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GLOBAL PROPERTIES OF THE PAINLEVÉ TRANSCENDENTS: NEW RESULTS AND OPEN QUESTIONS

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Abstract. We prove several lower estimates for the Nevanlinna characteristic functions and the orders of growth of the Painlevé transcendents I, II and IV. In particular it is shown that (a) $\limsup_{r\to\infty} T(r,w_1)/r^{5/2} > 0$ for any first transcendent, (b) $\varrho(w_2) \geq \frac{3}{2}$ for most classes of second transcendents, (c) $\varrho(w_4) \geq 2$ for several classes of fourth transcendents, and that (d) the poles with residues ± 1 are asymptotically equi-distributed.

1. Introduction

The solutions of Painlevé's differential equations

$$(1)\qquad \begin{cases} \text{(I)} & w'' = z + 6w^2, \\ \text{(II)} & w'' = \alpha + zw + 2w^3, \\ \text{(IV)} & 2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2-\alpha)w^2 + 2\beta \end{cases}$$

are meromorphic functions in the plane. For recent proofs see Hinkkanen and Laine [5] in cases (I) and (II), and the author [16] in all cases. In papers of Shimomura [13], [14] and the author [17] precise order estimates are proved with different methods: $\varrho(w) \leq \frac{5}{2}$, $\varrho(w) \leq 3$ and $\varrho(w) \leq 4$ in the respective cases. These results are also presented, at least in parts, in the recent monograph [3] by Gromak, Laine and Shimomura. The lower estimate $\varrho \geq \frac{5}{2}$ in case (I) is due to Mues and Redheffer [6].

Shimomura [15] extended his research on the Painlevé transcendents to prove lower estimates for the Nevanlinna characteristics of the first Painlevé transcendents, and, for particular parameters, also for the second transcendents. The aim of this paper is to make a comprehensive study of global properties of the Painlevé transcendents I, II and IV. In the first case we are able to show that every solution is of regular growth, while for equation (II) the well-known conjecture $\varrho \geq \frac{3}{2}$ is confirmed—except in one case. We also give independent proofs of known results, which might be of interest by themselves.

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The estimates of Nevanlinna functions are based on the *re-scaling method* developed in [17], which certainly gives not as precise results as are obtained in the theory of asymptotic integration, but avoids the well-known connection problem occurring there, and is therefore more suitable to study global aspects. Some of the problems, however, seem to be out of the range of these methods. Nevertheless, we also state and prove several results, which may be looked at being incomplete and preliminary, but point into the right direction.

2. Notation and auxiliary results

(a) Painlevé's equations I, II and IV. Each equation (1) has a *first integral*

$$\begin{cases} w'^2 = 4w^3 + 2zw - 2U, & U' = w, \\ w'^2 = w^4 + zw^2 + 2\alpha w - U, & U' = w^2, \\ w'^2 = w^4 + 4zw^3 + 4(z^2-\alpha)w^2 - 2\beta - 4wU, & U' = w^2 + 2zw. \end{cases} \tag{2}$$

Any transcendental solution has infinitely many poles p with Laurent series expansions

$$\begin{cases} w(z) = (z-p)^{-2} - \frac{1}{10}p(z-p)^2 - \frac{1}{6}(z-p)^3 + \mathsf{h}(z-p)^4 + \cdots, \\ w(z) = \varepsilon(z-p)^{-1} - \frac{1}{6}\varepsilon p(z-p) - \frac{1}{4}(\alpha+\varepsilon)(z-p)^2 + \mathsf{h}(z-p)^3 + \cdots, \\ w(z) = \varepsilon(z-p)^{-1} - p + \frac{1}{3}\varepsilon(p^2 + 2\alpha - 4\varepsilon)(z-p) + \mathsf{h}(z-p)^2 + \cdots \end{cases} \tag{3}$$

and

$$\begin{cases} U(z) = -(z-p)^{-1} - 14\mathsf{h} - \frac{1}{30}p(z-p)^3 - \frac{1}{24}(z-p)^4 + \cdots, \\ U(z) = -(z-p)^{-1} + 10\varepsilon\mathsf{h} - \frac{7}{36}p^2 - \frac{1}{3}p(z-p) - \frac{1}{4}(1+\varepsilon\alpha)(z-p)^2 + \cdots, \\ U(z) = -(z-p)^{-1} + 2\mathsf{h} + 2(\alpha-\varepsilon)p + \frac{1}{3}(4\alpha - p^2 - 2\varepsilon)(z-p) + \cdots \end{cases} \tag{4}$$

with $\varepsilon = \pm 1$; the coefficient h remains undetermined, and *free*: the pole p, the sign ε and h may be prescribed to define a unique solution in the same way as do initial values $w(z_0)$ and $w'(z_0)$.

(b) Nevanlinna theory. Let f be meromorphic and non-constant in the complex plane. Then $m(r,f)$, $N(r,f)$ and $T(r,f)$ denote the *Nevanlinna proximity function, counting function* of poles and *characteristic function* of f, respectively, while $n(r,f)$ denotes the number of poles of f in $|z| \le r$, see Hayman [4] or Nevanlinna [7]. In addition we will work with the L^1-norm of f on $|z| \le r$,

$$I(r,f) = \frac{1}{2\pi}\int_{|z|\le r} |f(z)|\,d(x,y),$$

where $d(x,y)$ denotes area element; the L^1-norm is defined for meromorphic functions f with simple poles. We also make use of the *Ahlfors–Shimizu characteristic*

$$T_0(r,f)=\int_0^r A(t,f)\frac{dt}{t} \quad \text{with} \quad A(t,f)=\frac{1}{\pi}\int_{|z|\le t}\big(f^{\#}(z)\big)^2\,d(x,y),$$

$f^{\#}(z)=|f'(z)|/\big(1+|f(z)|^2\big)$ being the *spherical derivative* of f; $T_0(r,f)$ differs from $T(r,f)$ by a bounded term.

The following facts are well known, and are only referred to for the convenience of the reader. Let f be any *canonical product* with simple zeros c_ν, and denote by $n(t)$ the number of zeros contained in $|z|\le t$. The *genus* of f is defined to be the least integer h, such that

$$\sum_{\nu=1}^{\infty}|c_\nu|^{-h-1}=\int_0^\infty t^{-h-1}\,dn(t)=(h+1)\int_0^\infty n(t)t^{-h-2}\,dt<+\infty.$$

The Nevanlinna characteristic of f then satisfies $n(r)\le T(er,f)$ and

$$T(r,f)\le K_h r^{h+1}\int_0^\infty \frac{n(t)}{t^{h+1}(t+r)}\,dt, \tag{5}$$

and hence the *order of growth*

$$\varrho=\varrho(f)=\varlimsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

coincides with the *exponent of convergence* $\inf\big\{\sigma>0:\sum_{\nu=1}^\infty |c_\nu|^{-\sigma}<\infty\big\}$, and satisfies $h\le\varrho\le h+1$.

The concept of genus may be extended to arbitrary meromorphic functions $f=e^Q f_1/f_2$ of finite order, where f_1 and f_2 are canonical products of genus h_1 and h_2, respectively, and Q is any polynomial. The genus of f then is defined by $\max\{h_1,h_2,\deg Q\}$.

(c) Counting poles of logarithmic derivatives. The *logarithmic derivative* $L=f'/f$, f entire of finite order with simple zeros, has Nevanlinna functions $m(r,L)=O(\log r)$ and $N(r,L)=N(r,1/f)$, and hence satisfies

$$T(r,L)\le T(r,f)+O(\log r).$$

Conversely, if Φ is meromorphic in the plane of finite order, with simple non-zero poles with residues 1 and satisfying $m(r,\Phi)=O(\log r)$, then there exists some polynomial Q such that $\Phi=Q+L$, where $L=f'/f$ and f is the canonical product with simple zeros exactly at the poles of Φ. If the order ϱ of Φ is not an integer, then $n(r,\Phi)=O(r^\varrho)$ and $n(r,\Phi)=o(r^\varrho)$ imply $T(r,f)=O(r^\varrho)$ and $T(r,f)=o(r^\varrho)$, respectively. This is no longer true for $\varrho\in\mathbf{N}$. If, however, $\int_0^\infty n(t,\Phi)t^{-\varrho-1}\,dt$ converges, then $T(r,f)=o(r^\varrho)$ holds.

Proposition 2.1. *Suppose* Φ *is meromorphic in the plane, having simple poles with residues* 1 *only. Then*

$$\int_0^R n(r,\Phi)\,dr \le I(R,\Phi). \tag{6}$$

Proof. By the Residue Theorem we have, for all but countably many radii $r>0$,

$$n(r,\Phi) = \left|\frac{1}{2\pi i}\int_{|z|=r}\Phi(z)\,dz\right| \le \frac{1}{2\pi}\int_0^{2\pi}|\Phi(re^{i\theta})|r\,d\theta, \tag{7}$$

and integrating with respect to r gives the assertion. □

For functions Ψ having only simple poles with both residues ± 1 we obtain in the same way

$$\int_0^R |n_+(r,\Psi) - n_-(r,\Psi)|\,dr \le I(R,\Psi), \tag{8}$$

where $n_\pm(r,\Psi)$ counts those poles of Ψ with residue ± 1.

