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REALIZING STEP FUNCTIONS AS HARMONIC MEASURE DISTRIBUTIONS OF PLANAR DOMAINS

Marie A. Snipes and Lesley A. Ward

University of Michigan, Department of Mathematics
Ann Arbor, MI 48109, U.S.A.; masnipes@umich.edu

Harvey Mudd College, Department of Mathematics
Claremont, CA 91711, U.S.A.; ward@math.hmc.edu

Abstract. The harmonic measure distribution function of a planar domain relates the geometry of the domain to the behaviour of Brownian motion in the domain. The value of the function $h(r)$ specifies the harmonic measure of the part of the boundary of the domain which lies within any given distance r of a fixed basepoint in the domain. A longterm goal is to realize all suitable functions as distribution functions, by explicit construction of appropriate domains. We show here that increasing step functions can be realized as distribution functions of discs with concentric circular arcs deleted from their interiors.

1. Introduction

We consider a function that captures some of the behaviour of Brownian motion in a planar domain. Let $D\subset \mathbf{C}$ be a bounded or unbounded domain, and fix a basepoint z_0 in D. For each $r\geq 0$, let $h(r)=h_{D,z_0}(r)$ denote the probability that the first hit on the boundary of D by a Brownian traveller starting at z_0 occurs within distance r of z_0. We call $h\colon [0,\infty)\to[0,1]$ the *harmonic measure distribution function* of D with respect to z_0.

By Kakutani's result [K] on the connection between Brownian motion and harmonic measure, $h(r)$ is equal to the harmonic measure in D at z_0 of the part of the boundary ∂D within distance r of z_0:

$$h(r)=\omega\bigl(z_0,\partial D\cap \overline{B(z_0,r)},D\bigr). \tag{1}$$

Here, given a subset E of the boundary ∂D, the harmonic function $u(z)=\omega(z,E,D)$ is the solution to the Dirichlet problem $\Delta u=0$ on D, with boundary values equal to 1 on E and 0 on the rest of the boundary.

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The current paper is motivated by these questions: Which functions $h(r)$ can arise as the harmonic measure distribution functions of domains? Given such a function, can one construct a domain which generates it?

It follows from the definition that $h(r)$ is a right-continuous function, increasing from 0 towards 1. In [WW1], it is shown that $h(r)$ does not uniquely determine the domain, and a lower bound for $h(r)$ (in the case of simply connected domains) is established using Beurling's solution to the Milloux problem [A]. Results on the possible asymptotic behaviours of $h(r)$ for $r \approx \operatorname{dist}(z_0, \partial D)$ are proved in [WW1] and [WW2]. A result on the effect of moving the basepoint z_0 is proved in [WW2]. Some of the questions posed in [WW1] are answered in [BS].

In this paper we identify a useful class of functions which do arise as harmonic measure distribution functions. A preliminary version of this work can be found in [S].

Theorem 1. *Every increasing step function on $[0, \infty)$ which increases from 0 to 1 by finitely many steps arises as the harmonic measure distribution function of some planar domain.*

A more precise version of Theorem 1 is stated as Theorem 2 in Section 2 below.

In [BH, Problem 6.116], Stephenson posed several questions about the related function

$$w(r) = w_D(r) = \omega\big(0, \partial D_r \cap \{|z| = r\}, D_r\big). \tag{2}$$

Here D is a domain in $\mathbf{C}$ containing 0, and D_r denotes the connected component of $D \cap \overline{B(0,r)}$ containing 0. For comparison, setting the basepoint for $h(r)$ at 0, we have

$$1 - w(r) = \omega\big(0, \partial D_r \cap B(0,r), D_r\big), \quad \text{while} \tag{3}$$

$$h(r) = \omega\big(0, \partial D \cap \overline{B(0,r)}, D\big). \tag{4}$$

The main difference is that $1 - w(r)$ asks about the location of a Brownian traveller's first exit from the bounded component D_r, while $h(r)$ asks about the location of the first exit from the whole domain D. In particular, the values of $1 - w(r)$ for small r, $0 < r < R$, are completely determined by the geometry of the part of the boundary within distance R of 0, while the values of $h(r)$ for $0 < r < R$ depend on the geometry of the full boundary of D, at all distances from 0.

