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ON AN ASYMPTOTICALLY SHARP VARIANT OF HEINZ'S INEQUALITY

In memory of Professor Shozo Koshi

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Abstract. In 1958, E. Heinz obtained a lower bound for $|\partial_x F|^2 + |\partial_y F|^2$, where F is a one-to-one harmonic mapping of the unit disc onto itself keeping the origin fixed. Assuming additionally that F is a K-quasiconformal mapping we aim at giving a variant of Heinz's inequality which is asymptotically sharp as K tends to 1. To this end we prove a variant of Schwarz's lemma for such a mapping F.

Introduction

Assume that F is a one-to-one harmonic mapping of the unit disc $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ onto itself normalized by $F(0) = 0$. In 1958, E. Heinz proved that the inequality

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{2}{\pi^2} \tag{0.1}$$

holds for every $z = x + \mathrm{i}y \in \mathbf{D}$; cf. [2]. Given a function $f\colon \mathbf{T} \to \mathbf{C}$ integrable on the unit circle $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$ we denote by $\mathrm{P}\,[f](z)$ the Poisson integral of f at $z \in \mathbf{D}$, i.e.

$$\mathrm{P}\,[f](z) := \frac{1}{2\pi} \int_{\mathbf{T}} f(u) \operatorname{Re} \frac{u+z}{u-z}\, |du|, \quad z \in \mathbf{D}. \tag{0.2}$$

Write $\mathrm{Hom}^+(\mathbf{T})$ for the class of all sense-preserving homeomorphic self-mappings of $\mathbf{T}$. In case $F = \mathrm{P}\,[f]$ for some $f \in \mathrm{Hom}^+(\mathbf{T})$ the estimation (0.1) may be improved as follows:

$$\inf_{z\in\mathbf{D}} \bigl(|\partial_x F(z)|^2 + |\partial_y F(z)|^2\bigr) \geq \frac{2}{\pi^2} + \frac{1}{2} d_f^2 + \frac{1}{2} \max\{d_f, 2d_f^3\}, \tag{0.3}$$

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where

$$d_f := \operatorname*{ess\,inf}_{z\in\mathbf{T}} |f'(z)|; \tag{0.4}$$

cf. [10, Theorem 0.4]. Here for every $z \in \mathbf{T}$

$$f'(z) := \lim_{u\to z} \frac{f(u)-f(z)}{u-z} \tag{0.5}$$

provided the limit exists and $f'(z) := 0$ otherwise. If the mapping F coincides with a rotation (around the origin), then the left-hand side in (0.3) equals 2, while the right-hand side equals $2/\pi^2 + 3/2$, because $d_f = 1$. As shown in [10, Lemma 0.1], for every $f \in \mathrm{Hom}^+(\mathbf{T})$, $d_f \le 1$, so the constant $2/\pi^2 + 3/2$ is the best possible, and thereby the estimation (0.3) is not so precise at least for F close to a rotation in the sense that d_f is close to 1.

It is a natural problem to study estimations of the type (0.1) in two cases:

(i) F is a K-quasiconformal mapping for some $K \ge 1$;
(ii) $F = \mathrm{P}[f]$ for some $f \in \mathrm{Hom}^+(\mathbf{T})$ which admits a K-quasiconformal extension to $\mathbf{D}$ for some $K \ge 1$.

In the first case F has a continuous extension F^* to a homeomorphic self-mapping of the closure $\overline{\mathbf{D}}$; cf. [5, Chapter I, Theorem 8.2]. Thus the limiting valued function $f := F^*_{|\mathbf{T}}$ belongs to $\mathrm{Hom}^+(\mathbf{T})$, and so $F = \mathrm{P}[f]$. Conversely, for each $K > 1$ there exists $f \in \mathrm{Hom}^+(\mathbf{T})$ which admits a K-quasiconformal extension to $\mathbf{D}$, but $F = \mathrm{P}[f]$ is not a quasiconformal mapping; cf. [6], [12], [4], [8], [9]. Therefore the second case yields the essentially wider class of F than the first one. In what follows we consider only the first, simpler case.

