annulus element, then all points of $|\xi|$ are λ-annulus points of G.

There are a λ_1-annulus point $x_1 \in |\xi|$ and a point $a \in \partial G$ such that G contains the annulus $B(a,t_1/\lambda_1)\setminus \overline{B}(a,\lambda_1 t_1)$ where $t_1 = |x_1 - a|$. Let $x \in |\xi|$ and set $t = |x-a|$. As $k(|\xi|) \le \mu$, we have

$$|x - x_1| \le (e^\mu - 1)\big(d(x)\wedge d(x_1)\big) \le (e^\mu - 1)(t\wedge t_1),$$

which implies that $e^{-\mu}t_1 \le t \le e^\mu t_1$. Hence $B(a,t/\lambda)\setminus\overline{B}(a,\lambda t) \subset G$, and Fact 3 is proved.

Fact 4. Let $\xi_1 < \xi_2$ be λ_1-arc elements of $\operatorname{str}\bar\alpha$ such that $d^*(\xi_2) \le 2e^\mu d^*(\xi_1) \le c_1 d^*(\xi)$ for all $\xi \in \operatorname{str}\bar\alpha$ with $\xi \ge \xi_1$. Then

$$l_k(\xi_1,\xi_2) \le c_2(\delta,\eta,\varepsilon,\mu,c_1).$$

Fix a member α_m of $\bar\alpha$ such that $[\xi_1,\xi_2] \subset \pi_m\alpha_m$. Let $x_i \in \alpha_m$ be the point with $\pi_m x_i = \xi_i$, $i = 1,2$. As these points are λ_1-arc points, we can choose c-anchors $\tau_i\colon a_i \curvearrowright b_i$ of x_i, $i = 1,2$, $c = c(\mu)$. We may assume that $|b_i - b_0| \le |a_i - b_0|$. Then $0 < |a_i - b_i| \le 2|a_i - b_0|$. Hence $\operatorname{cr}(Q,\mathrm{norm}) \le 2$ for the quadruple $Q = (a_2,b_2,b_0,\infty)$. Since the map $\psi^{-1}\colon \partial G \to \partial^* G$ is η'-quasimöbius with $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$, this yields $\operatorname{cr}(Q,d_{p,\varepsilon}) \le \eta'(2)$ for each $p \in G$. If $p \in |\bar\alpha|$, then (3.20) gives $(b_0\,|\,\infty) \le k(p,|\bar\alpha|) + \mu + h/2 \le \mu + 1$, which implies that $d_{p,\varepsilon}(b_0,\infty) \ge e^{-\mu-1}/2$ by (2.20). Since $d_{p,\varepsilon} \le 1$, we obtain $d_{p,\varepsilon}(a_2,b_2) \le 2e^{\mu+1}\eta'(2)d_{p,\varepsilon}(a_2,b_0)$, which gives

$$(a_2\,|\,b_0) \le (a_2\,|\,b_2) + K_3 \tag{3.44}$$

by 3.32 for each base point $p \in |\bar\alpha|$.

We have $|a_2 - x_2| \le cd(x_2)$ by 3.17(3) and $|x_2 - b_0| \le K_1 d(x_2)$ by 3.36. By 3.17(6) we get $d^*(\xi_1) \le d(x_1) \le c|a_1 - b_1|/2 \le c|a_1 - b_0|$, whence $d(x_2) \le e^{\mu} d^*(\xi_2) \le 2e^{2\mu} d^*(\xi_1) \le 2e^{2\mu} c|a_1 - b_0|$. Consequently,

$$|a_2 - b_0| \le |a_2 - x_2| + |x_2 - b_0| \le K_4 |a_1 - b_0|.$$

Setting $Q' = (b_0, a_2, a_1, \infty)$ we thus have $\operatorname{cr}(Q', \mathrm{norm}) \le K_4$, whence

$$\operatorname{cr}(Q', d_{p,\varepsilon}) \le \eta'(K_4). \tag{3.45}$$

To estimate this cross ratio we need a lower bound for $d_{p,\varepsilon}(a_1, \infty)$. We may assume that $\mu \ge 12\delta + 1$, and hence we can choose a (μ, h)-biroad $\bar{\alpha}_1\colon a_1 \curvearrowright \infty$; see 3.19. By the closeness lemma [Vä8, 6.9], there is an element $\varkappa \in \operatorname{str} \bar{\alpha}$ such that $\varkappa > \xi_2$ and $k(|\varkappa|, |\bar{\alpha}_1|) \le 7\delta + \mu + 1/2$. We fix a base point $p \in |\varkappa|$. Then (3.20) gives

$$(a_1 \,|\, \infty) \le k(p, |\bar{\alpha}_1|) + \mu + h/2 \le 7\delta + 2\mu + 1 = K_5.$$

Hence $d_{p,\varepsilon}(a_1, \infty) \ge e^{-K_5}/2$ and $\operatorname{cr}(Q', d_{p,\varepsilon}) \ge e^{-K_5} d_{p,\varepsilon}(b_0, a_2)/2 d_{p,\varepsilon}(b_0, a_1)$. By (3.45) and 3.32 we get

$$(a_1 \,|\, b_0) \le (a_2 \,|\, b_0) + K_6. \tag{3.46}$$

There is $n_0 \in \mathsf{N}$ such that $[\xi_1, \varkappa] \subset \pi_n \alpha_n$ for $n \ge n_0$. Let $n \ge n_0$ and let $x_1', x_2', p' \in \alpha_n$ be the points with $\pi_n x_i' = \xi_i$, $\pi_n p' = \varkappa$. Applying Lemma 3.23 with the substitution $(\tau, x_0, y_0, p, \alpha) \mapsto (\tau_1, x_1, x_1', p', \alpha_n[u_n, p'])$ we obtain $(a_1 \,|\, b_1)_{p'} \le (a_1 \,|\, u_n)_{p'} + K_7$. Since $k(p, p') \le \mu$, this gives $(a_1 \,|\, b_1) \le (a_1 \,|\, b_0) + K_8$ as $n \to \infty$. By (3.44) and (3.46) it follows that

$$(a_1 \,|\, b_1) \le (a_2 \,|\, b_2) + K_9. \tag{3.47}$$

We have $(a_2 \,|\, b_2)_{p'} \le k(p', x_2') + K_{10}$ by 3.25. If $x \in \alpha_n[x_1', p']$, and $\xi = \pi_n x$, then

$$d(x_1') \le e^{\mu} d^*(\xi_1) \le e^{\mu} c_1 d^*(\xi) \le e^{\mu} c_1 d(x).$$