Tsuji proved an upper estimate [T, Theorem III.67, p. 112] for a function $u_r(0)$ almost identical to Stephenson's $w(r)$.

In a remark after Theorem 2, we indicate how our results answer the analogues for our $h(r)$ of some of Stephenson's questions.

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2. Increasing step functions arise from circle domains

We prove that all step functions that increase from 0 to 1 through finitely many steps can be generated as the harmonic measure distribution functions of certain domains. In fact there are uncountably many essentially different domains for each such step function.

The simplest such step function takes the value 0 for small r, say for $0 < r < r_1$, and jumps to 1 at $r = r_1$. This step function is generated by the domain which is a disc of radius r_1 and has its basepoint at the origin. Further, the same step function arises from any domain whose boundary is an arc (or any subset of positive one-dimensional Hausdorff measure) of the circle of radius r_1 centred at the basepoint. (Brownian motion in the plane is recurrent, and a Brownian traveller will hit such a boundary with probability one, irrespective of whether the domain is bounded.)

This suggests our definition of *circle domains* as discs with deleted concentric boundary arcs. See Figure 1.

For convenience, by the *radius* of a circular arc we mean the radius of the circle on which the arc lies.

Definition 1. A *circle domain D with n arcs* is a bounded domain, containing the origin, whose boundary consists of finitely many concentric circular arcs $A_1, \dots, A_n$, of radii $r_1, \dots, r_n$ respectively, centred at the origin, together with a concentric boundary circle A_{n+1}, of radius r_{n+1}, enclosing the arcs. Here $0 < r_1 < \cdots < r_n < r_{n+1}$. Choose the origin as the basepoint of D. For $i = 1, \dots, n$, let x_i denote the angle subtended by A_i at 0; we call x_i the *length* of the arc A_i.

We assume that the midpoints of the boundary arcs all lie on the positive real axis, unless otherwise specified. Of course these midpoints could also be set at any preassigned angles from the positive real axis.

The harmonic measure distribution function $h_D(r)$ of a circle domain D is a right-continuous increasing step function, whose finitely many discontinuities may occur only at $r_1, \dots, r_{n+1}$. The step function has height 0 for $0 < r < r_1$, and height 1 for $r \geq r_{n+1}$. We denote by h_i the size of the jump in $h_D(r)$ at r_i, and by y_i the height of the step function for $r_i \leq r < r_{i+1}$, so that for $1 \leq i \leq n$,

$$h_i = h_D(r_i^+) - h_D(r_i^-) = h_D(r_i) - h_D(r_i^-) = \omega(0, A_i, D), \tag{5}$$

$$y_i = \sum_{1 \leq j \leq i} h_j = \omega(0, A_1 \cup \cdots \cup A_i, D). \tag{6}$$

Note that $0 \leq y_1 \leq \cdots \leq y_n \leq 1$.

We parametrize the space $\mathscr{X}$ of circle domains D with n arcs located at radii $r_1, \dots, r_{n+1}$ by the n-tuple of angles x_i subtended at 0 by the arcs.