An estimate in the other direction is given by

Proposition 2.2. *Let* f *be any canonical product* (*or a quotient of canonical products*) *with simple zeros* (*and poles*) *and counting function of zeros* (*and poles*) $n(r)$. *Then for* $L = f'/f$ *we have*

$$I(R,L) \le 8R\big[T(2R,f) + n(2R)\big]. \tag{9}$$

Proof. Let (c_ν) be the sequence of zeros (and poles) of f. We recall the inequality

$$|L(z)| \le 8\,T(2|z|,f)|z|^{-1} + \sum_{|c_\nu|\le 2|z|} 2|z-c_\nu|^{-1},$$

which is a simple consequence of the Poisson–Jensen formula, see Hayman [4]; f may be any meromorphic function with $f(0)=1$ and simple zeros and poles c_ν. Since

$$\int_{|z|\le R}|z-c|^{-1}\,d(x,y) \le \int_{|z|\le R+|c|}|z|^{-1}\,d(x,y) = 2\pi(R+|c|) \le 6\pi R$$

for $|c|\le 2R$, integration over the disk $|z|\le R$ yields, by monotonicity of $T(r,f)$,

$$\frac{1}{2\pi}\int_{|z|\le R}|L(z)|\,d(x,y) \le 8R\big[T(2R,f)+n(2R)\big],$$

and hence the assertion. □

We have also to deal with functions $L' = (f'/f)'$, f a canonical product with zeros p_ν. Differentiating the Poisson–Jensen formula twice gives the inequality

$$|L'(z)| \le 16\, T(2|z|, f)\, |z|^{-2} + 2 \sum_{|p_\nu| \le 2|z|} |z - p_\nu|^{-2}. \tag{10}$$

Since $|z-p_\nu|^{-2}$ and $|L'(z)|$ are not integrable, we proceed as follows: for $\delta > 0$ sufficiently small and some $\kappa > 0$ we consider the disks $\Delta_\nu : |z - p_\nu| < \delta |p_\nu|^{-\kappa}$ about the non-zero poles of L, multiply the above inequality by $r = |z|$ and integrate over

$$H(R) = \{z : 1 \le |z| \le R\} \setminus \bigcup_{|p_\nu| < 2R} \Delta_\nu. \tag{11}$$

Since

$$\int_{H(R)} \frac{|z|}{|z-p_\nu|^2}\, d(x,y) \le 6\pi R \int_{\delta |p_\nu|^{-\kappa}}^{R} \frac{dr}{r} = 6\pi R \log(R|p_\nu|^{\kappa}/\delta) = O(R \log R),$$

we obtain, keeping $\delta > 0$ and $\kappa > 0$ fixed and denoting

$$I_H(R, \Phi) = \frac{1}{2\pi} \int_{H(R)} |z|\, |\Phi(z)|\, d(x,y) : \tag{12}$$

Proposition 2.3. *Let f be a canonical product (or a quotient of canonical products) with zeros (and poles) p_ν and counting function $n(r)$. Then for $L = f'/f$ and $H(R)$ given by (11),*

$$I_H(R, L') = O\big(R\, [T(2R, f) + n(2R) \log R]\big)$$

holds.

3. Re-scaling Painlevé's equations

Some of the mystery of the Painlevé transcendents is hidden in the unknown coefficient h in the series expansion (3). The re-scaling method was developed in [17] only for one purpose, to estimate h in terms of p, and hence to obtain the growth estimates mentioned in the introduction. The method reminds of Painlevé's α-method [9], [10], and also the Zalcman method [21] and its refinement by Pang [11], [12], and is based on Poincaré's Theorem on analytic dependence on parameters and initial values, see, e.g., Bieberbach [1, p. 14]. We will describe the method and its results in case (II) in some detail, for full details in case (I) the reader is referred to [17].

(a) Re-scaling equation (II). Let w be any transcendental solution of (II) and set

$$r(z_0) = \min\bigl\{|z_0|^{-1/2}, |w(z_0)|^{-1}, |w'(z_0)|^{-1/2}\bigr\}$$

to re-scale $w(z) + \sqrt{z}$; $\sqrt{z}$ denotes any branch of the complex square-root. Let (z_n) be any sequence (z_n not a pole of w) tending to infinity, set $r_n = r(z_n)$ and

$$y_n(z) = r_n\bigl[w(z_n + r_n z) - \sqrt{z_n + r_n z}\,\bigr].$$

Then the differential equation for y_n has a formal limit as $n \to \infty$, and assuming that the limits $a^2 = \lim_{n\to\infty} r_n^2 z_n$, $y_0 = \lim_{n\to\infty} y_n(0) = \lim_{n\to\infty} r_n w(z_n) - a$ and $y_0' = \lim_{n\to\infty} y_n'(0) = \lim_{n\to\infty} r_n^2 w'(z_n)$ exist, we obtain the initial value problem

$$y'' = (y+a)(a^2 + 2(y+a)^2), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{13}$$

The solution is either a constant, a rational function, a simply periodic function or else an elliptic function. Constant solutions which may come from the re-scaling process are $y = -a$ and $y = -a \pm ia/\sqrt{2}$, this being only possible for $a \neq 0$. As a consequence of Poincaré's Theorem, *the main conclusion is*

$$y(z) = \lim_{n\to\infty} y_n(z) = \lim_{n\to\infty} r_n\bigl[w(z_n + r_n z) - \sqrt{z_n + r_n z}\,\bigr], \tag{14}$$

locally uniformly in $\mathbf{C}$, so that we can easily deduce properties of $w(z_n + r_n z)$ from properties of $y(z)$.

We denote by (p_ν) and (q_ν) the sequences of non-zero poles and zeros of $w^2(z) - z$, respectively, and set, for $\delta > 0$ fixed,

$$\Delta_\delta(c) = \{z : |z - c| < \delta r(c)\}.$$

Lemma 3.1. *Let (z_n) be any sequence such that $|z_n - q_n'| = o\bigl(r(z_n)\bigr)$ and $|z_n - q_n'| = o\bigl(r(q_n')\bigr)$ as $n \to \infty$, respectively, where (q_n') is some infinite subsequence of the sequence (q_n) of zeros of $w^2 - z$. Then $r(q_n') = O\bigl(r(z_n)\bigr)$ and $r(z_n) = O\bigl(r(q_n')\bigr)$, respectively.*

Proof. We will give the proof in the first case leaving the details in the second case to the reader. We set $r_n = r(z_n)$ and $y_n(z) = r_n\bigl[w(z_n + r_n z) - \sqrt{z_n + r_n z}\,\bigr]$, and assume that $r_n^2 z_n \to a^2$ and $y_n(z) \to y(z)$, locally uniformly in $\mathbf{C}$. On the other hand we consider $u_n(z) = r_n\bigl[w(q_n' + r_n z) - \sqrt{q_n' + r_n z}\,\bigr]$. Noting that $\varepsilon_n = (q_n' - z_n)/r_n \to 0$ we obtain by uniform convergence $u_n(z) = y_n(z + \varepsilon_n) \to y(z)$, $u_n'(z) \to y'(z)$ and also $r_n^2 q_n' \to a^2$. From this, $r(q_n')/r_n \to 1$ and hence $r(q_n') = O(r_n)$ follows. □

Proposition 3.2. *As $z \to \infty$ outside $\mathsf{Q}(\delta) = \bigcup_\nu \Delta_\delta(q_\nu)$ the following hold, for any choice of $\sqrt{z}$:*
(a) $|z|^{1/2} = O\big(\big|w(z) - \sqrt{z}\,\big|\big) = O(|w^2(z) - z|^{1/2})$,
(b) $|w'(z)| = O\big(\big|w(z) - \sqrt{z}\,\big|^2\big) = O(|w^2(z) - z|)$,
(c) $F^{\#}(z) = O(|z|^{-1/2})$ *for* $F(z) = w^2(z) - z$.

Proof. Suppose that (z_n) is any sequence tending to infinity such that $\big|w(z_n) - \sqrt{z_n}\,\big| = o(|z_n|^{1/2})$ or else $\big|w(z_n) - \sqrt{z_n}\,\big| = o\big(|w'(z_n)|^{1/2}\big)$ holds. Assuming, as above, that the limits $a^2 = \lim_{n\to\infty} r_n^2 z_n$, $y_0 = \lim_{n\to\infty} r_n w(z_n) - a$ and $y_0' = \lim_{n\to\infty} r_n^2 w'(z_n)$ exist, we obtain (13) by re-scaling $w(z) - \sqrt{z}$ (any branch of the square-root), with $y_0 = 0$. Hence, y is non-constant, and from (14) and Hurwitz' Theorem it follows that $w^2(z_n + r_n z) - (z_n + r_n z)$ has a zero z_n' with (z_n') tending to zero. Hence $z_n + r_n z_n' = q_n'$ is a zero of $w^2 - z$, and $|z_n - q_n'| = |z_n'| r_n = o(r_n) = o\big(r(q_n')\big)$ by Lemma 3.1. This proves (a) and (b). Assertion (c) then follows from

$$F^{\#}(z) = \frac{|2w(z)w'(z) - 1|}{1 + |w^2(z) - z|^2} = O\big(|w^2(z) - z|^{-1/2}\big)$$

and (a) and (b). ▫

Remark. Assertion (c) says that the value distribution of $w^2 - z$ takes place in very small neighbourhoods of the zeros of this function.

Proposition 3.3. *For δ sufficiently small, the set $\mathsf{Q}(\delta) = \bigcup_\nu \Delta_\delta(q_\nu)$ may be covered by the union of* disjoint *disks $\{z : |z - q_\nu'| < \theta_\nu \delta r(q_\nu')\}$, $1 \le \theta_\nu \le 3$, where (q_ν') is a subsequence of (q_ν).*

The *proof* is the same as the proof of the corresponding Lemma 2 in [17], see also [3]. It relies on the following fact, which says that, for δ sufficiently small, any disk $\Delta_\delta(q_\nu)$ meets at most one disk $\Delta_\delta(q_\mu)$:

If (q_n'), (q_n'') and (q_n''') are disjoint sub-sequences of (q_n), then

$$|q_n' - q_n''| + |q_n' - q_n'''| \ge c r(q_n')$$

for some $c > 0$, depending only on w.

Assuming $|q_n' - q_n''| + |q_n' - q_n'''| = o\big(r(q_n')\big)$, the re-scaling process

$$v_n(z) = r_n^2\big[w^2(q_n' + r_n z) - (q_n' + r_n z)\big], \quad r_n = r(q_n'),$$

for $w^2 - z$ leads to the differential equation

$$(v + a^2)v'' = \tfrac{1}{2}{v'}^2 + 4(v + a^2)^3, \quad a^2 = \lim_{n\to\infty} r_n^2 q_n',$$

with $v(0) = v'(0) = v''(0) = 0$, this following from Hurwitz' Theorem, and this implies $v(z) \equiv 0$ and $a = 0$. On the other hand we have $v(z) = y_+(z) \cdot y_-(z)$, where $y_\pm$ is the result of re-scaling $w(z) \pm \sqrt{z}$, and neither y_+ nor y_- vanishes identically. This contradiction proves the assertion. ▫

Remark. In particular Proposition 3.3 says that $\mathsf{Q}(\delta)$ is *porous* in the following sense: there exists some constant $K_0>1$, such that any two points $a,b\in\mathbf{C}\setminus\mathsf{Q}(\delta)$ may be joined by a path of integration in $\mathbf{C}\setminus\mathsf{Q}(\delta)$ of length $\le K_0|a-b|$.

Still now all results have been of local nature. To solve the *connection problem* we consider the function

$$V(z)=U(z)-w(z)w'(z)/\big(w^2(z)-z\big),$$

which has the remarkable property that

$$V(p)=10\varepsilon\mathsf{h}-7p^2/36$$

at every pole p of w with residue ε. Furthermore, V satisfies the linear differential equation

$$V'=\frac{w(w^2+3z)(zw+\alpha)}{(w^2-z)^2}-\frac{2w^3}{(w^2-z)^3}w'-\frac{z+w^2}{(w^2-z)^2}V.$$

To proceed further we need the following

Lemma 3.4. *Given $\sigma>0$, there exists $K>0$ such that*

$$\left|\frac{z+w^2(z)}{(w^2(z)-z)^2}V(z)\right|\le\sigma\frac{|V(z)|}{|z|}+K|z|,$$

and hence $|V'(z)|\le\sigma\big(|V(z)|/|z|\big)+K_1|z|$ holds outside $\mathsf{Q}(\delta)$.