For $K \ge 1$ set $\mathrm{QCH}(\mathbf{D}; K)$ for the class of all K-quasiconformal and harmonic self-mappings of $\mathbf{D}$. We wish to find a lower bound of the left-hand side in (0.1) by means of $K \ge 1$ only, provided $F \in \mathrm{QCH}(\mathbf{D}; K)$ satisfies $F(0) = 0$. According to [10, Theorem 0.6],

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \ge \frac{2}{\pi^2}\left(1 + \frac{1}{K}\right)^2, \quad z \in \mathbf{D}. \tag{0.6}$$

If $K = 1$, then the mapping F coincides with a rotation, so the left-hand side in (0.6) equals 2 for all $z \in \mathbf{D}$, while the right-hand side equals $8/\pi^2$. Thus there is a big gap between the both sides in (0.6), and in consequence, the estimation (0.6) is not satisfactory for small K close to 1.

In this work we are interested in finding a continuous decreasing function $M\colon [1; +\infty) \to (0; 1]$ such that

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \ge 2M(K), \quad z \in \mathbf{D}, \tag{0.7}$$

and $M(K) \to M(1) = 1$ as $K \to 1^+$. Theorem 2.2 in Section 2, which is our main result, gives a solution. In general, the proof of Theorem 2.2 borrows from techniques developed in [10]. The progress now is due to new results discussed in Section 1. The first result (cf. Lemma 1.1) deals with a variant of Schwarz's lemma for $F \in \mathrm{QCH}(\mathbf{D}; K)$ normalized by $F(0) = 0$. The estimation (1.4) improves essentially for z close to the boundary $\mathbf{T}$ and small K close to 1, the classical one ([2, Lemma])

$$|F(z)| \le \frac{4}{\pi} \arctan |z|, \quad z \in \mathbf{D}, \tag{0.8}$$

used by Heinz in the proof of (0.1). The second and third results (cf. Lemmas 1.3 and 1.4) give an asymptotically sharp lower bound of the value

$$\inf_{z\in\mathbf{T}} \liminf_{r\to 1} \frac{|F^*(z) - F(rz)|}{1-r}. \tag{0.9}$$

The estimation (1.15) also leads to Theorem 2.1 in Section 2, which is our second main result. It provides an asymptotically sharp lower estimation of d_f in terms of K for $f = F^*|_{\mathbf{T}}$.

1. Auxiliary results

In the theory of plane quasiconformal mappings the Hersch–Pfluger distortion function Φ_K, $K > 0$, plays important roles; cf. e.g. the book of Vuorinen [11]. It is defined by the equalities

$$\Phi_K(r) := \mu^{-1}\big(\mu(r)/K\big), \quad 0 < r < 1; \quad \Phi_K(0) := 0, \ \Phi_K(1) := 1, \tag{1.1}$$

where μ stands for the module of the Grötzsch extremal domain $\mathbf{D} \setminus [0, r]$; cf. [3] and [5, pp. 53 and 63]. The function μ can be expressed explicitly by means of the complete elliptic integral of the first kind

$$\mathscr{K}(r) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}}, \quad 0 < r < 1, \tag{1.2}$$

in the form

$$\mu(r) = \frac{\pi}{2} \frac{\mathscr{K}\big(\sqrt{1-r^2}\,\big)}{\mathscr{K}(r)}, \quad 0 < r < 1. \tag{1.3}$$

Lemma 1.1. *If $K \ge 1$, $F \in \mathrm{QCH}(\mathbf{D}; K)$ and $F(0) = 0$, then*

$$|F(z)| \le \mathrm{P}\,[\Psi_K](|z|), \quad z \in \mathbf{D}, \tag{1.4}$$

where

$$\Psi_K(\mathrm{e}^{\mathrm{i}t}) := \begin{cases} 2\Phi_K\bigl(\cos\tfrac{1}{2}t\bigr)^2 - 1, & 0 \le |t| \le \tfrac{1}{2}\pi, \\ 2\Phi_{1/K}\bigl(\cos\tfrac{1}{2}t\bigr)^2 + 4\Phi_K\bigl(1/\sqrt{2}\bigr)^2 - 3, & \tfrac{1}{2}\pi \le |t| \le \pi. \end{cases} \tag{1.5}$$

Proof. Defining $f := F^*|_{\mathbf{T}}$ we have

$$F(z) = \mathrm{P}\,[f](z), \quad z \in \mathbf{D}.$$

Hence, using the polar coordinates in (0.2), we obtain

$$F(r\mathrm{e}^{\mathrm{i}\varphi}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{\mathrm{i}t})\mathrm{P}_r(\varphi - t)\,\mathrm{d}t, \quad 0 \le r < 1,\ \varphi \in \mathbf{R}, \tag{1.6}$$

where

$$\mathrm{P}_r(\theta) := \operatorname{Re}\frac{1 + r\mathrm{e}^{\mathrm{i}\theta}}{1 - r\mathrm{e}^{\mathrm{i}\theta}} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \quad 0 \le r < 1,\ \theta \in \mathbf{R}.$$