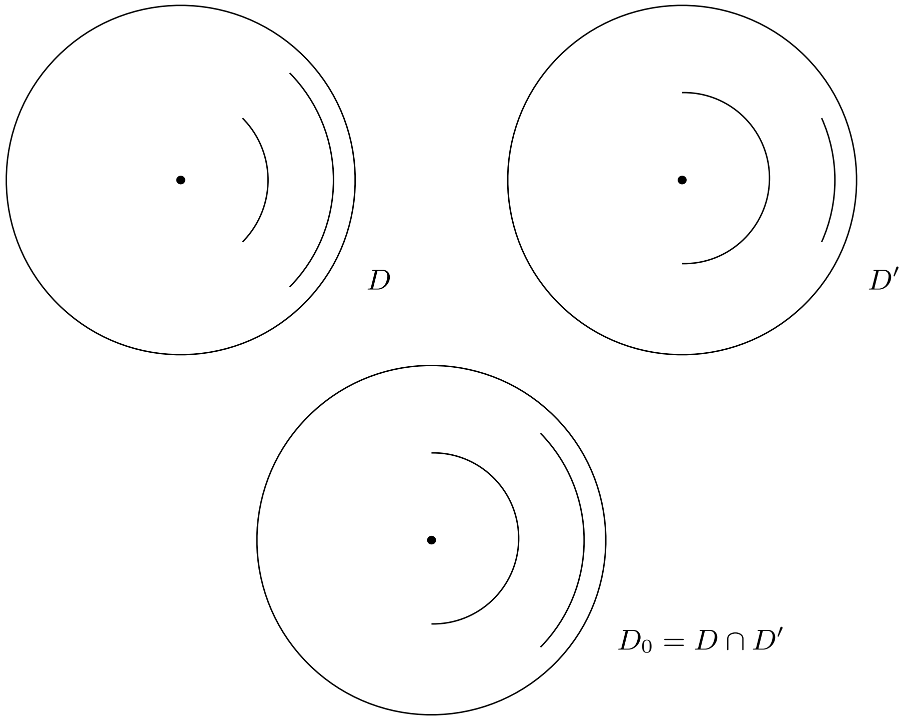

Figure 1. Circle domains with two arcs at fixed radii: D, D' with longer inner arc and shorter outer arc than in D, and $D_0 = D \cap D'$.

Definition 2. Fix an integer $n \geq 1$ and numbers r_i with $0 < r_1 < \cdots < r_{n+1}$. Let

$$\mathscr{X} = \mathscr{X}(n, r_1, \ldots, r_{n+1}) = \big\{(x_1, \ldots, x_n) : 0 \leq x_i \leq 2\pi,\ 1 \leq i \leq n\big\} \tag{7}$$

denote the *space of circle domains* with n arcs placed at radii r_i, $1 \leq i \leq n$, and with boundary circle of radius r_{n+1}.

We also consider the space $\mathscr{Y}$ of all right-continuous step functions, increasing from 0 to 1, with at most $n+1$ discontinuities, occurring only at the points $r_1, \ldots, r_{n+1}$. We parametrize $\mathscr{Y}$ by the n-tuple of values y_i taken by the step function on the intervals $[r_i, r_{i+1})$, for $1 \leq i \leq n$.

Definition 3. Fix an integer $n \geq 1$ and numbers r_i with $0 < r_1 < \cdots < r_{n+1}$. Let

$$\mathscr{Y} = \mathscr{Y}(n, r_1, \ldots, r_{n+1}) = \big\{(y_1, \ldots, y_n) : 0 \leq y_1 \leq \cdots \leq y_n \leq 1\big\} \tag{8}$$

denote the *space of step functions* associated with the radii $r_1, \ldots, r_{n+1}$.

The boundary $\partial\mathscr{X}$ consists of those elements $\vec{x} \in \mathscr{X}$ such that $x_i = 0$ or $x_i = 2\pi$ for at least one $i \in \{1, \ldots, n\}$. In other words, such an $\vec{x}$ represents a circle domain whose i^{th} arc A_i has length zero (so A_i is either the empty set or a single point) or is a full circle. The boundary $\partial\mathscr{Y}$ consists of those elements $\vec{y} \in \mathscr{Y}$ such that at least one of the following holds: $y_1 = 0$, or $y_n = 1$, or $y_i = y_{i+1}$ for some $i \in \{1, \dots, n-1\}$. In other words, such a $\vec{y}$ represents a step function in which at least two adjacent steps have the same height.

As noted above, the step function generated as the harmonic measure distribution function of a circle domain in $\mathscr{X}$ is an element of $\mathscr{Y}$. We now prove that all suitable step functions (in other words, all step functions in $\mathscr{Y}$) can be generated as harmonic measure distribution functions of circle domains. Here is a more precise version of Theorem 1.