Proof. Let (z_n) be any sequence tending to infinity outside $\mathsf{Q}(\delta)$. If $|z_n|=o\big(|w(z_n)|^2\big)$, then obviously

$$\left|\frac{z_n+w^2(z_n)}{(w^2(z_n)-z_n)^2}V(z_n)\right|=o\bigg(\frac{|V(z_n)|}{|z_n|}\bigg).$$

If, however, $|w^2(z_n)-z_n|=O(|z_n|)$, then from (2) and Proposition 3.2(b) follows $|U(z_n)|=O\big(|w^2(z_n)-z_n|^2\big)=O(|z_n|^2)$. From the same proposition and our assumption follows

$$\left|\frac{w(z_n)w'(z_n)}{w^2(z_n)-z_n}\right|=O(|z_n|^{1/2}),$$

and hence

$$\left|\frac{z_n+w^2(z_n)}{\big(w^2(z_n)-z_n\big)^2}V(z_n)\right|=\left|\frac{z_n+w^2(z_n)}{\big(w^2(z_n)-z_n\big)^2}\right|\left|U(z_n)-\frac{w(z_n)w'(z_n)}{w^2(z_n)-z_n}\right|=O(|z_n|).$$

This proves the lemma. □

Using Propositions 3.2, 3.3 and Lemma 3.4, it is not hard to show, using a Gronwall-like argument, see also [17] and [3] in case (I), that

$$V(z) = O(|z|^2) \quad \text{as } z \to \infty \text{ outside } \mathsf{Q}(\delta);$$

in particular, from $\varepsilon\mathsf{h} = V(p) + 7p^2/36$ at every pole of w it follows that $|\mathsf{h}| = O(|p|^2)$ as $p \to \infty$.

We now need some good *a priori* lower bound for the radius of convergence $r(p, \mathsf{h})$ of the Laurent series (3). It is not hard to show that, in our case (II), $r(p, \mathsf{h}) \geq K \min\{1, |p|^{-1/2}, |\mathsf{h}|^{-1/4}\}$ holds with K an absolute constant. Hence, for a fixed solution w and any pole p_ν, this radius is at least $K_1|p_\nu|^{-1/2}$, K_1 only depending on w. The proof is left to the reader, the corresponding estimate for the solutions of (IV) is proved in the appendix. This estimate also enables to re-scale w about poles p with re-scaling factor or *local unit of length* $r(p) = |p|^{-1/2}$. From these considerations follows

Proposition 3.5. *For any transcendental solution of* (II), *with sequence of poles* (p_ν) *and associated sequence* (h_ν), *the following is true*:

(a) $\mathsf{h}_\nu = O(|p_\nu|^2)$ *as* $\nu \to \infty$.
(b) $\sum_{0<|p_\nu|\leq r} |p_\nu|^{-1} = O(r^2)$ *as* $r \to \infty$.
(c) $w'(z) = O(|z|)$ *and* $U(z) = O(|z|^2)$ *as* $z \to \infty$ *outside* $\mathsf{P}(\delta) = \bigcup_\nu \Delta_\delta(p_\nu)$.
(d) $\left|w(z) - \sqrt{z}\,\right| \asymp |z|^{1/2}$ (*any branch*) *outside* $\mathsf{P}(\delta) \cup \mathsf{Q}(\delta)$, *which means that* $\left|w(z) - \sqrt{z}\,\right| = O(|z|^{1/2})$ *and* $|z|^{1/2} = O\big(\left|w(z) - \sqrt{z}\,\right|\big)$.
(e) $r(z) \asymp |z|^{-1/2}$ *for* z *outside* $\mathsf{P}(\delta)$.

The main application of (b), of course, is the estimate

$$T(r, w) = O(r^3).$$

(b) Re-scaling equation (IV). We will briefly describe the procedure in case (IV), which is quite similar to its counterpart in case (II). We set

$$r(z_0) = \min\big\{|z_0|^{-1}, |w(z_0)|^{-1}, |w'(z_0)|^{-1/2}\big\}$$

to re-scale $w(z) + z$ rather than w itself. Let (z_n) tend to infinity, set $r_n = r(z_n)$ and $y_n(z) = r_n[w(z_n + r_n z) + z_n + r_n z]$. Again, assuming the limits $a = \lim_{n\to\infty} r_n z_n$, $y_0 = \lim_{n\to\infty} y_n(0) = \lim_{n\to\infty} r_n w(z_n) + a$ and $y_0' = \lim_{n\to\infty} y_n'(0) = \lim_{n\to\infty} r_n^2 w'(z_n)$ to exist, we obtain the limit differential equation

$$2(y-a)y'' = y'^2 + 3(y-a)^4 + 8a(y-a)^3 + 4a^2(y-a)^2.$$

Again we have $y \not\equiv 0$, this following from $|a| + |y_0| + |y_0'| > 0$. Constant solutions are $y = \pm a$ and $y = \frac{1}{3}a$.

We denote by (p_ν) and (q_ν) the sequence of non-zero poles and zeros of $w(z) + z$, respectively, and set $\mathsf{Q}(\delta) = \bigcup_\nu \Delta_\delta(q_\nu)$, where again $\Delta_\delta(c) = \{z : |z-c| < \delta r(c)\}$, $\delta > 0$ arbitrarily small, but fixed. Then we obtain, similarly to case (II), but now using the key auxiliary function

$$V(z) = U(z) - w(z)w'(z)/\big(w(z)+z\big)^2 \quad \text{with} \quad V(p) = -3\varepsilon p + 2\alpha p + 2\mathsf{h}:$$

Proposition 3.6. *t For any solution of* (IV) *the following holds*:

(a) $z = O\big(|w(z)+z|\big)$ *and* $w'(z) = O\big(|w(z)+z|^2\big)$ *as* $z \to \infty$ *outside* $\mathsf{Q}(\delta)$.
(b) $V(z) = O(|z|^3)$ *as* $z \to \infty$ *outside* $\mathsf{Q}(\delta)$; *in particular,* $\mathsf{h}_\nu = O(|p_\nu|^3)$.

Again the proof is based on asymptotic integration of the linear differential equation

$$V' = Q(z,w) + \frac{2zw(z+5w)}{(w+z)^5}w' + \frac{2w(3z-w)}{(w+z)^3}V$$

with

$$Q(z,w) = -z^2 + \frac{8\alpha z + 2z^3}{w+z} - \frac{2\beta + 16\alpha z^2 - z^4}{(w+z)^2} + \frac{4\beta z + 8\alpha z^3 - 2z^5}{(w+z)^3}.$$

Similarly to Lemma 3.4 one can show that given $\sigma > 0$ there exists $K > 0$ such that

$$\left|\frac{2w(z)\big(3z - w(z)\big)}{\big(w(z)+z\big)^3}V(z)\right| \le \sigma\frac{|V(z)|}{|z|} + K|z|^2$$

outside $\mathsf{Q}(\delta)$; since the set $\mathsf{Q}(\delta)$ is *porous*, the same technique as was used in case (II) yields $V(z) = O(|z|^3)$, and, in particular, $|\mathsf{h}| = O(|p|^2)$. From this result and the appropriate lower estimate for the radius of convergence $r(p,\mathsf{h})$, see the appendix, it follows that we may re-scale about any pole $p \neq 0$ with *local unit of scale* $r(p) = |p|^{-1}$. Again by setting $\mathsf{P}(\delta) = \bigcup_\nu \Delta_\delta(p_\nu)$ we obtain

Proposition 3.7. *For any solution of* (IV) *the following holds*:

(a) $|w(z)+z| \asymp |z|$ *as* $z \to \infty$ *outside* $\mathsf{P}(\delta) \cup \mathsf{Q}(\delta)$.
(b) $w'(z) = O(|z|^2)$ *and* $U(z) = O(|z|^3)$ *as* $z \to \infty$ *outside* $\mathsf{P}(\delta)$; *in particular it is allowed to replace* $r_n = r(z_n)$ *by* $|z_n|^{-1}$ *for* (z_n) *outside* $\mathsf{P}(\delta)$.

In this case, too, the main application is the estimate

$$T(r,w) = O(r^4).$$

Final remark. To each equation (I), (II) and (IV) there corresponds in a canonical way a Riemannian metric $ds = |z|^{\lambda/4}|dz|$, $\lambda = 1, 2, 4$; distances are denoted by $\mathsf{d}(a, b) = \mathsf{d}_\lambda(a, b)$. The euclidian disk $\Delta_\delta(c) = \{z : |z - c| < \delta|c|^{-\lambda/4}\}$ obviously may be replaced by $\{z : \mathsf{d}_\lambda(z, c) < \delta\}$, for $|c|$ large compared with δ. Hence, for any fixed solution the corresponding Laurent series (3) converges in $\mathsf{d}_\lambda(z, p) \le K_\lambda(w)$, $\lambda = 1, 2, 4$, where $K_\lambda(w) > 0$ is a constant not depending on the pole p.

4. Value distribution of the second transcendents

Let w be any transcendental solution of equation (II), with non-zero poles p_ν and $\operatorname{Res}_{p_\nu} w = \varepsilon_\nu$, and let g_ε be the canonical product with simple zeros exactly at the non-zero poles of w with residue $\varepsilon = \pm 1$. We set $g(z) = z^{|\varepsilon_0|} g_1(z) g_{-1}(z)$ and $f(z) = z^{\varepsilon_0} g_1(z)/g_{-1}(z)$, with $\varepsilon_0 = \operatorname{Res}_0 w$. Then g and f have genus $h \ge 0$, and from $m(r, w) = O(\log r)$ and $m(r, U) = O(\log r)$ it follows that there exist unique polynomials Q_w and Q_U such that

$$w(z) = Q_w(z) + \frac{f'(z)}{f(z)} \quad \text{and} \quad U(z) = Q_U(z) - \frac{g'(z)}{g(z)}. \tag{15}$$

In the sequel we will discuss how the polynomials Q_w and Q_U associated with w are related to h and to each other. Clearly, for $\varepsilon_0 = 0$, Q_w and Q_U are the Taylor polynomials of w and U, respectively, of degree $h - 1$, *plus higher terms*!

Before proceeding further we prove a surprising result, which at first glance seems to show that each second Painlevé transcendent has order of growth $\varrho \le 2$.