Setting for any $\varphi \in \mathbf{R}$, $f_\varphi(t) := f\bigl(\mathrm{e}^{\mathrm{i}(\varphi+t)}\bigr) + f\bigl(\mathrm{e}^{\mathrm{i}(\varphi-t)}\bigr)$, $t \in \mathbf{R}$, we conclude from (1.6) that for a given $z = r\mathrm{e}^{\mathrm{i}\varphi} \in \mathbf{D}$,

$$\begin{aligned} 2\pi F(z) &= \int_{-\pi}^{\pi} \mathrm{P}_r(t - \varphi) f(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t \\ &= \int_{-\pi}^{\pi} \mathrm{P}_r(t) f(\mathrm{e}^{\mathrm{i}(\varphi+t)})\,\mathrm{d}t \\ &= \int_{0}^{\pi} \mathrm{P}_r(t) f_\varphi(t)\,\mathrm{d}t \\ &= \int_{0}^{\pi/2} \mathrm{P}_r(t) f_\varphi(t)\,\mathrm{d}t + \int_{0}^{\pi/2} \mathrm{P}_r(\pi - t) f_\varphi(\pi - t)\,\mathrm{d}t \\ &= \int_{0}^{\pi/2} \bigl(\mathrm{P}_r(t) - \mathrm{P}_r(\pi - t)\bigr) f_\varphi(t)\,\mathrm{d}t \\ &\quad + \int_{0}^{\pi/2} \mathrm{P}_r(\pi - t)[f_\varphi(t) + f_\varphi(\pi - t)]\,\mathrm{d}t. \end{aligned}$$

Hence

$$\begin{aligned} |F(z)| \le{}& \frac{1}{2\pi}\int_{0}^{\pi/2} \bigl(\mathrm{P}_r(t) - \mathrm{P}_r(\pi - t)\bigr)|f(\mathrm{e}^{\mathrm{i}(\varphi+t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi-t)})|\,\mathrm{d}t \\ &+ \frac{1}{2\pi}\int_{0}^{\pi/2} \mathrm{P}_r(\pi - t)|f(\mathrm{e}^{\mathrm{i}(\varphi+t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi-\pi+t)})|\,\mathrm{d}t \\ &+ \frac{1}{2\pi}\int_{0}^{\pi/2} \mathrm{P}_r(\pi - t)|f(\mathrm{e}^{\mathrm{i}(\varphi-t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi+\pi-t)})|\,\mathrm{d}t. \end{aligned} \tag{1.7}$$

If $z, w \in \mathbf{T}$ and if $w = z\mathrm{e}^{\mathrm{i}s}$ for some $s \in \mathbf{R}$, then

$$
\begin{aligned}
|z+w| &= |z + z\mathrm{e}^{\mathrm{i}s}| = |z|\,|\mathrm{e}^{\mathrm{i}s/2}|\,|\mathrm{e}^{\mathrm{i}s/2} + e^{-is/2}| \\
&= 2\bigl|\cos\tfrac{1}{2}s\bigr| = 2\bigl|2\bigl(\cos\tfrac{1}{4}s\bigr)^2 - 1\bigr|.
\end{aligned} \tag{1.8}
$$

Since F is a K-quasiconformal mapping normalized by $F(0) = 0$, we see by the quasi-invariance of the harmonic measure that

$$
\Phi_{1/K}\bigl(\cos\tfrac{1}{4}(\beta-\alpha)\bigr) \le \cos\tfrac{1}{4}\gamma \le \Phi_K\bigl(\cos\tfrac{1}{4}(\beta-\alpha)\bigr), \tag{1.9}
$$

provided $\alpha, \beta, \gamma \in \mathbf{R}$ are any numbers such that $\alpha \le \beta < \alpha + 2\pi$, $0 \le \gamma < 2\pi$ and

$$
f(\mathrm{e}^{\mathrm{i}\beta}) = \mathrm{e}^{\mathrm{i}\gamma} f(\mathrm{e}^{\mathrm{i}\alpha}); \tag{1.10}
$$