Theorem 2. *Let $f(r)$ be a right-continuous step function, increasing from 0 to 1, with its discontinuities at $r_1, \dots, r_{n+1}$, where $0 < r_1 < \dots < r_{n+1}$. Then there exists a circle domain D with n arcs whose harmonic measure distribution function $h_D(r)$ is equal to $f(r)$. The radii of the n arcs and of the boundary circle in D are given by $r_1, \dots, r_n$ and by r_{n+1} respectively.*

Proof. Fix the number n of concentric circular arcs, and the radii $r_1, \dots, r_{n+1}$. Let $F\colon \mathscr{X} \to \mathscr{Y}$ be the function which takes the point $(x_1, \dots, x_n) \in \mathscr{X}$ representing the circle domain D with arc lengths $x_1, \dots, x_n$ to the point $(y_1, \dots, y_n) \in \mathscr{Y}$ representing the harmonic measure distribution function $h_D(r)$ of D. Here $h_D(r)$ is an increasing step function with step heights 0, $y_1, \dots, y_n$, 1.

(Numerical evidence [C] suggests that F is nonlinear. Specifically, it suggests that in the case of circle domains with two arcs at fixed radii, F is not the restriction to $\mathscr{X}$ of a linear transformation of $\mathbf{R}^2$.)

The map F is not one-to-one on the boundary $\partial\mathscr{X}$ at points where any of the lengths x_i are equal to 2π, for $i < n+1$. For once the i^{th} arc A_i becomes a full circle, the outer arcs are no longer accessible to a Brownian traveller from the origin. Therefore changing the lengths of any arcs with radii larger than r_i does not change $h_D(r)$.

We observe that F maps the boundary of $\mathscr{X}$ into the boundary of $\mathscr{Y}$, and the interior $\operatorname{Int}\mathscr{X}$ into the interior $\operatorname{Int}\mathscr{Y}$. First, if $\vec{x} \in \partial\mathscr{X}$ and $x_i = 0$, then the harmonic measure h_i of A_i is zero, and so $y_{i-1} = y_i$ and $\vec{y} = F(\vec{x})$ is in $\partial\mathscr{Y}$. If $\vec{x} \in \partial\mathscr{X}$ and $x_i = 2\pi$, then $1 = \omega(0, A_1 \cup \dots \cup A_i, D) = y_i = y_{i+1} = \dots = y_n$, and so $\vec{y} = F(\vec{x})$ is in $\partial\mathscr{Y}$. Second, if $\vec{x} \in \operatorname{Int}\mathscr{X}$, then each arc A_j has length strictly between 0 and 2π, so each arc has a positive (but less than one) probability of being hit by a Brownian traveller from 0, so the jumps h_j in the step function $\vec{y} = F(\vec{x})$ are all greater than zero. Therefore $\vec{y} \notin \partial\mathscr{Y}$, and so $F(\operatorname{Int}\mathscr{X}) \subset \operatorname{Int}\mathscr{Y}$.

(We show below, in part (iii) of the proof, that in fact F maps $\partial\mathscr{X}$ onto $\partial\mathscr{Y}$ and $\operatorname{Int}\mathscr{X}$ onto $\operatorname{Int}\mathscr{Y}$.)

We show that F is a homeomorphism between the interiors of $\mathscr{X}$ and $\mathscr{Y}$, and that F is a continuous map of $\mathscr{X}$ onto $\mathscr{Y}$. The main part of the proof is to show that F is onto.

(i) $F\colon \mathscr{X} \to \mathscr{Y}$ *is continuous.* One can easily see that, since harmonic measure is continuous up to the boundary, F is continuous up to and including the boundary.