Theorem 4.1. *Any transcendental solution of equation* (II) *with* $w(0) \ne \infty$ *may be represented in the form*

$$w(z) - w(0) = \lim_{r\to\infty} \sum_{0<|p_\nu|\le r} \frac{\varepsilon_\nu z}{(z - p_\nu)p_\nu} = \sum_{(p_\nu)}^{\star} \frac{\varepsilon_\nu z}{(z - p_\nu)p_\nu}. \tag{16}$$

If w *has a pole at* $z = 0$ *with residue* ε_0, *then* $w(0)$ *has to be replaced by* ε_0/z.

Remark. We note that convergence is locally uniform, but $\sum^{\star}_{(p_\nu)}$, being defined by (16), has to be understood as (Cauchy) *principal value*, obtained by exhausting the plane, and hence the sequence (p_ν), by disks $|z| \le r$.

Proof. Let $r > 0$ be sufficiently large; we construct a closed path of integration Γ_r of length $O(r)$ with the following properties: the interior of Γ_r contains exactly those poles of w which are contained in $|z| \le r$, and $\Gamma_r \cap \Delta_\nu = \emptyset$ for each ν, where $\Delta_\nu = \{z : |z - p_\nu| < \delta|p_\nu|^{-1/2}\}$; $\delta > 0$ is chosen in such a way that $\Delta_\nu \cap \Delta_\mu = \emptyset$ for $\mu \ne \nu$. We start with the positively oriented circle $C_r : |z| = r$. If $c_\nu = C_r \cap \Delta_\nu \ne \emptyset$, we replace this sub-arc of C_r by the corresponding sub-arc

d_ν of $\partial\Delta_\nu$ *inside* $|z| = r$ if $|p_\nu| > r$, and *outside* $|z| = r$ if $|p_\nu| \le r$, to obtain Γ_r after finitely many steps. Since $\mathrm{length}(d_\nu) \le \pi \times \mathrm{length}(c_\nu)$, the length of Γ_r is at most $2\pi^2 r$.

We assume $w(0) \ne \infty$ for simplicity. Then for z inside Γ_r, the Residue Theorem gives

$$S(z,r) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{w(\zeta)}{\zeta(\zeta - z)}\,d\zeta = \frac{w(z) - w(0)}{z} + \sum_{0<|p_\nu|\le r} \frac{\varepsilon_\nu}{p_\nu(p_\nu - z)},$$

and from $|w(\zeta)| = O(|\zeta|^{1/2}) = O(r^{1/2})$ on Γ_r follows $S(z,r) = O(r^{-1/2})$ as $r \to \infty$, uniformly with respect to $|z| \le \frac{1}{2}r$, say. □

Remark. This result is surprising insofar as it is supposed that, in general, $\sum_{\nu=1}^{\infty} |p_\nu|^{-3}$ diverges. We note that $w'(0) = -\sum^\star_{(p_\nu)} \varepsilon_\nu p_\nu^{-2}$ in the first case, and $\sum^\star_{(p_\nu)} \varepsilon_\nu p_\nu^{-2} = 0$ if w has a pole at $z = 0$.

We may also consider

$$S^{(2)}(z,r) = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{w^2(\zeta)}{\zeta(\zeta - z)}\,d\zeta$$

to obtain

Theorem 4.2. *For any transcendental solution of* (II) *with* $w(0) \ne \infty$

$$\begin{aligned} w^2(z) &= w^2(0) + bz + \lim_{k\to\infty} \sum_{|p_\nu|\le r_k} \big[(z - p_\nu)^{-2} - p_\nu^{-2}\big] \\ &= w^2(0) + bz + \sum_{(p_\nu)}^{\star\star} \big[(z - p_\nu)^{-2} - p_\nu^{-2}\big] \end{aligned} \tag{17}$$

holds for some sequence $r_k \to \infty$, *with*

$$b = \lim_{k\to\infty} S^{(2)}(z, r_k) = 2w(0)w'(0) - 2\sum_{(p_\nu)}^{\star\star} p_\nu^{-3};$$

if w *has a pole at* $z = 0$ *with residue* ε_0, *then the terms* $w^2(0)$ *and* $2w(0)w'(0)$ *have to be replaced by* z^{-2} *and* $-\frac{1}{2}(1 + \varepsilon_0\alpha)$, *respectively.*

Remark. We call $\sum^{\star\star}_{(p_\nu)}$ *principal value of the second kind*, obtained by the exhaustion $|p_\nu| \le r_k \to \infty$; again convergence is locally uniform with respect to z. Considering the integral

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{w^2(\zeta)}{\zeta^2(\zeta - z)}\,d\zeta$$

instead of $S^{(2)}(z,r)$ yields

$$\begin{aligned} w^2(z) &= w^2(0) + 2w(0)w'(0)z + \sum_{\nu=1}^{\infty} \frac{z^2(3p_\nu - 2z)}{(z-p_\nu)^2 p_\nu^3} \\ &= w^2(0) + 2w(0)w'(0)z + \sum_{\nu=1}^{\infty} \big[(z-p_\nu)^{-2} - p_\nu^{-2} - 2zp_\nu^{-3}\big], \end{aligned}$$

which converges absolutely and locally uniformly.

Proof of Proposition 4.2. Again from the Residue Theorem follows

$$S^{(2)}(z,r) = \frac{w^2(z) - w^2(0)}{z} + \sum_{0<|p_\nu|\le r} \frac{z - 2p_\nu}{p_\nu^2 (p_\nu - z)^2}.$$

Since $w^2(\zeta) = O(r)$ on Γ_r, we may, however, only conclude that $S^{(2)}(z,r)$ is uniformly bounded, for $|z| \le \frac{1}{2}r$, say, independent of r. For some appropriate sequence $r_k \to \infty$ we thus have $\lim_{k\to\infty} S^{(2)}(z, r_k) = b$, locally uniformly in the plane. ◻

Theorem 4.3. *In any case* $\deg Q_w \le \max\{0, h-1\} \le 2$ *and* $\deg Q_U \le 2$ *hold.*

Remark. If $w(0) \ne \infty$ and $h \ge 1$, then $Q_w(z) = T_{h-1}(z; w)$ is the Taylor polynomial of w about $z = 0$, of degree $h-1$. In case $w(0) = \infty$ we have $Q_w(z) = 0$ for $1 \le h \le 2$, and $Q_w(z) = -\frac{1}{4}(\alpha + \varepsilon_0)z^2$ for $h = 3$.

Things are different for Q_U. Writing $b_k = \sum_{\nu=1}^{\infty} p_\nu^{-k-1}$ (the series converges absolutely for $k \ge h$) we obtain $Q_U(z) = 10\varepsilon_0 \mathsf{h}_0 - \frac{1}{2}(1 + \varepsilon_0\alpha)z^2 - \sum_{k=h}^{2} b_k z^k$ if $z = 0$ is a pole with residue ε_0, and $Q_U(z) = T_2(z; U) - \sum_{k=h}^{2} b_k z^k$ if $w(0) \ne \infty$.

Proof. We assume $w(0) \ne \infty$ for simplicity. Then, on one hand, (15) gives

$$w(z) = Q_w(z) + \sum_{\nu=1}^{\infty} \frac{\varepsilon_\nu z^h}{(z - p_\nu)p_\nu^h}, \tag{18}$$

while (16) continues to hold. From this we may conclude that $\deg Q_w < \max\{h, 1\}$, and hence $Q_w(z) = T_{h-1}(z; w)$ for $h \ge 1$. If $z = 0$ is a pole of w with residue ε_0, then we also have $\deg Q_w < \max\{h, 1\}$, and from (3) follows $Q_w(z) = 0$ for $1 \le h \le 2$, and $Q_w(z) = -\frac{1}{4}(\alpha + \varepsilon_0)z^2$ in case $h = 3$.

The representation of U easily follows from (17) and $U' = w^2$. Comparison with the ordinary series expansion

$$U(z) = Q_U(z) - \frac{f'(z)}{f(z)} = Q_U(z) - \sum_{\nu=1}^{\infty} \frac{z^h}{(z - p_\nu)p_\nu^h}$$

yields $\deg Q_U \le 2$, and $Q_U(z) = T_2(z, U) - \sum_{k=h}^{2} b_k z^k$, if $w(0) \ne \infty$, and $Q_U(z) = 10\varepsilon_0 \mathsf{h}_0 - \frac{1}{2}(1 + \varepsilon_0\alpha)z^2 - \sum_{k=h}^{2} b_k z^k$ else. ◻

In [15] Shimomura has shown:

For $2\alpha\in\mathbf{Z}$ every transcendental solution of (II) *has order of growth*

$$\varrho\ge\tfrac{3}{2}.$$

More precisely, it was shown that $T(r,w)\ge C_\varepsilon r^{3/2-\varepsilon}$ for every $\varepsilon>0$ and any transcendental solution of equation (II) with parameter $\alpha=0$. This result then may be extended to any α with $2\alpha\in\mathbf{Z}$ by applying the Bäcklund transformation.

To prove Shimomura's result, we have only to deal with the case where Q_w is constant, since $\varrho(w)\ge h\ge\deg Q_w+1\ge 2$ in all other cases, and thus may write $w=\Phi'/\Phi$ with $\varrho(\Phi)=\sigma\le\max\{1,\varrho(w)\}$ (note that, in case $h=0$, $\Phi(z)=e^{cz}f(z)$ might contain an extra factor e^{cz} to represent w). For $\alpha=0$, a simple computation gives

$$z=\frac{w''}{w}-2w^2=\frac{\Phi''}{\Phi'}\left(\frac{\Phi'''}{\Phi''}-3\frac{\Phi'}{\Phi}\right)=\frac{\Phi''}{\Phi'}\frac{\Psi'}{\Psi}$$

with $\Psi=\Phi''/\Phi^3$. Since the order of Ψ is at most σ, we obtain from the *lemma on the logarithmic derivative*, in the form due to Ngoan and Ostrovskii [8], that

$$\log r=m(r,z)\le m(r,\Phi''/\Phi')+m(r,\Psi'/\Psi)\le 2\big(\sigma-1+o(1)\big)^+\log r,$$

and hence $\sigma\ge\frac{3}{2}$, which implies $\varrho=\sigma\ge\frac{3}{2}$. □

Remark. For arbitrary α the same proof shows that

$$2(\varrho-1)^+\ge\limsup_{r\to\infty}m(r,z+\alpha/w)/\log r.$$

In most cases the order of growth of any solution w has turned out to satisfy $\varrho\ge 2$, the only exemption occurring when Q_w is constant. We will now prove several lower estimates depending on $\deg Q_U$.