see e.g. [7, (2.3.9)]. Applying now (1.8), (1.9), (1.10) and the identity ([1, Theorem 3.3])

$$
\Phi_K(t)^2 + \Phi_{1/K}\bigl(\sqrt{1-t^2}\,\bigr)^2 = 1, \quad 0 \le t \le 1, \tag{1.11}
$$

we obtain for every $t \in \bigl[0; \tfrac{1}{2}\pi\bigr]$,

$$
\begin{aligned}
\frac{1}{2}|f(\mathrm{e}^{\mathrm{i}(\varphi+t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi-\pi+t)})| &\le 2\Phi_K\left(\cos\frac{\pi}{4}\right)^2 - 1 = 2\Phi_K\left(\frac{1}{\sqrt{2}}\right)^2 - 1, \\
\frac{1}{2}|f(\mathrm{e}^{\mathrm{i}(\varphi-t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi+\pi-t)})| &\le 2\Phi_K\left(\cos\frac{\pi}{4}\right)^2 - 1 = 2\Phi_K\left(\frac{1}{\sqrt{2}}\right)^2 - 1, \\
\frac{1}{2}|f(\mathrm{e}^{\mathrm{i}(\varphi+t)}) + f(\mathrm{e}^{\mathrm{i}(\varphi-t)})| &\le 2\Phi_K\left(\cos\frac{t}{2}\right)^2 - 1.
\end{aligned}
$$

Hence by (1.7) we have

$$
\begin{aligned}
|F(z)| &\le \frac{1}{\pi}\int_0^{\pi/2} \bigl(\mathrm{P}_r(t) - \mathrm{P}_r(\pi-t)\bigr)\bigl(2\Phi_K\bigl(\cos(\tfrac{1}{2}t)\bigr)^2 - 1\bigr)\,\mathrm{d}t \\
&\quad + \frac{2}{\pi}\int_0^{\pi/2} \mathrm{P}_r(\pi-t)\bigl(2\Phi_K\bigl(1/\sqrt{2}\,\bigr)^2 - 1\bigr)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^{\pi/2} \mathrm{P}_r(t)\bigl(2\Phi_K\bigl(\cos(\tfrac{1}{2}t)\bigr)^2 - 1\bigr)\,\mathrm{d}t \\
&\quad + \frac{1}{\pi}\int_{\pi/2}^{\pi} \mathrm{P}_r(t)\bigl(4\Phi_K\bigl(1/\sqrt{2}\,\bigr)^2 - 1 - 2\Phi_K\bigl(\sin(\tfrac{1}{2}t)\bigr)^2\bigr)\,\mathrm{d}t.
\end{aligned}
$$

Applying now (1.11) and (1.5) we obtain

$$
|F(z)| \le \frac{1}{\pi}\int_0^{\pi} \mathrm{P}_r(t)\Psi_K(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t. \tag{1.12}
$$

Since $\Psi_K(\mathrm{e}^{\mathrm{i}t}) = \Psi(\mathrm{e}^{-\mathrm{i}t})$ for $t \in \mathbf{R}$, (1.12) leads to (1.4), which ends the proof. □

Remark 1.2. From (1.5) it follows that $|\Psi_K(\mathrm{e}^{\mathrm{i}t})| \leq 1$ for all $K \geq 1$ and $\mathrm{e}^{\mathrm{i}t} \in \mathbf{T}$. This combined with (1.4) yields $0 \leq \mathrm{P}[\Psi_K](|z|) < 1$ for all $K \geq 1$ and $z \in \mathbf{D}$. Thus the estimation (1.4) gives a good control for $|z|$ close to 1. In the case where $K = 1$ we have $\Psi_1(\mathrm{e}^{\mathrm{i}t}) = \cos t$ for $t \in [0; \pi]$ and hence

$$|F(z)| = |z| = \mathrm{P}[\Psi_1](|z|), \quad z \in \mathbf{D}.$$

Thus the estimation (1.4) is asymptotically sharp for K close to 1.