Hence we may apply Lemma 3.26 with $(\tau, x_0, y_0, p, \alpha, c_1) \mapsto (\tau_1, x_1, x_1', p', \alpha_n[x_1', p'], e^{\mu} c_1)$ and obtain the estimate $k(p', x_1) \le (a_1 \,|\, b_1)_{p'} + C(\delta, \eta, \varepsilon, \mu, c_1)$. Since $k(p, p') \le \mu$ and since

$$l_k(\xi_1, \xi_2) = l_k(\alpha_n[x_1', x_2']) \le k(x_1', p') - k(x_2', p') + h,$$

these estimates and (3.47) imply Fact 4.

Fact 5. $l_k(\zeta_{n-1},\zeta_n)\le K_2$ for all $n\in\mathsf{Z}$.

We consider two cases.

Case 1. ζ_{n-1} and ζ_n are λ_1-arc elements. Since $d^*(\zeta_n)\le 2^n\le 2e^\mu d^*(\zeta_{n-1})$ by (3.42) and since $d^*(\xi)\ge d^*(\zeta_{n-1})$ for all $\xi>\zeta_{n-1}$ by the choice of ζ_{n-1}, the desired estimate follows from Fact 4.

Case 2. ζ_{n-1} or ζ_n is a λ_1-annulus element. By Fact 2 there is M_1 such that $d(x)\le M_1$ whenever $x\in|\xi|$ for some $\xi\le\zeta_n$. Pick $m\in\mathsf{N}$ such that $\pi_m\alpha_m$ covers $[\zeta_{n-1},\zeta_n]$ and such that $d(v_m)\ge 9e^{2\mu}M_1$. Choose points $x_j\in\pi_m^{-1}\zeta_j$, $j=n-1,n$ and set $r_j=d(x_j)$. By (3.43) we have

$$e^{-\mu}2^j\le r_j\le e^\mu 2^j,\quad e^{-2\mu}/2\le r_{n-1}/r_n\le e^{2\mu}/2. \tag{3.48}$$

If ζ_j is a λ_1-annulus element, then x_j is a λ-annulus point by Fact 3. Since $d(g_{mi}v_m)\ge e^{-\mu}d(v_m)\ge 9e^\mu d(x_j)$, we may apply Lemma 3.33 to the (μ,h)-road $(\alpha_i[u_i,g_{mi}v_m])_{i\ge m}$ with $i=m$, $x=x_j$ and obtain

(1) $B\big(b_0,(\lambda^{-1}-1)r_j\big)\setminus\overline{B}(b_0,3\lambda r_j)\subset G$,

(2) $\alpha_m[u_m,x_j]\subset B(b_0,9r_j)$,

(3) $r_j\le|x_j-b_0|\le(1+4\lambda)r_j<2r_j$.

Subcase 2a. ζ_n is a λ_1-annulus element. By (2) we have $|x_{n-1}-b_0|\le 9r_n$. Since $\lambda^{-1}-1\ge 19e^{2\mu}$ and $3\lambda\le e^{-2\mu}/6$, it follows that $d(x_{n-1})\ge|x_{n-1}-b_0|-r_n/6$. By (3.48) this yields $|x_{n-1}-b_0|\le e^{2\mu}r_n$. In the other direction (3.48) gives $|x_{n-1}-b_0|\ge r_{n-1}\ge e^{-2\mu}r_n/2$. In view of (3) it follows that the points x_{n-1} and x_n lie in the closed annulus $\overline{B}(b_0,2e^{2\mu}r_n)\setminus B(b_0,e^{-2\mu}r_n/2)$. By 3.37 we see that $k(x_{n-1},x_n)\le 30e^{4\mu}$. Hence $l_k(\zeta_{n-1},\zeta_n)\le k(x_{n-1},x_n)+h\le 30e^{4\mu}+1$.

Subcase 2b. ζ_{n-1} is a λ_1-annulus element and ζ_n is a λ_1-arc element. Choose a point $a_1\in\partial G$ with $|a_1-x_n|\le 2r_n$. If $|a_1-b_0|\le r_{n-1}/6$, then (3.48) gives

$$|x_n-b_0|\le|x_n-a_1|+|a_1-b_0|\le 2r_n+r_{n-1}/6\le 5e^{2\mu}r_{n-1}$$

and $|x_n-b_0|\ge r_n\ge 2e^{-2\mu}r_{n-1}$. Applying again Lemma 3.37 we get an upper bound $l_k(\zeta_{n-1},\zeta_n)\le K_{11}$.

If $|a_1-b_0|>r_{n-1}/6$, then (1) yields $|a_1-b_0|\ge(\lambda^{-1}-1)r_{n-1}\ge 19e^{2\mu}r_{n-1}$. By connectedness we find points $x',z'\in\alpha_m[x_{n-1},x_n]$ in the order x_{n-1},x',z',x_n such that

(i) z' is a λ_1-arc point,

(ii) all points of $\alpha_m[x_{n-1},x']$ are λ-annulus points,

(iii) $l_k(\alpha_m[x',z'])\le 1$.