Theorem 4.4. *Let w be any transcendental solution of* (II), *with associated polynomials Q_w and Q_U. Then if $\varrho<3$ and $\deg Q_U=2$, the following is true*:

(a) $\varrho\ge\frac{3}{2}$ *and* $Q_U(z)=-\frac{1}{4}z^2+\cdots$,

(b) $w(z)\sim\sqrt{-z/2}$, $U(z)\sim -z^2/4$ *and* $w'(z)=o(|z|)$ *as* $z\to\infty$ *on some set* D *satisfying* $\operatorname{area}(D\cap\{z:|z|\le r\})\sim\pi r^2$ *as* $r\to\infty$.

Example. The solutions of $w'=z/2+w^2$ have order of growth $\varrho=\frac{3}{2}$ and solve equation (II) with parameter $\alpha=\frac{1}{2}$. In this case $U'=w^2=w'-\frac{1}{2}z$ and $Q_w(z)=w(0)$, hence $U(z)=w(z)-z^2/4+U(0)-w(0)$ and $Q_U(z)=-z^2/4+U(0)$. We note that $w^2(0)=\sum_{\nu=1}^\infty p_\nu^{-2}$ and $w^3(0)=w(0)w'(0)=\sum_{\nu=1}^\infty p_\nu^{-3}-\frac{1}{4}$. In case $w(0)=\infty$ we have $Q_U(z)=-z^2/4+10\varepsilon_0\mathsf{h}_0$ and $\sum_{\nu=1}^\infty p_\nu^{-2}=0$.

Proof of Theorem 4.4. We assume $\varrho(w) < 3$ and set $Q_w(z) = az + a_0$ (note that $\deg Q_w = 2$ implies $\varrho = 3$) and $Q_U(z) = \frac{1}{2}bz^2 + \cdots$. Then from Propositions 2.2 and 2.3 and $U' = w^2$ follows

$$I(R, U - Q_U) + I_H(R, w^2 - Q_U') + I_H(R, w' - Q_w') = O(R^{4-2\lambda})$$

for some $\lambda > 0$. Let the set $E \subset \mathbf{C}$ consist of all points z, such that at least one of the inequalities

$$\left|U(z) - \tfrac{1}{2}bz^2\right| > |z|^{2-\lambda}, \quad |w^2(z) - bz| > |z|^{1-\lambda}, \quad |w'(z)| > |z|^{1-\lambda}$$

holds, and set $E_R = E \cap \left\{z : \frac{1}{2}R \le |z| \le R\right\}$. Then, having $I(R, z) + I_H(R, 1) = O(R^3)$ in mind,

$$\begin{aligned} CR^{4-2\lambda} &\ge \int_{E_R} \left(\left|U(z) - \tfrac{1}{2}bz^2\right| + |z|\,|w^2(z) - bz| + |z|\,|w'(z)|\right) d(x, y) \\ &\ge \left(\tfrac{1}{2}R\right)^{2-\lambda} \text{area}(E_R), \end{aligned}$$

and hence $\text{area}(E_R) = O(R^{2-z\lambda})$ follows, this implying

$$\text{area}(D_R) = \text{area}(D \cap \{z : |z| \le R\}) = \pi R^2 - o(R^2)$$

for $D = \mathbf{C} \setminus E$.

Re-scaling equation (2) on any sequence $(z_n) \subset D$, with local unit of length $r_n = z_n^{-1/2}$, i.e., taking the limit $n \to \infty$ for $y_n(z) = z_n^{-1/2} w(z_n + z_n^{-1/2} z)$ then yields

$$y'^2 = y^4 + y^2 - \tfrac{1}{2}b, \quad y(0) = \sqrt{b}, \quad y'(0) = 0,$$

for some choice of $\sqrt{b}$, from which $b = -\frac{1}{2}$ and $y(z) \equiv \sqrt{-\frac{1}{2}}$, and hence $w(z_n) \sim \sqrt{-\frac{1}{2}z_n}$ and $U(z_n) \sim -\frac{1}{4}z_n^2$ follows. This proves $w(z) \sim \sqrt{-\frac{1}{2}z}$ and $U(z) \sim -\frac{1}{4}z^2$ as $z \to \infty$ on D.

Since $\varrho \ge 2$ for $\deg Q_w \ge 1$, we have only to deal with the case $a = 0$. Then $\varrho(w) \ge \frac{3}{2}$ follows from

$$2\pi I(R, w) \ge \int_{D_R} |w(z)|\, d(x, y) \ge \text{const} \cdot R^{5/2}$$

and Proposition 2.2. □

Theorem 4.5. *Let w be any transcendental solution of* (II), *with associated polynomials Q_w and Q_U. Then $\deg Q_U = 1$ implies $\varrho \ge 2$.*

Proof. We assume $Q_U(z) = bz + b_0$ and $\varrho < 2$, hence Q_w is a constant. Then from Propositions 2.2 and 2.3 follows

$$I_H(R, w^2 - b) + I_H(R, w') = O(R^{3-2\lambda})$$

for some $\lambda > 0$, and as in the proof of Theorem 4.4 the set

$$E = \{z : |w^2(z) - b| > |z|^{-\lambda} \text{ or } |w'(z)| > |z|^{-\lambda}\}$$

satisfies $\operatorname{area}(E \cap \{z : |z| \le R\}) = o(R^2)$. Again we set $D = \mathbf{C} \setminus E$ and $D_R = D \cap \{z : |z| \le R\}$, and again $\operatorname{area}(D_R) \sim \pi R^2$ holds. From equation (1) then follows $|w''(z)| \ge |b|^{1/2}|z|/2$ for $z \in D$ and $|z| \ge r_0$, say. Together with $|w'(z)| = o(1)$ this implies $(w')^{\#}(z) \ge |b|^{1/2}|z|/4$, and hence $A(r, w') \ge c_1 r^4$ for some $c_1 > 0$, this contradicting our assumption $\varrho < 2$ (and even $\varrho \le 3$). □

Concluding remarks. Theorems 4.2, 4.3 and 4.4 together show that $\varrho(w) \ge \frac{3}{2}$ is true except when Q_U and Q_w are constants. Thus the only case left is $w = a + f'/f$ and $w^2 = -(g'/g)'$ with $f = g_1/g_{-1}$ and $g = g_1 g_{-1}$, where $g_{\pm 1}$ are canonical products of genus ≤ 1. In the sequel we will discuss several ideas which could or could not help to prove $\varrho \ge \frac{3}{2}$.

(a) It seems promising trying to prove

$$\limsup_{r\to\infty} \frac{m(r, z + \alpha/w)}{\log r} \ge 1.$$

This, however, is far beyond the scope of our method, since it requires analyzing solutions on circles $|z| = r$. We note also that for $\alpha \in \mathbf{Z}$ there exist rational solutions with $z + \alpha/w(z) = O(|z|^{-1})$, and hence any proof had to distinguish between different parameters, and also between rational and transcendental solutions. Also this method would not work in case of equation (IV).

(b) It also seems hopeless trying to prove that $\deg Q_w = 0$ implies $\deg Q_U = 2$, though several hints indicate that this might be true. It might, however, be fruitful to consider the following problem: let $g_{\pm 1}$ denote the canonical products with zeros at the non-zero poles of w with residues ± 1, and assume $Q_w(z) = a$ and $Q_U(z) = b$. Replacing $g_{\pm 1}$ by $f_{\pm 1} = e^{\pm az/2} g_{\pm 1}$ and writing $g = f_1 f_{-1}$ and $f = f_1/f_{-1}$ we obtain $w = f'/f$ and $w^2 = U' = -(g'/g)'$. Thus

$$-(f_1'/f_1)' - (f_{-1}'/f_{-1})' = (f_1'/f_1 - f_{-1}'/f_{-1})^2,$$

or, equivalently,

$$f_{-1} f_1'' - 2f_{-1}' f_1' + f_{-1}'' f_1 = 0$$

has to be disproved for (essentially) canonical products $f_{\pm 1}$ without common zeros.

(c) Our third proposal seems to be more promising, namely, to prove some estimate $|w(z)| \geq c|z|^{1/2}$ outside small disks $|z - c_\nu| < \delta|c_\nu|^{-1/2}$ about the zeros c_ν of w, and then apply Proposition 2.2. In contrast to equation (I), however, it is not possible (although I believed in [17] it would be) to prove an asymptotic relation like $|w(z)| \asymp |z|^{1/2}$ outside the set $\mathsf{P}(\delta) \cup \mathsf{C}(\delta)$, by using re-scaling methods only. The reason for this is that (II) also has rational solutions (for parameters $\alpha \in \mathbf{Z}$) satisfying $w(z) \sim -\alpha/z$ as $z \to \infty$. It is a weakness of the re-scaling method that it cannot distinguish between different parameters nor between rational and transcendental solutions. All results which may be obtained by this method, must be true for *all parameters* and *all solutions.* A similar remark holds for equation (IV). Thus, some additional argument has to be introduced, which excludes rational solutions from consideration.

Nevertheless I believe that the following is true: *Any transcendental solution of* (II) *has order of growth either* $\varrho = 3$ *or else* $\varrho = \frac{3}{2}$, *this occurring exactly for particular solutions, called Airy Solutions.* These solutions are characterized by the fact that they also solve first order algebraic differential equations $P(z, w, w') = 0$, and are obtained by successive application of the so-called Bäcklund transformation, starting from the solutions of the *Riccati Equation* $u' = \pm(z/2 + u^2)$. For details the reader is referred to [3].

5. Value distribution of fourth transcendents

Equations (II) and (IV) are in many respects similar to each other. For certain parameters they admit rational solutions, or solutions which solve also some first order algebraic differential equations, and the residues ε_ν alternate.

We use the same notation as was used in the previous section to represent transcendental solutions w of (IV). Let g denote the canonical product, of genus h, with simple zeros exactly at the non-zero poles of w. Then $g = g_1 g_{-1}$, where $g_{\pm 1}$ has zeros exactly at poles with residue ± 1. If w has a pole at $z = 0$ with residue ε_0, we replace g and $f = g_1/g_{-1}$ by $zg(z)$ and $z^{\varepsilon_0} f(z)$, respectively. Then as in case (II) we have the representations (15).