Let $\mathrm{L}^1(\mathbf{T})$ denote the space of all complex-valued functions Lebesgue integrable on $\mathbf{T}$. For every $f \in \mathrm{L}^1(\mathbf{T})$ define

$$f_{\mathbf{T}} := \frac{1}{2\pi} \int_{\mathbf{T}} f(u)\, |\mathrm{d}u| \tag{1.13}$$

and write $\mathrm{C}_{\mathbf{T}}[f]$ for the Cauchy singular integral of f, i.e. for every $z \in \mathbf{T}$,

$$\mathrm{C}_{\mathbf{T}}[f](z) := \mathrm{P.V.}\, \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(u)}{u - z}\, \mathrm{d}u := \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\mathbf{T} \setminus \mathbf{T}(z,\varepsilon)} \frac{f(u)}{u - z}\, \mathrm{d}u \tag{1.14}$$

whenever the limit exists and $\mathrm{C}_{\mathbf{T}}[f](z) := 0$ otherwise. Here and subsequently, $\mathbf{T}(\mathrm{e}^{\mathrm{i}x}, \varepsilon) := \{\mathrm{e}^{\mathrm{i}t} \in \mathbf{T} : |t - x| < \varepsilon\}$ and integration along any arc $I \subset \mathbf{T}$ is understood under counterclockwise orientation.

Lemma 1.3. *If $K \geq 1$, $F \in \mathrm{QCH}(\mathbf{D}; K)$ and $F(0) = 0$, then the inequality*

$$\liminf_{r \to 1} \frac{|F^*(z) - F(rz)|}{1 - r} \geq \max\left\{\frac{2}{\pi}, L_K\right\}, \tag{1.15}$$

holds for every $z \in \mathbf{T}$, where

$$L_K := \frac{2}{\pi} \int_0^{\Phi_{1/K}\left(1/\sqrt{2}\right)^2} \frac{\mathrm{d}t}{\Phi_K\left(\sqrt{t}\right)\Phi_{1/K}\left(\sqrt{1-t}\right)}. \tag{1.16}$$

Proof. By [10, (2.11)] the left-hand side in (1.15) is not less than $2/\pi$, thus we have to show only that it is equal to or greater than L_K. Fix $K \geq 1$. By (1.1), (1.2) and (1.3) we see that the function Φ_K is continuously differentiable on $(0; 1)$. Moreover, Φ_K is continuous and increasing on $[0; 1]$. Thus the function Φ_K is absolutely continuous on $[0; 1]$, and so is Ψ_K. Setting $\psi(t) := \Psi_K(\mathrm{e}^{\mathrm{i}t})$, $t \in [-\pi; \pi]$, we have $\psi'(t) = \mathrm{i}\Psi_K'(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}t}$ for $t \in [-\pi; \pi]$. Hence

$$\begin{aligned} 2\,\mathrm{C}_{\mathbf{T}}[\Psi_K'](1) &= \frac{1}{\pi\mathrm{i}} \lim_{\varepsilon \to 0^+} \int_{\mathbf{T} \setminus \mathbf{T}(1,\varepsilon)} \frac{\Psi_K'(z)}{z - 1}\, \mathrm{d}z \\ &= \frac{1}{\pi\mathrm{i}} \lim_{\varepsilon \to 0^+} \int_{\varepsilon < |t| \leq \pi} \frac{\psi'(t)}{\mathrm{e}^{\mathrm{i}t} - 1}\, \mathrm{d}t. \end{aligned} \tag{1.17}$$

Since $\psi(t) = \psi(-t)$ for $t \in [-\pi;\pi]$, we have $\psi'(t) = -\psi'(-t)$ for $t \in [-\pi;\pi]$. Then by (1.17),

$$
\begin{aligned}
2\,\mathrm{C}_{\mathbf{T}}[\Psi_K'](1) &= \frac{1}{\pi \mathrm{i}} \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon}^{\pi} \frac{\psi'(t)}{\mathrm{e}^{\mathrm{i}t}-1}\,\mathrm{d}t + \int_{-\pi}^{-\varepsilon} \frac{\psi'(t)}{\mathrm{e}^{\mathrm{i}t}-1}\,\mathrm{d}t \right) \\
&= \frac{1}{\pi \mathrm{i}} \lim_{\varepsilon\to 0^+} \left(\int_{\varepsilon}^{\pi} \frac{\psi'(t)}{\mathrm{e}^{\mathrm{i}t}-1}\,\mathrm{d}t - \int_{\varepsilon}^{\pi} \frac{\psi'(t)}{\mathrm{e}^{-\mathrm{i}t}-1}\,\mathrm{d}t \right) \\
&= \frac{1}{\pi \mathrm{i}} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\pi} \left(\frac{1}{\mathrm{e}^{\mathrm{i}t}-1} - \frac{1}{\mathrm{e}^{-\mathrm{i}t}-1} \right) \psi'(t)\,\mathrm{d}t \qquad (1.18)\\
&= \frac{1}{\pi \mathrm{i}} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\pi} \frac{\mathrm{e}^{-\mathrm{i}t}-\mathrm{e}^{\mathrm{i}t}}{|\mathrm{e}^{\mathrm{i}t}-1|^2} \psi'(t)\,\mathrm{d}t \\
&= -\frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^{\pi} \psi'(t) \cot\frac{t}{2}\,\mathrm{d}t.
\end{aligned}
$$