Set $\xi'=\pi_m x'$, $\zeta'=\pi_m z'$, $r'=d(x')$. Applying Lemma 3.33 to the points of $\alpha_m[x_{n-1},x']$ we see that G contains the annulus $B(b_0,19e^{2\mu}r')\setminus\overline{B}(b_0,e^{-2\mu}r_{n-1}/6)$, whence $|a_1-b_0|\ge 19e^{2\mu}r'$. By 3.36 and (3.48) we obtain

$$19e^{2\mu}r'\le|a_1-x_n|+|x_n-b_0|\le 2r_n+K_1r_n\le 2e^{2\mu}(2+K_1)r_{n-1}. \tag{3.49}$$

Moreover, (3) gives $r_{n-1} \le |x_{n-1} - b_0| \le 2r_{n-1}$ and similarly $r' \le |x' - b_0| \le 2r'$. Assuming $\xi' \ne \zeta_{n-1}$ we have $r' \ge d^*(\xi') > 2^{n-1} \ge e^{-\mu} r_{n-1}$. By Lemma 3.37 these estimates yield $l_k(\zeta_{n-1}, \xi') \le k(x_{n-1}, x') + h \le K_{12}$.

For each $\xi \ge \zeta'$ we have $d^*(\xi) \ge 2^{n-1}$. In particular, $d^*(\zeta_n) \le 2^n \le 2d^*(\zeta')$. Moreover, by (iii), (3.48) and (3.49) we have

$$d^*(\zeta') \le d(z') \le er' \le (2 + K_1)e^{\mu} 2^{n-1}.$$

Since $d^*(\xi) \ge 2^{n-1}$ for all $\xi \ge \zeta'$, we may apply Fact 4 with $\xi_1 \mapsto \zeta'$, $\xi_2 \mapsto \zeta_n$, $c_1 \mapsto 2e^{2\mu}(2 + K_1)$ and obtain $l_k(\zeta', \zeta_n) \le K_{13}$. Hence $l_k(\zeta_{n-1}, \zeta_n) \le K_{12} + 1 + K_{13}$. Fact 5 and the lemma are proved. □

We next show that a biroad with finite endpoints cannot go too far from the boundary.

3.50. Lemma. *Let $G \in Q(\delta, \eta, \varepsilon)$ and let $\bar{\alpha}$: $a_1 \curvearrowright a_2$ be a (μ, h)-biroad in G with $\mu = 12\delta + 1$, $h = 1/10$, $a_1 \ne \infty \ne a_2$. Then $d(z) \le K(\delta, \eta, \varepsilon)|a_1 - a_2|$ for all $z \in |\bar{\alpha}|$.*

Proof. We let $K_1, K_2, \dots$ denote positive constants depending only on $(\delta, \eta, \varepsilon)$. Set $\lambda^{-1} = 1 + 64e^{\mu}$.

Case 1. z is a λ-annulus point. There is a point $a \in \partial G$ such that G contains the annulus $A = B(a, t/\lambda) \setminus \overline{B}(a, \lambda t)$ where $t = |z - a|$. If a_1 and a_2 lie in different components of $E \setminus A$, then $d(z) \le |z - a| = t$, $|a_1 - a_2| \ge (\lambda^{-1} - \lambda)t > t$, and the lemma holds with $K = 1$.

Suppose that a_1 and a_2 lie in a component of $E \setminus A$. We show that this leads to a contradiction.

If $|a_1 - a| \vee |a_2 - a| \le \lambda t$, we choose a member α_m: $u_m \curvearrowright v_m$ of $\bar{\alpha}$ such that $|u_m - a| \vee |v_m - a| \le 4\lambda t$ and such that α_m contains a point y with $k(y, z) \le \mu$. By Lemma 3.5(2) we have $d(y) \le |y - a| < 64\lambda t$. Since $d(z) \ge (1 - \lambda)t$, we get the contradiction

$$\mu \ge k(y, z) \ge \log \frac{d(z)}{d(y)} > \log \frac{\lambda^{-1} - 1}{64} = \mu.$$

If $|a_1 - a| \wedge |a_2 - a| \ge t/\lambda$, we choose α_n and a point $y \in \alpha_n$ such that $|u_n - a| \wedge |v_n - a| \ge t/2\lambda$ and $k(y, z) \le \mu$. Now Lemma 3.5(1) with $t \mapsto t/16\lambda$ gives $|y - a| > t/16\lambda$. Since $|y - z| \le (e^{\mu} - 1)d(z) \le (e^{\mu} - 1)t$, we get $t/16\lambda < |y - z| + |z - a| \le e^{\mu} t$, which contradicts the definition of λ.

Case 2. z is a λ-arc point. By the anchor lemma 3.18 there is an c-anchor τ: $b_1 \curvearrowright b_2$ of z; now $c = c(\delta)$. By [Vä8, 6.13] we can choose (μ, h)-biroads $\bar{\alpha}_i$: $a_i \curvearrowright \infty$ and $\bar{\beta}_i$: $b_i \curvearrowright \infty$, $i = 1, 2$. By the extended Rips condition [Vä8, 6.24] we may assume that $k(z, |\bar{\alpha}_1|) \le K_1(\delta) = 46\delta + 11\mu + 3$. Hence there is an element $\xi_1 \in \operatorname{str} \bar{\alpha}_1$ with $d(z, |\xi_1|) \le K_1 + \mu$. By the closeness lemma [Vä8, 6.9] we can find an element $\zeta_1 > \xi_1$ in $\operatorname{str}\bar\alpha_1$ with $k(|\zeta_1|, |\bar\beta_i|) \le 7\delta + \mu + 1/2$, $i = 1,2$. Pick a member α_{1m} of $\bar\alpha_1$ whose natural image $\pi_{1m}\alpha_{1m}$ covers $[\xi_1, \zeta_1]$, and choose points $x_1, p \in \alpha_{1m}$ with $\pi_{1m}x_1 = \xi_1$, $\pi_{1m}p = \zeta_1$. We consider p as the base point of G.