From Section 3(b) we obtain the estimates $w(z) = O(|z|)$ and $U(z) = O(|z|^3)$ as $z \to \infty$ outside $\mathsf{P}(\delta) = \bigcup_\nu \Delta_\delta(p_\nu)$, with $\Delta_\delta(p_\nu) = \{z : |z - p_\nu| < \delta|p_\nu|^{-1}\}$. We also construct the closed curve Γ_r as in the proof of Theorem 4.1; it contains in its interior exactly those poles with $|p_\nu| \leq r$, while w satisfies $|w(\zeta)| = O(|\zeta|) = O(r)$ on Γ_r. Then the Residue Theorem applies to

$$S_m(z, r) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{w(\zeta)}{\zeta^m(\zeta - z)}\, d\zeta, \quad m = 1, 2,$$

with $S_m(z, r) = O(r^{-m+1})$ as $r \to \infty$, locally uniformly with respect to z. Hence, for $w(0) \neq \infty$, we obtain

$$w(z) = w(0) + w'(0)z + \sum_{(p_\nu)}^{\star} \frac{\varepsilon_\nu z^2}{(z - p_\nu)p_\nu^2} \tag{19}$$

(principal value with exhaustion $|p_\nu| \le r \to \infty$) in case $m = 2$, and, for $m = 1$,

$$w(z) = w(0) + bz + \sum_{(p_\nu)}^{\star\star} \frac{\varepsilon_\nu z}{(z - p_\nu)p_\nu} \tag{20}$$

for some sequence $r_k \to \infty$, where $b = \lim_{k\to\infty} S_1(z, r_k)$ is constant; note that $b = w'(0) + \sum_{(p_\nu)}^{\star\star} \varepsilon_\nu p_\nu^{-2}$. Equations (19) and (20) have to be modified if $z = 0$ is a pole with residue ε_0 as follows: $w(0)$ and $w'(0)z$ have to be replaced by ε_0/z and $\frac{1}{3}(2\varepsilon_0\alpha + 4)z$, respectively. Similarly, by considering

$$\widetilde{S}_m(z, r) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{w^2(\zeta) + 2\zeta w(\zeta)}{\zeta^{m+1}(\zeta - z)}\, d\zeta$$

and noting that $w^2(\zeta) + 2\zeta w(\zeta) = O(|\zeta|^2) = O(r^2)$ on Γ_r, we obtain

$$w^2(z) + 2zw(z) = T_2(z) + \sum_{(p_\nu)}^{\star} \big[(z - p_\nu)^{-2} - p_\nu^{-2} - 2zp_\nu^{-3}\big] \tag{21}$$

(principal value) and

$$w^2(z) + 2zw(z) = T_1(z) + \tilde{b}z^2 + \sum_{(p_\nu)}^{\star\star} \big[(z - p_\nu)^{-2} - p_\nu^{-2}\big] \tag{22}$$

(principal value of the second kind, obtained by the exhaustion $|p_\nu| \le r_k$ for some sequence $r_k \to \infty$), with $\tilde{b} = \lim_{k\to\infty} \widetilde{S}_1(z, r_k)$, locally uniformly, and T_m being the Taylor polynomial of $w^2 + zw$ of degree m about $z = 0$.

Then (19), (20) have to be compared with the Mittag-Leffler series expansions

$$w(z) = Q_w(z) + \sum_{\nu=1}^{\infty} \frac{\varepsilon_\nu z^h}{(z - p_\nu)p_\nu^h}, \quad 0 \le h \le 4, \tag{23}$$

and, similarly, (21), (22) have to be compared with

$$w^2(z) + 2zw(z) = Q_U'(z) + \sum_{\nu=1}^{\infty} \big[(z - p_\nu)^{-2} - K_h(z, p_\nu)\big], \tag{24}$$

with $K_h(z, p) = \sum_{k=0}^{h-2} (k+1) z^k p^{-k-2}$ for $0 \le h \le 4$. The case $w(0) = \infty$ requires obvious modifications, it does not make any sense to write this down. We thus obtain, similarly to case (II):

Theorem 5.1. *In any case* $\deg Q_w \le \max\{h - 1, 1\} \le 3$ *and* $\deg Q_U \le 3$ *hold.*

As in case (II) we next prove several lower estimates depending on $\deg Q_U$, noting that $\varrho \geq \deg Q_w + 1$ for $\deg Q_w \geq 2$ is already known.

Theorem 5.2. *Let w be any transcendental solution of* (IV), *with associated polynomials Q_w and Q_U. If $\varrho < 4$ and $\deg Q_U = 3$, then the following is true*:

(a) $Q_U(z) = -\frac{8}{27}z^3 + \cdots$,
(b) $w(z) \sim -\frac{2}{3}z$, $U(z) \sim -\frac{8}{27}z^3$ *and* $w'(z) = o(|z|^2)$ *as* $z \to \infty$ *on some set* D *satisfying* $\text{area}(D \cap \{z : |z| \leq r\}) \sim \pi r^2$ *as* $r \to \infty$,
(c) $\varrho \geq 2$, *provided* $Q_w(z) \not\equiv -\frac{2}{3}z + a_0$.

Remark. We note that certain equations (IV) have rational solutions with principal part $-\frac{2}{3}z$ at infinity.

Proof. There is almost no difference to the proof of Theorem 4.4. We set $Q_U = \frac{1}{3}bz^3 + \cdots$ and $Q_w(z) = \frac{1}{2}az^2 + \cdots$ (note that $\deg Q_w = 3$ implies $\varrho = 4$), and assume $\varrho(w) < 4$, hence

$$I(R, U - Q_U) + I_H(R, w^2 + 2zw - Q_U') + I_H(R, w' - Q_w') = O(R^{5-2\lambda}),$$

for some $\lambda > 0$ (note that $I(R, z^2) + I_H(R, z) = O(R^4)$). Consider the set $E \subset \mathbf{C}$, such that for $z \in E$ at least one of the inequalities

$$\left|U(z) - \tfrac{1}{3}bz^3\right| > |z|^{3-\lambda}, \quad |w^2(z) + 2zw(z) - bz^2| > |z|^{2-\lambda}, \quad |w'(z) - az| > |z|^{2-\lambda}$$

holds, and set $E_R = E \cap \left\{z : \frac{1}{2}R \leq |z| \leq R\right\}$. Then as in Section 4 we conclude that $\text{area}(E_R) = O(R^{2-\lambda})$, and hence $\text{area}(D_R) \sim \pi R^2$ for $D_R = D \cap \{z : |z| \leq R\}$ and $D = \mathbf{C} \setminus E$.

Re-scaling the corresponding equation (2) on any sequence $(z_n) \subset D$, $z_n \to \infty$, with local unit of length $r_n = z_n^{-1}$ then yields

$$y'^2 = y^4 + 4y^3 + 4y^2 - \tfrac{4}{3}by, \quad y(0)^2 + 2y(0) = b, \quad y'(0) = 0,$$

from which $b = -\frac{8}{9}$ and $y \equiv -\frac{2}{3}$ follows (any other constant solution is ruled out by the assumption $b \neq 0$.) This proves $w(z) \sim -\frac{2}{3}z$, $U(z) \sim -\frac{8}{27}z^2$ and $w'(z) = o(|z|^2)$ as $z \to \infty$ in D. For $Q_w(z) \not\equiv -\frac{2}{3}z + a_0$ we have $|w(z) - Q_w(z)| \geq c|z|$ for $z \in D$, $|z| \geq r_0$ and some $c > 0$, and thus $\varrho \geq 2$ follows from

$$2\pi I(r, w - Q_w) \geq 2\pi \int_{D_r} c|z|\, d(x,y) \geq c_1 r^3$$

and Proposition 2.2. □

Theorem 5.3. *Let w be any transcendental solution of* (IV), *with associated polynomials Q_w and Q_U, and assume $\deg Q_U = 2$. Then either $\varrho \geq \frac{14}{5}$ holds, or else there exists some set D and some sequence $r_n \to \infty$, such that* $\text{area}(D \cap \{z : |z| \leq r_n\}) \sim \pi r_n^2$ *and $w(z) \sim -2z$ as $z \to \infty$ on D. Moreover, $Q_w(z) \not\equiv -2z + a_0$ implies $\varrho \geq 2$.*

Remark. We note that certain equations (IV) have rational solutions satisfying $w(z) \sim -2z$.

Proof. We write $Q_U(z) = \frac{1}{2}bz^2 + \cdots$, $b \neq 0$, and assume $\varrho < 3$. Then $Q_w(z) = az + a_0$, and for some $\lambda > 0$, to be determined later, we have

$$I_H(R, w^2 + 2zw - bz) + I_H(R, w' - a) = O(R^{4-\lambda}).$$

Thus, for every ε, $0 < \varepsilon < \lambda$, the set

$$E(\varepsilon) = \bigl\{z : |w^2(z) + 2zw(z) - bz| > |z|^{1-\lambda+\varepsilon} \text{ or } |w'(z) - a| > |z|^{1-\lambda+\varepsilon}\bigr\}$$

satisfies $\operatorname{area}\bigl(E(\varepsilon) \cap \bigl\{z : \frac{1}{2}r \le |z| \le R\bigr\}\bigr) = O(R^{2-\varepsilon})$. Thus, for $D(\varepsilon) = \mathbf{C} \setminus E(\varepsilon)$ we have

$$\operatorname{area}\bigl(D(\varepsilon) \cap \{z : |z| \le R\}\bigr) \sim \pi R^2,$$

while

$$|w^2(z) + 2zw(z) - bz| < |z|^{1-\lambda+\varepsilon} \quad \text{and} \quad |w'(z) - a| < |z|^{1-\lambda+\varepsilon}$$

hold on $D(\varepsilon)$.

Now $w^2 + 2zw - bz = o(|z|)$ has two solutions, $w_1 = b/2 + o(1)$ and $w_2 = -2z - b/2 + o(1)$ as $z \to \infty$. We set

$$D_1(\varepsilon) = \bigl\{z \in D(\varepsilon) : \bigl|w(z) - \tfrac{1}{2}b\bigr| < \tfrac{1}{4}|b|\bigr\} \quad \text{and} \quad D_2(\varepsilon) = D(\varepsilon) \setminus D_1(\varepsilon).$$

If for some $\varepsilon > 0$ and some sequence $r_n \to \infty$

$$\operatorname{area}\bigl(D_1(\varepsilon) \cap \{z : |z| \le r_n\}\bigr) = o(r_n^2)$$

holds, then $Q_w(z) \not\equiv -2z + a_0$ implies $|w(z) - Q_w(z)| \ge c_1|z|$ on D_2, $|z|$ sufficiently large, and from

$$2\pi I(r_n, w - aQ_w) \ge 2\pi \int_{D_2 \cap \{|z| \le r_n\}} c_1|z|\, d(x, y) \ge c r_n^3$$

and Proposition 2.2 then follows $\varrho \ge 2$.