Fix $\varepsilon \in \left(0;\tfrac{1}{2}\pi\right]$. From (1.5) it follows that

$$
-\frac{1}{\pi}\int_{\varepsilon}^{\pi/2} \psi'(t)\cot\frac{t}{2}\,\mathrm{d}t = \frac{2}{\pi}\int_{\varepsilon}^{\pi/2} \Phi_K\left(\cos\tfrac{1}{2}t\right)\Phi_K'\left(\cos\tfrac{1}{2}t\right)\cos\tfrac{1}{2}t\,\mathrm{d}t. \tag{1.19}
$$

Applying the identity (1.11) and substituting $x := \Phi_K\left(\cos\tfrac{1}{2}t\right)$ we conclude from (1.19) that

$$
\begin{aligned}
\frac{2}{\pi}\int_{\varepsilon}^{\pi/2} \Phi_K\left(\cos\tfrac{1}{2}t\right)\Phi_K'\left(\cos\tfrac{1}{2}t\right)\cos\tfrac{1}{2}t\,\mathrm{d}t &= \frac{4}{\pi}\int_{\Phi_K(\cos\pi/4)}^{\Phi_K(\cos\varepsilon/2)} \frac{x\Phi_{1/K}(x)}{\sqrt{1-\Phi_{1/K}(x)^2}}\,\mathrm{d}x \\
&= \frac{4}{\pi}\int_{\Phi_K(1/\sqrt{2})}^{\Phi_K(\cos\varepsilon/2)} \frac{x\Phi_{1/K}(x)}{\Phi_K\left(\sqrt{1-x^2}\right)}\,\mathrm{d}x. \qquad (1.20)
\end{aligned}
$$

Substituting $t := 1-x^2$ in the last integral we have

$$
\frac{4}{\pi}\int_{\Phi_K(1/\sqrt{2})}^{\Phi_K(\cos\varepsilon/2)} \frac{x\Phi_{1/K}(x)}{\Phi_K\left(\sqrt{1-x^2}\right)}\,\mathrm{d}x = \frac{2}{\pi}\int_{\Phi_{1/K}(\sin\varepsilon/2)^2}^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{\Phi_{1/K}\left(\sqrt{1-t}\right)}{\Phi_K\left(\sqrt{t}\right)}\,\mathrm{d}t. \tag{1.21}
$$

By the Hübner inequality (cf. [1, (3.2)] or [5, p. 65, (3.6)])

$$
x^{1/K} \le \Phi_K(x) \le 4^{1-1/K}x^{1/K}, \quad 0 \le x \le 1,\ K \ge 1, \tag{1.22}
$$

we get $\Phi_K\left(\sqrt{t}\right)^{-1} \le t^{-1/2K}$ for $t \in (0;1]$. Thus the last integral in (1.21) is convergent as $\varepsilon \to 0$, and combining (1.19), (1.20) and (1.21) we obtain

$$
-\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{\pi/2} \psi'(t)\cot\frac{t}{2}\,\mathrm{d}t = \frac{2}{\pi}\int_{0}^{\Phi_{1/K}(1/\sqrt{2})^2} \frac{\Phi_{1/K}\left(\sqrt{1-t}\right)}{\Phi_K\left(\sqrt{t}\right)}\,\mathrm{d}t. \tag{1.23}
$$