By the extended standard estimate (3.20) we have

$$(b_i \,|\, \infty) \le k(p, |\bar\beta_i|) + \mu + h/2 \le K_2(\delta) = 7\delta + 2\mu + 1.$$

As $\varepsilon \le 1$, this yields

$$d_{p,\varepsilon}(b_i, \infty) \ge e^{-K_2}/2, \quad i = 1,2. \tag{3.51}$$

Furthermore,

$$d(z) \le |b_i - z| \le l(\tau) \le c|b_1 - b_2|, \quad i = 1,2. \tag{3.52}$$

Since $|x_1 - a_1| \le K_3 d(x_1)$ by 3.36 and since $k(x_1, z) \le K_1 + 2\mu$, we obtain

$$\begin{aligned}|b_i - a_1| &\le |b_i - z| + |z - x_1| + |x_1 - a_1| \\ &\le c|b_1 - b_2| + e^{K_1+2\mu} d(z) + K_3 e^{K_1+2\mu} d(z) \le K_4 |b_1 - b_2|.\end{aligned}$$

Let Q_i be the quadruple $(b_i, a_1, b_{3-i}, \infty)$ in ∂G, $i = 1,2$. Then $\operatorname{cr}(Q_i, \mathrm{norm}) = |b_i - a_1|/|b_1 - b_2| \le K_4$, whence $\operatorname{cr}(Q_i, d_{p,\varepsilon}) \le \eta'(K_4)$. As $d_{p,\varepsilon} \le 1$, this and (3.51) yield $d_{p,\varepsilon}(b_i, a_1) \le 2e^{K_2}\eta'(K_4) d_{p,\varepsilon}(b_1, b_2)$, which gives

$$(b_1 \,|\, b_2) \le (b_i \,|\, a_1) + K_5 \tag{3.53}$$

by 3.32.

Choose an h-short arc $\sigma\colon z \curvearrowright p$ and let $x \in \sigma$. Since $k(z, x_1) \le K_1 + 2\mu$, we may apply the second ribbon lemma [Vä8, 2.18] to find a point $x' \in \alpha_{1m}[x_1, p]$ with $k(x, x') \le K_6(\delta) = 8\delta + 5(K_1 + 2\mu) + 1$. If $|x_1 - x'| \le d(x_1)/2$, then $d(x') \ge d(x_1)/2$. If $|x_1 - x'| \ge d(x_1)/2$, it follows from the length carrot lemma 3.40 that $d(x_1) \le 2|x_1 - x'| \le 2K_7 d(x')$. In both cases these estimates imply that $d(z) \le K_8 d(x)$ for all $x \in \sigma$ with $K_8 = 2K_7 e^{K_1+2\mu+K_6}$. By 3.26 and (3.53) this yields $k(p, z) \le (b_i \,|\, a_1) + K_9$, $i = 1,2$. By (3.20) we get

$$(a_1 \,|\, a_2) \le k(p, |\bar\alpha|) + \mu + h/2 \le (b_i \,|\, a_1) + K_{10},$$

whence $d_{p,\varepsilon}(b_i, a_1) \le K_{11} d_{p,\varepsilon}(a_1, a_2)$. Setting $Q_i' = (a_1, b_i, a_2, \infty)$ we obtain by (3.51) $\operatorname{cr}(Q_i', d_{p,\varepsilon}) \le 2K_{11}e^{K_2} = K_{12}$, whence

$$|a_1 - b_i|/|a_1 - a_2| = \operatorname{cr}(Q_i', \mathrm{norm}) \le \eta(K_{12}), \quad i = 1,2.$$

By (3.52) this gives the desired estimate $d(z) \le 2c\eta(K_{12})|a_1 - a_2|$. □

The next lemma shows that a (μ, h)-biroad in a domain $G \in Q(\delta, \eta, \varepsilon)$ has properties close to uniformity.

3.54. Lemma. *Let $G \in Q(\delta, \eta, \varepsilon)$ and let $\bar{\alpha}$: $a_1 \curvearrowright a_2$ be a (μ, h)-biroad with $a_1 \neq \infty \neq a_2$, $\mu = 12\delta + 1$, $h = 1/10$. Then*:

(1) *There is an element $\xi_\alpha \in \operatorname{str} \bar{\alpha}$ such that if $x_1, x_2 \in \alpha_m$ and if either $\pi_m x_1 \leq \pi_m x_2 \leq \xi_\alpha$ or $\pi_m x_1 \geq \pi_m x_2 \geq \xi_\alpha$, then $l(\alpha_m[x_1, x_2]) \leq C(\delta, \eta, \varepsilon) d(x_2)$.*

(2) *If $y_1, y_2 \in \alpha_m$ and if $d(y_1) \vee d(y_2) \leq 2|y_1 - y_2|$, then $l(\alpha_m[y_1, y_2]) \leq C'(\delta, \eta, \varepsilon)|y_1 - y_2|$.*

Proof. (1) We let $C_1, C_2, \ldots$ denote positive constants depending only on $(\delta, \eta, \varepsilon)$. Choose (μ, h)-biroads $\bar{\alpha}_i$: $a_i \curvearrowright \infty$, $i = 1, 2$. By the extended tripod lemma [Vä8, 6.25] we can find elements $\xi_\alpha \in \operatorname{str} \bar{\alpha}$ and $\xi_i \in \operatorname{str} \bar{\alpha}_i$, $i = 1, 2$, such that the bijective length maps f_1: $(-\infty, \xi_\alpha] \to (-\infty, \xi_1]$ and f_2: $[\xi_\alpha, \infty) \to (-\infty, \xi_2]$ satisfy $k(|f_i \xi|, |\xi|) \leq C_1(\delta)$ for all possible ξ.

It suffices to consider the case $\pi_m x_1 \leq \pi_m x_2 \leq \xi_\alpha$. Choose a member α_{1n} of α_1 whose natural image $\pi_{1n}\alpha_{1n}$ covers the interval $f_1 \pi_m \alpha_m$. The length map π_{1n}^{-1} is defined in $f_1 \pi_m \alpha_m$ and we obtain a quasihyperbolic length map $g = \pi_{1n}^{-1} f_1 \pi_m$: $\alpha_m \to \alpha_{1n}$ satisfying $k(gx, x) \leq C_1 + 2\mu = C_2(\delta)$. Then $d(gx_2) \leq e^{C_2} d(x_2)$ and $l(\alpha_m[x_1, x_2]) \leq e^{C_2} l(\alpha_{1n}[gx_1, gx_2])$ by 3.3, and (1) follows from 3.40.