We now assume that for every sufficiently small $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$, such that

$$\operatorname{area}\bigl(D_1(\varepsilon) \cap \{z : |z| \le r\}\bigr) \ge cr^2.$$

Then from the corresponding equation (1) follows $|w''(z)| \ge c_1|z|^2$ for some $c_1 > 0$ and $z \in D_1(\varepsilon)$, $|z|$ sufficiently large. Thus $|w'(z) - a| < |z|^{1-\lambda+\varepsilon}$ gives

$$(w')^{\#}(z) \ge c_2|z|^{2\lambda - 2\varepsilon}$$

on $D_1(\varepsilon)$, this implying

$$A(r, w') \ge c_3 r^{2+4\lambda-4\varepsilon} \quad \text{and} \quad T(r, w) \ge c_4 r^{2+4\lambda-4\varepsilon},$$

and hence $3 - \lambda > \varrho \ge 2 + 4\lambda$, since $\varepsilon > 0$ was arbitrary. The only restriction on λ, however, is thus $0 < 5\lambda < 1$, and hence the inequality $\varrho \ge \frac{14}{5}$ follows by letting λ tend to $\frac{1}{5}$. □

Theorem 5.4. *Let w be any transcendental solution of* (IV), *with associated polynomials Q_w and $Q_U(z) = bz + b_0$, $b \neq 0$. Then, either $\varrho \geq 2$ holds, or else there exist some set $D = D_1 \cup D_2$ satisfying* $\operatorname{area}(D \cap \{z : |z| \leq r\}) \sim \pi r^2$ *and such that one of the following assertions holds*:

(a) $\operatorname{area}\big(D_1 \cap \{z : |z| \leq r\}\big) \sim \pi r^2$ *and $w(z) \sim b/2z$ on D_1,*
(b) $\operatorname{area}(D_2 \cap \{z : |z| \leq r_n\}) \geq c r_n^2$ *for some $c > 0$ and some sequence $r_n \to \infty$, and $w(z) \sim -2z$ on D_2.*

Moreover, $\varrho \geq 2$ also holds in case (a) *if either $b^2 + 2\beta \neq 0$ or else Q_w is non-constant, and in case* (b) *if $Q_w(z) \not\equiv -2z + a_0$.*

Proof. The arguments are easier than in the proof of Theorem 5.3. We assume $\varrho < 2$, hence $Q_w(z) = az + a_0$. Then

$$I_H(R, U' - b) + I_H(R, w' - a) = O(R^{3-2\lambda})$$

for some $\lambda > 0$, this again implying that

$$|w^2(z) + 2zw(z) - b| < |z|^{-\lambda} \quad \text{and} \quad |w'(z) - a| < |z|^{-\lambda}$$

on some set D with $\operatorname{area}\big(D \cap \{z : |z| \leq R\}\big) \sim \pi R^2$.

As before the equation $w^2 + 2zw - b = o(1)$ has two different solutions $w_1 \sim b/2z$ and $w_2 \sim -2z$. According to these solutions we set

$$D_1 = \{z \in D : |w(z)| < 1\} \quad \text{and} \quad D_2 = D \setminus D_1.$$

Then, if $b^2 + 2\beta \neq 0$ and

$$\operatorname{area}(D_1 \cap \{z : |z| \leq r\}) \geq cr^2$$

for $r \geq r_0$ and some $c > 0$, we obtain from equation (1) $w''(z) \sim (b + 2\beta/b)z$. Since w' is bounded on D_1, this implies $(w')^{\#}(z) \geq c_1|z|$ on D_1, $A(r, w') \geq c_2 r^4$, and hence $\varrho = 4$ against our assumption.

Similarly, from $a \neq 0$, $|w(z)| < 1$ and $w'(z) \sim a$ on D_1 follows $w^{\#}(z) \sim |a|$, this implying $A(r, w) \geq c_1 r^2$ and $T(r, w) \geq c_2 r^2$, without restriction on $b^2 + 2\beta$.

We thus may assume that

$$\operatorname{area}\big(D_2 \cap \{z : |z| \leq r_n\}\big) \sim \pi r_n^2$$

on some sequence $r_n \to \infty$, and $w(z) \sim -2z$ on D_2. Then, if $Q_w(z) \not\equiv -2z + a_0$, the inequality $\varrho \geq 2$ follows from $|w(z) - Q_w(z)| \geq c_1|z|$ on D_2 and Proposition 2.2. ▫

Remark. We note that, for certain parameters α and β, equation (IV) has particular solutions of order $\varrho = 2$. These are also solutions of certain first order algebraic differential equations $P(z, w, w') = 0$, and are obtained by repeated application of the Bäcklund transformation, starting with solutions of the Riccati equations $\pm w' = \mp 2 - 2\alpha + 2zw + w^2$. For details the reader is referred to [3]. The Riccati equation $w' = -2 - 2\alpha + 2zw + w^2$ also serves as an

Example for Theorem 5.4. Here $U(z) = w(z) + (2 + 2\alpha)z - w(0) + U(0)$, and Q_w and $Q_U(z) = Q_w(z) + (2 + 2\alpha)z + U(0) - w(0)$ have degree ≤ 1.

The remarks at the end of Section 4 remain valid. Since for particular parameters α and β, equation (IV) has rational solutions $w \sim -2z$, $w \sim -\frac{2}{3}z$ and $w \sim \mathrm{const}/z$ as $z \to \infty$, it seems impossible to prove the desired estimate $|w(z)| \asymp |z|$, outside the neighbourhood $\mathsf{C}(\delta)$ of zeros of w, by using only rescaling methods. This asymptotic result, of course, would be very helpful, but not sufficient to prove $\varrho \ge 2$.

Nevertheless, *it is quite natural to believe that the transcendental solutions have order of growth $\varrho = 4$, except for those solutions satisfying also first order algebraic differential equations, in which case $\varrho = 2$.*

6. Distribution of poles with residues ± 1

In [3, Theorem 18.1 and 18.2] it was shown that every transcendental solution of (II) has infinitely many zeros with residue $+1$ and also infinitely many zeros with residue -1, except when $w = u$ is some particular solution of (II) with parameter $\alpha = \pm\frac{1}{2}$, also solving the Riccati equation

$$\pm u' = z/2 + u^2. \tag{25}$$

It is, however, not hard to prove a sharp quantitative result.

Theorem 6.1. *Let w be any transcendental solution of* (II)*, but not a solution of* (25)*. Then $N_+(r, w) \le 2N_-(r, w) + O(\log r)$ and conversely, $N_-(r, w) \le 2N_+(r, w) + O(\log r)$ hold.*

Proof. By our assumption the meromorphic function $\Phi(z) = w'(z) - w^2(z) - \frac{1}{2}z$ does not vanish identically. From (3) it is easily seen that Φ has a double pole at any pole of w with residue $+1$, and has a zero at any pole of w with residue -1. Hence, by Nevanlinna's *first fundamental theorem*,

$$N_-(r, w) \le N(r, 1/\Phi) \le T(r, \Phi) + O(1) = 2N_+(r, w) + O(\log r)$$

follows. Of course, $+$ and $-$ may be permuted. □

Remark. Any so-called *Airy solution*, already mentioned in Section 4, may be represented in the form $w(z) = R\big(z, u(z)\big)$, where R is rational and has degree $2n+1$ with respect to the second variable, and u is a solution of (25). It is easily seen that in this case either $N_-(r,w) = c_n N_+(r,w) + O(\log r)$ or else $N_+(r,w) = c_n N_-(r,w) + O(\log r)$ holds with $c_n = 1 + 1/n$. This example shows, by choosing $n = 1$, that Theorem 6.1 is sharp.

In general we have

Theorem 6.2. *For any transcendental solution w of* (II)

$$|n_+(r,w) - n_-(r,w)| = O(r^{3/2})$$

holds.

Obviously, Theorem 6.2 says that for solutions of order $\varrho > \frac{3}{2}$, the poles with residue $+1$ and the poles with residue -1 are *asymptotically equi-distributed.*

Proof. Using the path of integration Γ_r constructed in Section 4 the Residue Theorem gives

$$|n_+(r,w) - n_-(r,w)| = \left| \frac{1}{2\pi i} \int_{\Gamma_r} w(z)\, dz \right| = O(r^{3/2}),$$

and hence the assertion. □

Similar results, with the same proofs, are obtained for solutions of (IV). The only difference is that one has to work with $|w(z)| = O(|z|)$ rather than $|w(z)| = O(|z|^{1/2})$.

Theorem 6.3. *Let w be any transcendental solution of* (IV), *but not a solution of equation* (26) *below. Then $N_+(r,w) \le 2N_-(r,w) + O(\log r)$ and conversely, $N_-(r,w) \le 2N_+(r,w) + O(\log r)$ hold.*

Theorem 6.4. *For any transcendental solution w of* (IV)

$$|n_+(r,w) - n_-(r,w)| = O(r^2)$$

holds.

Equation (IV) has particular transcendental solutions $u = w$ for parameters $\beta = -2(1 \pm \alpha)^2$, α arbitrary. These solutions also solve the Riccati equation

$$\pm u' = -2(\alpha \pm 1) + 2zu + u^2. \tag{26}$$

By applying the Bäcklund transformation repeatedly to any solution u of (26), see [3, Theorem 25.1], we obtain the so-called *Weber–Hermite solutions*, which take the form $w(z) = R\big(z, u(z)\big)$. They are solutions of certain first order algebraic differential equations $P(z, w, w') = 0$, and have order of growth $\varrho = 2$. The substitution $u = \mp v'/v$, transforms (26) into the *Weber–Hermite* linear differential equation

$$v'' \mp 2zv' - 2(\alpha \pm 1)v = 0.$$

For more information the reader is again referred to [3].

7. Value distribution of the first transcendents

In [15] Shimomura obtained the lower estimate $n(r,w) \geq \text{const} \cdot r^{5/2}/\log r$ for any solution of Painlevé's first equation. We will give a different proof of a slightly weaker result, and then prove Theorem 7.3 below, which sharpens both results considerably; in both cases f denotes the canonical product having simple zeros exactly at the non-zero poles p_ν of w.

First of all we will give a new proof of

Theorem 7.1. *Every solution w of* (I) *has the representation*

$$w(z) = a_0 + \varepsilon_0 z^{-2} + \sum_{\nu=1}^{\infty} \left[(z-p_\nu)^{-2} - p_\nu^{-2}\right],$$

and hence

$$U(z) = a_0 z + a_1 - \varepsilon_0 z^{-1} - \sum_{\nu=1}^{\infty} z^2 (z-p_\nu)^{-1} p_\nu^{-2},$$

where either $\varepsilon_0 = 0$ and $a_0 = w(0)$, or else $\varepsilon_0 = 1$ and $a_0 = 0$, hold. The series $\sum_{\nu=1}^{\infty} |p_\nu|^{-\kappa}$ converges for $\kappa > \frac{5}{2}$, but diverges for $\kappa = \frac{5}{2}$.