(ii) F *is one-to-one on the interior of* $\mathscr{X}$. Consider two distinct circle domains D and D' in $\mathscr{X}$. Let S be the collection of arcs in ∂D that are strictly *longer* than their counterparts (at the same radii) in the collection $S' \subset \partial D'$. We may assume S is non-empty, exchanging D and D' if necessary. Let T be the collection of those arcs (if any) in ∂D that are strictly *shorter* than their counterparts in $T' \subset \partial D'$. Construct a new domain $D^* \in \mathscr{X}$ using at each radius r_i the longer of the boundary arcs A_i and A_i'. In particular $S^* = S$ and $T^* = T'$. Then by the monotonicity of harmonic measure,

$$\omega(0, S, D) \geq \omega(0, S^*, D^*) \geq \omega(0, S', D'). \tag{9}$$

Since S is non-empty, the second inequality is actually strict. But then D and D' cannot have the same step function $h(r)$, since for instance $\omega(0, S, D)$ is the sum of the jump heights in $h(r)$ at the values r_i corresponding to the arcs in S, and this is strictly greater than the analogous sum for D'.

(iii) $F\colon \mathscr{X} \to \mathscr{Y}$ *is onto.* We use induction on the dimension n of $\mathscr{X}$ (and $\mathscr{Y}$).

When D has only one boundary arc, then $\mathscr{X} = [0, 2\pi]$, $\mathscr{Y} = [0, 1]$, $F(0) = 0$, and $F(2\pi) = 1$, and so by continuity F is onto.

Suppose that F maps $\mathscr{X}$ onto $\mathscr{Y}$ when there are n arcs. Consider the case of $n+1$ arcs, so $\mathscr{X} = \mathscr{X}(n+1, r_1, \dots, r_{n+2})$ and $\mathscr{Y} = \mathscr{Y}(n+1, r_1, \dots, r_{n+2})$. We show first that F maps the interior $\operatorname{Int}\mathscr{X}$ onto $\operatorname{Int}\mathscr{Y}$, and then that F maps the boundary $\partial\mathscr{X}$ onto the boundary $\partial\mathscr{Y}$.

We have shown that $F\colon \mathscr{X} \to \mathscr{Y}$ is continuous and injective on the open subset $\operatorname{Int}\mathscr{X}$ of $\mathbf{R}^{n+1}$. Therefore the image $F(\operatorname{Int}\mathscr{X})$ is an open subset of $\mathbf{R}^{n+1}$, by Brouwer's theorem on invariance of domain. (See for example [M, p. 207].) Since $F(\operatorname{Int}\mathscr{X}) \subset \operatorname{Int}\mathscr{Y}$, we have that $F(\operatorname{Int}\mathscr{X})$ is an open subset of $\operatorname{Int}\mathscr{Y}$.

Suppose that $F(\operatorname{Int}\mathscr{X})$ is not the whole of $\operatorname{Int}\mathscr{Y}$. We claim there is a point $\vec{y} \in \partial F(\mathscr{X}) \cap \operatorname{Int}\mathscr{Y}$. For by assumption, there is a point $\vec{y}_1 \in \operatorname{Int}\mathscr{Y} \setminus F(\mathscr{X})$. Take any point $\vec{y}_2$ in $F(\operatorname{Int}\mathscr{X}) \subset \operatorname{Int}\mathscr{Y}$. Since $\mathscr{Y}$ is convex, the line segment γ joining $\vec{y}_1$ to $\vec{y}_2$ lies entirely in $\operatorname{Int}\mathscr{Y}$. Since $\vec{y}_2$ is in $F(\operatorname{Int}\mathscr{X}) \subset F(\mathscr{X})$ and $\vec{y}_1$ is not in $F(\mathscr{X})$, there is some point $\vec{y}$ on γ such that $\vec{y} \in \partial F(\mathscr{X})$. So $\vec{y} \in \partial F(\mathscr{X}) \cap \operatorname{Int}\mathscr{Y}$.

Next, $F(\mathscr{X})$ is compact, because $\mathscr{X}$ is compact and F is continuous. Therefore $\vec{y} \in F(\mathscr{X})$. So there is some $\vec{x} \in \mathscr{X}$ such that $\vec{y} = F(\vec{x})$. Since $F(\partial\mathscr{X}) \subset \partial\mathscr{Y}$ and $\vec{y} \in \operatorname{Int}\mathscr{Y}$, $\vec{x}$ must lie in $\operatorname{Int}\mathscr{X}$, and so $\vec{y} \in F(\operatorname{Int}\mathscr{X})$.