(2) We may assume that $\pi_m y_1 \leq \pi_m y_2$. If $\pi_m y_2 \leq \xi_\alpha$, then (1) gives $l(\alpha_m[y_1, y_2]) \leq C d(y_2) \leq 2C|y_1 - y_2|$, and the case $\pi_m y_1 \geq \xi_\alpha$ is treated similarly. Assume that $\pi_m y_1 < \xi_\alpha < \pi_m y_2$.

Let $z \in \alpha_m$ be the point in $\alpha_m[y_1, y_2]$ with $\pi_m z = \xi_\alpha$. By (1) we have

$$l(\alpha_m[y_1, y_2]) \leq 2C d(z). \tag{3.55}$$

Now the extended tripod lemma gives points $z_i \in |\bar{\alpha}_i|$ with $k(y_i, z_i) \leq C_2$. Then $d(z_i) \leq e^{C_2} d(y_i)$ and $|y_i - z_i| \leq e^{C_2} d(y_i)$. By the distance carrot lemma 3.36 we have $|a_i - z_i| \leq C_3 d(z_i)$, whence

$$|a_i - y_i| \leq |a_i - z_i| + |z_i - y_i| \leq C_4 d(y_i)$$

where $C_4 = e^{C_2}(C_3 + 1)$. By Lemma 3.50 we have

$$d(z) \leq C_5 |a_1 - a_2|. \tag{3.56}$$

Consequently, if $d(y_1) \vee d(y_2) \leq d(z)/4C_4C_5$, then $|a_i - y_i| \leq |a_1 - a_2|/4$, whence $|a_1 - a_2| \leq 2|y_1 - y_2|$, and (2) follows from (3.55) and (3.56). Finally, if $d(y_1) \vee d(y_2) \geq d(z)/4C_4C_5$, then $d(z) \leq 8C_4C_5|y_1 - y_2|$, and (2) follows from (3.55). □

3.57. *Proof of Theorem* 3.27. Let $G \in Q(\delta, \eta, \varepsilon)$. By 3.30 it suffices to show that G is $C(\delta, \eta, \varepsilon)$-uniform. We let $C_1, C_2, \ldots$ denote positive constants depending only on $(\delta, \eta, \varepsilon)$. Let $x_1, x_2 \in G$. If $|x_1 - x_2| \leq d(x_1) \vee d(x_2)$, then the line segment $[x_1,x_2]$ is a 1-uniform arc. Assume that $|x_1-x_2|\ge d(x_1)\vee d(x_2)$. Let $h=1/10$ and let $\gamma\colon x_1 \curvearrowright x_2$ be an h-short arc. We show that γ is C-uniform.

By 3.22 and by [Vä8, 6.35] there is a (μ,h)-biroad $\bar\alpha\colon a_1\curvearrowright a_2$ with $\mu=12\delta+1$ such that $k(x_i,\bar\alpha)\le C_1(\delta)$, $i=1,2$. By the ribbon lemma [Vä8, 2.17] we find a member α_m of $\bar\alpha$ and a quasihyperbolic length map $f\colon\gamma\to\alpha_m$ such that $k(fx,x)\le C_2(\delta)$ for all $x\in\gamma$. We may assume that $\pi_m fx_1\le\pi_m fx_2$ on $\operatorname{str}\bar\alpha$.

We first verify the cigar condition. Let $x\in\gamma$. If $a_2=\infty$, we have $l(\alpha_m[fx_1,fx])\le C_3 d(fx)$ by the length carrot lemma 3.40. By 3.3 this yields $l(\gamma[x_1,x])\le C_3e^{2C_2}d(x)$, which implies the cigar condition. The case $a_2=\infty$ is similar.

Assume that $a_1\ne\infty\ne a_2$. Let $\xi_\alpha\in\operatorname{str}\bar\alpha$ be the element given by 3.54(1) and let $z\in\alpha_m$ be the point with $\pi_m z=\xi_\alpha$. We have either $\pi_m fx_1\le\pi_m fx\le\xi_\alpha$ or $\xi_\alpha\le\pi_m fx\le\pi_m fx_2$. In the first case, Lemma 3.54(1) gives $l(\alpha_m[fx_1,fx])\le C_4 d(fx)$, and in the second case, we similarly get $l(\alpha_m[fx,fx_2])\le C_4d(fx)$. As above, these estimates yield

$$l(\gamma[x_1,x])\wedge l(\gamma[x,x_2])\le C_4e^{2C_2}d(x),$$

which is the cigar condition.

To prove the turning condition, we first assume that $d(fx_1)\vee d(fx_2)\le 2|fx_1-fx_2|$. If $a_1\ne\infty\ne a_2$, then 3.54(2) gives $l(\alpha_m[fx_1,fx_2])\le C_5|fx_1-fx_2|$. But this is also true if $a_1=\infty$ or $a_2=\infty$, because then 3.40 implies that $l(\alpha_m[fx_1,fx_2])\le C_3\big(d(fx_1)\vee d(fx_2)\big)\le 2C_3|fx_1-fx_2|$. Since $|fx_i-x_i|\le e^{C_2}d(x_i)\le e^{C_2}|x_1-x_2|$, we obtain the turning condition

$$l(\gamma)\le e^{C_2}C_5|fx_1-fx_2|\le e^{C_2}C_5(1+2e^{C_2})|x_1-x_2|.$$

Finally, assume that $d(fx_1)\vee d(fx_2)\ge 2|fx_1-fx_2|$. Now $k(fx_1,fx_2)\le 1$, whence $k(x_1,x_2)\le 1+2C_2$. As γ is h-short, this gives $l_k(\gamma)\le 2+2C_2=C_6$. By 3.2 we obtain $l(\gamma)\le e^{C_6}d(x_1)\le e^{C_6}|x_1-x_2|$. □

4. A counterexample

In [BHK, 7.12] it was proved that a domain in S^n is uniform if and only if it is hyperbolic and linearly locally connected (LLC). Every C-uniform domain $G\subset E$ is $\delta(C)$-hyperbolic by 2.12 and $3C$-LLC by Theorem 4.2 below. However, we give in 4.3 an example of a domain $G\subset l_2$ that is hyperbolic and LLC but not uniform.