Proof. The Residue Theorem applies to

$$S(z,r) = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{w(\zeta)}{\zeta(\zeta - z)} \, d\zeta;$$

the path of integration Γ_r is constructed in the same manner as in Section 4, such that the following holds: the interior domain of Γ_r contains the same poles of w as does the closed disk $|\zeta| \leq r$, the length of Γ_r is $O(r)$, and $w(\zeta) = O(|\zeta|^{1/2}) = O(r^{1/2})$ holds on Γ_r. Then

$$S(z,r) = \frac{w(z) - w(0)}{z} + \sum_{|p_\nu| \leq r} \frac{z - 2p_\nu}{(z-p_\nu)^2} = O(r^{-1/2})$$

as $r \to \infty$, uniformly for $|z| \leq \frac{1}{2} r$, say (to be modified for $w(0) = \infty$, replace $w(0)$ by $1/z$), which gives the desired result, since the Mittag-Leffler series converges locally uniformly. □

In the next step we prove a slightly weaker result than Shimomura's.

Let (q_ν) denote the sequence of zeros of w, and set $\{c_\nu\} = \{p_\nu\} \cup \{q_\nu\}$ with counting function $n(r) = n(r,w) + n(r,1/w)$. Then Proposition 2.3 applies with $L' = w$ and the set $H(R)$ defined in (11) with $\kappa = \frac{1}{4}$. For $\delta > 0$ fixed, and noting that $I_H(R,1) = O(R^3)$, we obtain

$$I_H(R,w) = O\big(R\left[T(2R,f) + n(2R)\log R\right]\big).$$

On the set $H(R)$ we have $|w(z)| \geq K_0|z|^{1/2}$, and hence the left-hand side of the above inequality is

$$I_H(R,w) \geq K_1 R^{7/2} - K_2 \sum_{|c_\nu|\leq 2R} |c_\nu|^{1/2}(|c_\nu|^{-1/4})^2 = K_1 R^{7/2} - K_2 n(2R).$$

This gives $r^{5/2} \leq O\big(T(r,f) + n(r)\log r\big)$ as $r \to \infty$, and, in particular,

$$\limsup_{r\to\infty} n(r,w)/(r^{5/2}\log r) > 0.$$

The assumption $n(r,w) = o(r^{5/2})$ has strong implications, one being that $U(z) = o(|z|^{3/2})$ as $z \to \infty$, outside a set E of planar density zero.

If only

$$\liminf_{r\to\infty}\,[n(r,w) + T(r,f)]/r^{5/2} = 0$$

is assumed, then there exists some sequence R_n tending to infinity, such that

$$\big(n(2R_n,w) + T(2R_n,f)\big)/R_n^{5/2} \leq \eta_n \to 0$$

as $n \to \infty$. We may assume that (η_n) is decreasing, and define a continuous, decreasing and piecewise linear function η: $(0,\infty) \to (0,\infty)$, which interpolates η_n at R_n. Then

$$I(R_n,U) \leq I(R_n, U - a_0 z - a_1) + O(R_n^3) \leq C R_n^{7/2}\eta(R_n)$$

holds for some constant $C > 0$. Defining the sets

$$E = \big\{z : |U(z)| \geq |z|^{3/2}\sqrt{\eta(|z|)}\,\big\} \quad \text{and} \quad E_n = E \cap \big\{z : R_n/2 \leq |z| \leq R_n\big\},$$

we obtain from

$$\begin{aligned} C R_n^{7/2}\eta(R_n) &\geq \int_{|z|\leq R_n} |U(z)|\,d(x,y) \geq \int_{E_n} |z|^{3/2}\sqrt{\eta(|z|)}\,d(x,y) \\ &\geq \big(\tfrac{1}{2}R_n\big)^{3/2}\sqrt{\eta(R_n)}\,\mathrm{area}(E_n) \end{aligned}$$

that $\mathrm{area}(E_n) \leq C_1 R_n^2\sqrt{\eta(R_n)}$. Clearly w has no poles on $D = \mathbf{C}\setminus E$. Let P denote the set of poles of w, and assume that D contains some sequence (z_n) satisfying $\mathrm{dist}(z_n,\mathsf{P}) \to \infty$, distance measured with respect to the metric $ds = |z|^{1/4}\,|dz|$. Then from [17, Proposition 9] follows

$$z_n^{-1/2} w(z_n) \to i\sqrt{1/6} \quad \text{and} \quad z_n^{-3/2} U(z_n) \to i\sqrt{2/27}$$

as $n \to \infty$, for some suitably chosen branch of the square-root. This, however, contradicts $z_n \in D$, hence we have proved that there exists $K > 0$, such that every disk $\{z : \mathsf{d}(z, c) < K\}$, $c \in D$, contains some pole of w.

Again let (z_n) denote any sequence in D, $z_n \to \infty$. Re-scaling equation (2) with local unit $r_n = z_n^{-1/4}$ (it does not matter that r_n is complex), and noting that $U(z_n) z_n^{-3/2} \to 0$ as $n \to \infty$, then leads to $y'^2 = 4y^3 + 2y$. Since w has some pole p_n with $\mathsf{d}(p_n, z_n) < K$, the limit function y is non-constant, and hence is a Weierstrass $\wp$-function with invariants $g_2 = -2$ and $g_3 = 0$, and period module $\Lambda = h(\mathbf{Z} + i\mathbf{Z})$, $h > 0$, independent of the sequence (z_n).

By [17, Proposition 8], the following *local* result holds:

Given $R > 0$ and $\sigma > 0$, there exists $r_0 > 0$, such that for any z_0 with $|z_0| \geq r_0$ there exists some lattice L, such that the Hausdorff distance with respect to the metric d between the image of $L \cap \{\zeta : |\zeta| < R\}$ under the map $\phi(\zeta) = \big(z_0^{5/4} + \frac{5}{4}\zeta\big)^{4/5} \sim z_0 + z_0^{-1/4}\zeta$ and the set $\mathsf{P} \cap \{z : |z - z_0| < R|z_0|^{-1/4}\}$ is less than σ.

In our case, however, this is a *global* result, since the lattice $L = \Lambda$ is always the same for any $z_0 \in D$. From this it is not hard to conclude that the number of poles of w in $|z| \leq r$ is at least $cr^{1/2}\text{area}\big(D \cap \{z : |z| \leq r\}\big)$, $c > 0$ an absolute constant. In particular we have $n(2R_n, w) \geq c_1 R_n^{5/2}$, which contradicts our assumption.

Remark. To derive this contradiction, it is actually not necessary to know the true distribution of poles. The union of disks $\{z : \mathsf{d}(z, p_\nu) < K\}$ covers D, and hence we have, for $D_n = D \cap \big\{z : \frac{1}{2}R_n \leq |z| \leq R_n\big\}$,

$$\text{area}(D_n) \leq \sum_{R_n/4 \leq |p_\nu| \leq 2R_n} 2K^2\pi(|p_\nu|^{-1/4})^2 = O\big(n(R_n, w)R_n^{-1/2}\big),$$

which gives $n(R_n, w) \geq \text{const} \cdot R_n^{5/2}$.

Anyway, we thus have proved

Theorem 7.2. *Any solution w of* (I) *satisfies $T(r, f) \geq \text{const} \cdot r^{5/2}$, and, in particular,*

$$\limsup_{r \to \infty} n(r, w)/r^{5/2} > 0.$$

Appendix: Radii of convergence

We deduce uniform lower bounds for the radii of convergence $r(p, \mathsf{h})$ of the series expansions about any pole p, $|p| \geq 1$, of the Painlevé transcendents. In [17] it was shown that $r(p, \mathsf{h}) \geq K \min\{|p|^{-1/4}, |\mathsf{h}|^{-1/6}\}$ for the first transcendents, $K > 0$ independent of p and h. In case (II) we leave it to the reader to prove that $r(p,\mathsf{h}) \geq K\min\{|p|^{-1/2}, |\mathsf{h}|^{-1/4}\}$, and focus on case (IV). From $w'^2 = w^4 + 4zw^3 + 4(z^2-\alpha)w^2 - 2\beta - 4wU$ and (IV) follows

$$w'' = 2w^3 + 6zw^2 + 4(z^2-\alpha)w - 2U. \tag{27}$$

Consider $w(z) = \sum_{n=0}^{\infty} c_n(z-p)^{n-1}$, with $c_0 = \varepsilon = \pm 1$. Then

$$U(z) = \sum_{n=0}^{\infty} a_n(z-p)^{n-1}$$

has coefficients $a_0 = -1$, $a_1 = 2\alpha p + 2\mathsf{h} - 2\varepsilon p$ (= local constant of integration) and

$$a_{n-2} = \frac{1}{n-3} \sum_{i+j=n-2} (c_i c_j + 2pc_{n-3} + 2c_{n-4}) \quad \text{for } n \geq 4. \tag{28}$$

Inserting the series expansions into (27) we obtain for $n \geq 4$, by equating coefficients,

$$\begin{aligned}(n-2)(n-1)c_n = &\sum_{i+j+k=n} 2c_ic_jc_k + \sum_{i+j=n-1} 6pc_ic_j + \sum_{i+j=n-2} 6c_ic_j \\ &+ 4(p^2-\alpha)c_{n-2} + 8pc_{n-3} + 4c_{n-4} - 2a_{n-2}.\end{aligned} \tag{29}$$

Let $0 < \theta < 1$ be arbitrary and set $M = K\max\{|p|, |\mathsf{h}|^{1/3}\}$, where $K \geq \theta^{-1}$ is chosen in such a way that $|c_n| \leq \theta M^n$ holds for $1 \leq n \leq 4$; this is possible with K only depending on θ, α and β. We just note that

$$45c_4 = -32\alpha + 20p^2 + \varepsilon(26 + 14\alpha^2 + 9\beta - 4\alpha p^2 - p^4 - 36p\mathsf{h}).$$

We also assume $|p| \geq 1$, and hence $M^{-1} \leq K^{-1} \leq \theta$. Starting from $|c_k| \leq \theta M^k$ for $0 < k < n$ and some integer $n \geq 4$, we obtain from (28)

$$|a_n| \leq K_1\theta M^{n-2} \leq K_1\theta^3 M^n$$

for some absolute constant $K_1 > 0$. Thus, noting that $|p| \leq \theta M$ and that the term $2c_n = 2c_0^2c_n$ appears three times in the first sum in (29), we may conclude that

$$(n^2 - 3n + 2 - 6)|c_n| \leq K_2 n^2\theta^2 M^n$$

holds with some absolute constant $K_2 > 0$. This proves $|c_n| \leq \theta M^n$ for all $n \geq 1$, provided θ is chosen sufficiently small, and hence the radius of convergence is

$$r(p,\mathsf{h}) \geq K^{-1}\min\{|p|^{-1}, |\mathsf{h}|^{-1/3}\}.$$

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Remark added in proof. In the meanwhile it was proved independently by S. Shimomura, by A. Hinkkanen and I. Laine, and also by the author, that the non-rational solutions of (II) and (IV) have lower order of growth at least $\frac{3}{2}$ and 2, respectively.

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