As noted above, $F(\operatorname{Int}\mathscr{X})$ is an open subset of $\mathscr{Y}$. So there is a neighbourhood U of $\vec{y}$ such that $\vec{y} \in U \subset F(\operatorname{Int}\mathscr{X})$. This contradicts $\vec{y} \in \partial F(\mathscr{X})$, since U does not intersect the complement of $F(\mathscr{X})$.

Therefore F maps $\operatorname{Int}\mathscr{X}$ onto $\operatorname{Int}\mathscr{Y}$, as required.

Next we consider the boundary. Let $\vec{y}$ be a step function in the boundary $\partial\mathscr{Y}(n+1, r_1, \ldots, r_{n+2})$. Denote by h_j the height of the j^{th} step, i.e., $h_1 = y_1$, $h_j = y_j - y_{j-1}$ for $2 \le j \le n+1$, and $h_{n+2} = 1 - y_{n+1}$. Then for some $j \in \{1, \ldots, n+2\}$, the jump $h_j = 0$ in $\vec{y}$. Therefore, $\vec{y}$ is also an element of the new space of step functions $\mathscr{Y}'(n, r'_1, \ldots, r'_{n+1})$, where $\{r'_1, \ldots, r'_{n+1}\} = \{r_1, \ldots, r_{n+2}\} \setminus \{r_j\}$. By the induction hypothesis, there exists a circle domain $\vec{x}$ with n arcs in the space $\mathscr{X}'(n, r'_1, \ldots, r'_{n+1})$ such that $F(\vec{x}) = \vec{y}$. The circle domain $\vec{x}$ is also an element of the boundary $\partial\mathscr{X}(n+1, r_1, \ldots, r_{n+2})$, with $x_j = 0$, where x_j is the length of the arc A_j at radius r_j. Thus F maps $\partial\mathscr{X}$ onto $\partial\mathscr{Y}$.

(iv) $F^{-1}\colon \operatorname{Int}\mathscr{Y} \to \operatorname{Int}\mathscr{X}$ *is continuous.* Since F is a continuous function from the compact set $\mathscr{X}$ to the compact subset $\mathscr{Y}$ of $\mathbf{R}^n$, it maps closed subsets of $\mathscr{X}$ to closed subsets of $\mathscr{Y}$.

We have shown that $F\colon \operatorname{Int}\mathscr{X} \to \operatorname{Int}\mathscr{Y}$ is a homeomorphism, and that F maps $\mathscr{X}$ onto $\mathscr{Y}$. This completes the proof of Theorem 2. □

Remark 1. *Non-uniqueness*: In fact, there are uncountably many circle domains that generate the given step function. The proof above establishes the existence of such a domain with the midpoints of the arcs located on the positive real axis. The same arguments show that there is such a domain with the arc midpoints set at any preassigned angles. One could also construct examples in which several boundary arcs lie on the same circle.

Remark 2. *Analogues of Stephenson's questions*: First, the circle domains discussed in Remark 1 give examples, different from those in [WW1], which answer negatively the analogue for $h(r)$ of Stephenson's question (b) on uniqueness. Namely, there are essentially different domains (differing on sets of positive capacity, in particular) whose harmonic measure distribution functions are all the *same* step function. So h_D, or h_D restricted to some interval $R_1 < r < R_2$, do not uniquely determine the domain.

Next, Theorem 2 is related to the analogue of question (c): Given a function h_D, can one 'reconstruct' D? However, our theorem refers only to the special case of suitable step functions, and the reconstruction is not completely explicit, in that the lengths of the boundary arcs are not specified.

Finally, one cannot infer the connectivity of D (analogue of question (d)) from h_D, at least within the class of step functions. For example, the step function with a single jump from 0 to 1 at r_1 arises as the harmonic measure distribution function of *every* domain D whose boundary lies entirely on the circle $|z| = r_1$ and has positive harmonic measure (or, equivalently, $\partial D \subset \{|z| = r_1\}$ and ∂D has positive length). Here ∂D can have any number of components.

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