4.1. Definition. Let $c\ge1$. We recall that a domain G is *c-linearly locally connected* or briefly c-LLC if the following conditions hold for all $x\in G$ and $r>0$:

(LLC_1) Each pair of points in $G \cap B(x,r)$ can be joined by an arc in $G \cap B(x,cr)$.

(LLC_2) Each pair of points in $G \setminus \overline{B}(x,r)$ can be joined by an arc in $G \setminus \overline{B}(x,r/c)$.

It is well known that uniform domains are LLC; we recall the easy argument in 4.2. The planar domain $G = \mathsf{R}^2 \setminus \{(n,0) : n \in \mathsf{Z}\}$ is 1-LLC but not uniform.

4.2. Theorem. *Every C-John domain is c-LLC_2 and every C-uniform domain is c-LLC with $c = 3C$.*

Proof. Let $x \in G$ and let $a, b \in G \setminus \overline{B}(x,r)$. We may assume that $x = 0$. Suppose that G is a C-John domain. Choose a number s with $r < s < |a| \wedge |b|$. Let α: $a \curvearrowright b$ be an arc satisfying the C-cigar condition 2.4(1). If $\alpha \cap S(s/3C) = \emptyset$, there is nothing to prove. If there is a point $z \in \alpha \cap S(s/3C)$, we have

$$d(z) \geq (|z-a| \wedge |z-b|)/C > (s - s/3C)/C \geq 2s/3C$$

by the cigar condition. Thus $d(0) \geq d(z) - |z| > s/3C$, whence $S(s/3C) \subset G$. We can therefore join a and b by an arc $\gamma \subset G \setminus B(s/3C)$ consisting of two subarcs of α and an arc $\beta \subset S(s/3C)$.

Suppose that G is C-uniform. Let $x \in G$ and let $a, b \in G \cap B(x,r)$. Choose a C-uniform arc α: $a \curvearrowright b$ in G. For each $y \in \alpha$ we have

$$|y - x| \leq |y - a| + |a - x| < l(\alpha) + r < 3Cr$$

by the turning condition. Hence $\alpha \subset B(x, 3Cr)$. □

4.3. Example. Let E be an infinite-dimensional separable Hilbert space. Let $W \subset E$ be a *broken tube*, considered first in [Vä2, 2.12]; a detailed treatment is given in [Vä7]. We recall the construction.

Choose an orthonormal basis $(e_j)_{j \in \mathsf{Z}}$ of E, indexed by all integers, and set $u_j = \sqrt{2}\, e_j$. Let α be the line spanned by e_0, and set $a_j = 2je_0$ for $j \in \mathsf{Z}$. Let f: $\alpha \to E$ be the map with $fa_j = u_j$, $j \in \mathsf{Z}$, such that f is affine on each line segment $\alpha_j = [a_{j-1}, a_j]$. Then $f \,|\, \alpha_j$ is an isometry onto $\beta_j = [u_{j-1}, u_j]$. Let U be the tubular neighborhood $\{x \in E : d(x,\alpha) < 1/5\}$ of α. In [Vä7] we give a detailed construction of a locally bilipschitz extension F: $U \to W$ of f onto a domain W satisfying the conditions

(1) W is hyperbolic,
(2) W is LLC,
(3) W is not a John domain and hence not uniform.

5. A lower bound to the hyperbolicity constant

5.1. *A lower bound for* δ. There are 0-hyperbolic metric spaces (metric trees). However, we show in 5.3 that the constant δ cannot be arbitrarily close to 0 for a δ-hyperbolic domain $G \subset E$. Remember that $\dim E \geq 2$.

5.2. Lemma. *If E is a normed plane, then there are unit vectors x, y, z such that*

$$|x-y| = |y-z|, \qquad |x-z| \geq |x-y| + 1/2.$$

Proof. Choose a unit vector x and then a unit vector y such that $|x-y| = |y+x| = s$.

Case 1. $s \leq 3/2$. Now x, y and $z = -x$ are the desired points, because $|x-y| = |y-z| = s$ and $|x-z| = 2 \geq s + 1/2$.

Case 2. $s \geq 3/2$. Choose unit vectors u and v such that the points x, u, y, v, $-x$ are in this order in the unit circle and such that $|x-u| = |u-y| = t_1$, $|y-v| = |v+x| = t_2$. We may assume that $t_1 \leq t_2$. Since the length of a unit semicircle is at most 4 by Theorem 3.4, we have $2t_1 + 2t_2 \leq 4$, whence $t_1 \leq 1$. Since

$$|x-y| = s \geq 3/2 \geq t_1 + 1/2,$$

the desired points are x, u, y. □

5.3. Theorem. *If G is a δ-hyperbolic domain, then $\delta \geq 0.0027$.*

Proof. Fix a point $p \in G$. We normalize the situation by $d(p) = 1$. Let $0 < t \leq 1/2$. By 5.2 there are points $x, y, z \in S(p,t)$ such that

$$|x-y| = |y-z|, \quad |x-z| \geq |x-y| + t/2.$$

For all $a, b \in \overline{B}(p,t)$ we have

$$\frac{|a-b|}{1+2t} \leq k(a,b) \leq \frac{|a-b|}{1-t}$$

by [Vä6, 3.7]. Hence

$$2(x\,|\,y)_p = k(x,p) + k(y,p) - k(x,y) \geq \frac{2t}{1+2t} - \frac{|x-y|}{1-t},$$

and a similar lower bound holds for $2(y\,|\,z)_p$. Moreover,

$$2(x\,|\,z)_p = k(x,p) + k(z,p) - k(x,z) \leq \frac{2t}{1-t} - \frac{|x-z|}{1+2t}.$$

As $|x-z| \leq 2t$, we obtain by combining the estimates that

$$\delta \geq (x\,|\,y)_p \wedge (y\,|\,z)_p - (x\,|\,z)_p \geq \frac{t(1-22t)}{4(1-t)(1+2t)}.$$

For $t = 0.02$ this gives $\delta \geq 0.0027$. □

5.4. Remark. For Hilbert spaces one can get the bound $\delta \geq 0.005$. These bounds are presumably far from the best possible. In the other direction, we know that the half plane is δ-hyperbolic with $\delta = \log 3 = 1.098\dots$ [CDP, p. 12]. I conjecture that 5.3 holds with this δ.

Appendix. Quasiconvexity in normed spaces

In this appendix I give some results on quasiconvexity needed in Section 3. The main goal is Lemma A.6, which gives a rigorous and detailed proof for the quasiconvexity of the crosscut γ in Lemma 3.18. Moreover, Lemmas A.3 and A.4 were used in 3.9.

Throughout the appendix, we let E denote a normed space. We start with a useful elementary inequality.

A.1. Lemma. *Let $x, y, z, u \in E$ with $|x-y| \leq |x-z|$ and $u \in [x, y]$. Then*

$$|y-z| \leq 2|u-z|.$$

Proof. If $|u-y| \leq |y-z|/2$, then $|u-z| \geq |y-z| - |u-y| \geq |y-z|/2$. If $|u-y| \geq |y-z|/2$, then

$$|u-z| \geq |x-z| - |x-u| \geq |x-y| - |x-u| = |u-y| \geq |y-z|/2. \ \square$$

A.2. *Deviations.* The *deviation* between nonzero vectors $x, y \in E$ is defined by

$$\operatorname{dev}(x, y) = \big| x/|x| - y/|y| \big| \in [0, 2].$$

If α and β are rays with common vertex $v \in E$, then $\operatorname{dev}(x-v, y-v)$ is independent of the points $x \in \alpha$ and $y \in \beta$, and we set

$$\operatorname{dev}(\alpha, \beta) = \operatorname{dev}(x-v, y-v).$$

In an inner product space we have $\operatorname{dev}(\alpha, \beta) = 2\sin(\varphi/2)$, where φ is the angle between the rays α and β.

A.3. Lemma. *Let α and β be rays from the origin and let $x \in \alpha$, $y \in \beta$.*

(1) *If $|x| = |y| = r$, then $|x-y| = r \operatorname{dev}(\alpha, \beta)$.*

(2) $|x-y| \geq (|x| \vee |y|) \operatorname{dev}(\alpha, \beta)/2$.

(3) *Let $x, y, z \in E$ with $0 < |y-z| \leq |x-y| \leq |x-z|$. Then* $\operatorname{dev}(x-y, z-y) \geq 1$.

Proof. Part (1) is trivial, and (2) follows from (1) and A.1. To prove (3), set $r = |x-y|$ and let u be the point on the ray from y through z with $|u-y| = r$. Then $|u-x| \geq r$ by the convexity of the norm. Hence $\operatorname{dev}(x-y, z-y) = |u-x|/r \geq 1$. $\square$

A.4. Lemma. *Let $\alpha \neq \beta$ be rays with a common vertex. Then $\alpha \cup \beta$ is c-quasiconvex with $c = 4/\operatorname{dev}(\alpha,\beta) - 1$. The bound is sharp.*

Proof. We may assume that the common vertex is the origin. Set $r = \operatorname{dev}(\alpha,\beta)$ and let $x \in \alpha$, $y \in \beta$. It suffices to show that $|x|+|y| \leq (4/r-1)|x-y|$.

We may assume that $0 < |y| \leq |x| = 1$. Let $b \in \beta$ be a unit vector and set $t = |y-b| = 1-|y|$. We must show that

$$\frac{2-t}{|x-y|} \leq \frac{4}{r} - 1.$$

If $t \leq r/2$, then $|x-y| \geq |x-b| - |y-b| = r-t$, whence

$$\frac{2-t}{|x-y|} \leq \frac{2-t}{r-t} \leq \frac{2-r/2}{r-r/2} = \frac{4}{r} - 1.$$

If $t \geq r/2$, then $|x-y| \geq r/2$ by A.3, and we obtain

$$\frac{2-t}{|x-y|} \leq \frac{2-r/2}{r/2} = \frac{4}{r} - 1.$$

To prove the sharpness, let E be the plane with the norm $\|x\| = |x_1| \vee |x_2|$ and let α and β be the rays from the origin through the points $(1,0)$ and $(1,1)$, respectively. Now $\operatorname{dev}(\alpha,\beta) = 1$. For $x = (2,0)$ and $y = (1,1)$ we have $\|x\| + \|y\| = 3 = 3\|x-y\|$. Hence $\alpha \cup \beta$ is not c-quasiconvex for any $c < 3 = 4/\operatorname{dev}(\alpha,\beta) - 1$. □

A.5. Lemma. *Suppose that $\alpha\colon x \curvearrowright y$ is a c-quasiconvex arc and that $a \in E$ with $|a-y| = d(a,\alpha)$. Then $\beta = \alpha \cup [y,a]\colon x \curvearrowright a$ is a $(2c+1)$-quasiconvex arc.*

Proof. Let $u \in \alpha$, $v \in [y,a]$. Then $|v-y| = d(v,\alpha) \leq |u-v|$, and we get

$$\begin{aligned} l(\beta[u,v]) &= l(\alpha[u,y]) + |y-v| \leq c|u-y| + |y-v| \\ &\leq c|u-v| + c|v-y| + |y-v| \leq (2c+1)|u-v|. \ \square \end{aligned}$$

A.6. Lemma. *The arc $\tau\colon b \curvearrowright b'$ constructed in the proof of Lemma* 3.18 *is c-quasiconvex in norm with $c = c(\lambda)$.*

Proof. We recall from 3.18 the basic inequalities

$$(1 - s/2)t \leq d \leq t \leq (1+s/2)d, \quad r \leq d' \leq (1+s/2)r, \tag{A.7}$$

where $d = d(x_0)$, $d' = d(x_0')$, $t = |x_0|$, $s = \lambda/100$, $r = \lambda t/3$, $\lambda \leq 1/2$.

As in 3.9, we let x_1 denote the unique point in $[x_0,0] \cap S(x_0,d)$, and similarly $x_1' \in [x_0', a'] \cap S(x_0', d')$. Then $[x_0,x_1] \subset \gamma$ and $\gamma_0 = \gamma \setminus [x_0,x_1] \subset B(x_1,sd) \subset B(2sd)$ by 3.9(4). Similarly, $[x_0',x_1'] \subset \gamma'$ and $\gamma_0' = \gamma' \setminus [x_0',x_1'] \subset B(x_1', sd')$.

Let $u, v \in \tau$ be points with $u \in \tau[b, v]$. We must get an estimate

$$l(\tau[u,v]) \le c|u-v| \tag{A.8}$$

with $c = c(\lambda)$. The arcs γ and γ' are 4-quasiconvex by 3.9(6), and α is 2-quasiconvex by 3.4. Applying Lemma A.5 we see that the arcs $[x_1, x_0] \cup \alpha$ and β_0 are 5-quasiconvex and that $\beta_0 \cup [x_0', x_1']$ is 11-quasiconvex. Consequently, if $\{u, v\}$ is contained in one of the arcs

$$\gamma,\ \gamma',\ [x_1, x_0] \cup \alpha,\ \beta_0 \cup [x_0', x_1'],$$

then (A.8) holds with $c = 11$.

There remain 4 cases and some subcases for u and v. We proved in 3.18 that τ satisfies condition 3.17(3), whence $l(\tau) \le c_1 d \le c_1 t$ with $c_1 = c_1(\lambda)$. Hence it suffices to get a lower bound

$$|u-v| \ge qt \tag{A.9}$$

with $q = q(\lambda) > 0$.

Case 1. $u \in \gamma_0$, $v \notin \gamma$. Now $|u| \le 2sd \le 2st$. Let $w \in [x_0', x_1']$. We show that

$$|w| \ge \lambda t. \tag{A.10}$$

We have $|w| \ge |x_0'| - d'$, where $d' < (1 + s/2)r < \lambda t$ and $d' \le |x_0 - y'|$. If $x_0' \in \alpha$, then $|w| \ge t - \lambda t \ge \lambda t$. If $x_0' \in [y, z]$, then $|w| \ge |x_0'| - |x_0' - y| = |y| \ge \lambda t$, and (A.10) is proved. It follows that $|v| \ge \lambda t - sd' \ge (\lambda - s)t$, whence $|u - v| \ge (\lambda - 3s)t > \lambda t/2$, and (A.9) holds with $q = \lambda/2$.

Case 2. $u \in [x_1, x_0]$, $v \in [z, x_0']$. This case does not occur if $x_0' \in \alpha$. We consider two subcases.

If $|y| \ge t$ we apply A.3 and obtain $|u - v| \ge |v| \operatorname{dev}(u, v)/2 \ge |x_0 - z|/2$. Since α is 2-quasiconvex, this implies that

$$l(\tau[u,v]) \le |u - x_0| + 2|x_0 - z| + |z - v| = 2|x_0 - z| + |v| - |u| \le 5|u - v|.$$

Next assume that $|y| < t$, and let $y_1 \in [0, x_0]$ be the point with $|y_1| = |y| \ge \lambda t$. Since $|y - x_0| \ge d \ge t(1 - s/2)$, we have

$$|y| + t(1 - s/2) \le |y| + |y - x_0| \le |y_1| + |y - y_1| + |y_1 - x_0| = t + |y - y_1|,$$

whence $|y - y_1| \ge |y| - st/2$. Consequently, $\operatorname{dev}(x_0, z) = |y - y_1|/|y| \ge 1 - st/2|y|$. Here $st/2|y| \le s/2\lambda = 1/200$. By A.3 we get

$$|u - v| \ge |v| \operatorname{dev}(u, v)/2 \ge |v|/4 \ge |y|/4 \ge \lambda t/4.$$

Case 3. $u \in [x_1, x_0]$, $v \in [x_0', x_1']$. We again consider two subcases.

First assume that $|x_0'| \geq t$. By (A.7) we get

$$|x_0 - x_0'| \geq d - d' \geq (1 - s/2)t - (1 + s/2)r.$$

By A.1 we have $|u - x_0'| \geq |x_0 - x_0'|/2$. Since $|v - x_0'| \leq d' \leq (1 + s/2)r$ and since $r = \lambda t/3 \leq t/6$, these estimates yield (A.9) with $q = 1/4 - 3s/8 > 1/5$.

Next assume that $|x_0'| < t$. Now $|y| < t$ and $x_0' \in [y, z]$. As in Case 2 we get $\operatorname{dev}(x_0, z) \geq 1 - 1/200$. Since

$$|x_0'| \geq |y| + r \geq \lambda t + \lambda t/3 = 4\lambda t/3,$$

Lemma A.3 gives $|x_0' - u| \geq 2\lambda t(1 - 1/200)/3$. As $|x_0' - v| \leq d' \leq (1 + s/2)r \leq (1 + 1/400)\lambda t/3$, this yields (A.9) with $q = \lambda/4$.

Case 4. $u \in \tau[x_1, x_0'] = [x_1, x_0] \cup \beta_0$, $v \in \gamma_0'$. If $u \in [x_1, x_0]$, then $|u - x_1'| \geq \lambda t/4$ by Case 3. If $u \in \beta_0$, then $|u - x_1'| \geq d(x_1', \beta_0) = |x_1' - x_0'| = d' \geq r = \lambda t/3$. Since

$$|v - x_1'| \leq sd' \leq s(1 + s/2)t/6 < \lambda t/500,$$

we obtain (A.9) with $q = \lambda t/5$. □

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