Similar calculations to that in (1.19), (1.20) and (1.21) lead, in view of (1.5), to

$$(1.24)\qquad \begin{aligned} -\frac{1}{\pi}\int_{\pi/2}^{\pi}\psi'(t)\cot\frac{t}{2}\,\mathrm{d}t &= \frac{2}{\pi}\int_{\Phi_K(1/\sqrt{2})^2}^{1}\frac{\Phi_K\big(\sqrt{1-t}\,\big)}{\Phi_{1/K}\big(\sqrt{t}\,\big)}\,\mathrm{d}t \\ &= \frac{2}{\pi}\int_0^{\Phi_{1/K}(1/\sqrt{2})^2}\frac{\Phi_K\big(\sqrt{t}\,\big)}{\Phi_{1/K}\big(\sqrt{1-t}\,\big)}\,\mathrm{d}t. \end{aligned}$$

Combining (1.18) with (1.23) and (1.24) we see by (1.11) and (1.16) that

$$(1.25)\qquad 2\,\mathrm{C}_{\mathbf{T}}[\Psi_K'](1) = L_K.$$

From [8, (1.3)] it follows that

$$(1.26)\qquad \partial\mathrm{P}\,[\Psi_K](z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{T}}\frac{\mathrm{d}\Psi_K(u)}{u-z} = \frac{1}{2\pi\mathrm{i}}\int_{\mathbf{T}}\frac{\Psi_K'(u)}{u-z}\,\mathrm{d}u,\quad z\in\mathbf{D}.$$

Given $\delta\in\big(0;\frac{1}{2}\pi\big)$ and $r\in[0;1)$ we have

$$(1.27)\qquad \begin{aligned} \left|\frac{1}{2\pi i}\int_{\mathbf{T}(1,\delta)}\frac{\Psi_K'(u)}{u-r}\,\mathrm{d}u\right| &\le \frac{1}{2\pi}\int_{\mathbf{T}(1,\delta)}\frac{|u-1|}{|u-r|}\frac{|\Psi_K'(u)|}{|u-1|}\,|\mathrm{d}u| \\ &\le \frac{1}{\pi}\int_{-\delta}^{\delta}\frac{|\psi'(t)|}{2|\sin(\frac{1}{2}t)|}\,\mathrm{d}t \\ &\le \frac{\sqrt{2}}{\pi}\int_0^{\delta}|\psi'(t)|\cot\frac{t}{2}\,\mathrm{d}t. \end{aligned}$$

Since $\psi'(t)\le 0$ for $t\in\big[0;\frac{1}{2}\pi\big]$, we conclude from (1.23) that

$$\frac{\sqrt{2}}{\pi}\int_0^{\pi/2}|\psi'(t)|\cot\frac{t}{2}\,\mathrm{d}t < +\infty.$$

Combining this with (1.27) and (1.25) with (1.14) we see that for a given $\varepsilon>0$ there exists $\delta\in\big(0;\frac{1}{2}\pi\big)$ such that

$$(1.28)\qquad \left|\frac{1}{2\pi i}\int_{\mathbf{T}(1,\delta)}\frac{\Psi_K'(u)}{u-r}\,\mathrm{d}u\right| < \frac{\varepsilon}{3},\quad r\in[0;1),$$

and

$$(1.29)\qquad \left|\frac{1}{2\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(1,\delta)}\frac{\Psi_K'(u)}{u-1}\,\mathrm{d}u - \mathrm{C}_{\mathbf{T}}[\Psi_K'](1)\right| < \frac{\varepsilon}{3}$$

as well as for $r \in [0;1)$ sufficiently close to 1,

$$\left|\frac{1}{2\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(1,\delta)} \frac{\Psi_K'(u)}{u-r}\,\mathrm{d}u - \frac{1}{2\pi i}\int_{\mathbf{T}\setminus\mathbf{T}(1,\delta)} \frac{\Psi_K'(u)}{u-1}\,\mathrm{d}u\right| < \frac{\varepsilon}{3}. \tag{1.30}$$

Combining (1.28) with (1.29) and (1.30) we see that for $r \in [0;1)$ sufficiently close to 1,

$$\left|\frac{1}{2\pi i}\int_{\mathbf{T}} \frac{\Psi_K'(u)}{u-r}\,\mathrm{d}u - \mathrm{C}_{\mathbf{T}}[\Psi_K'](1)\right| < \varepsilon.$$

Then by (1.26),

$$\lim_{r\to 1^-} \partial\mathrm{P}\,[\Psi_K](r) = \mathrm{C}_{\mathbf{T}}[\Psi_K'](1).$$

Since $\Psi_K'(1) = 0$, $\Psi_K(1) = 1$ and the limit $\lim_{r\to 1^-}\partial\mathrm{P}\,[\Psi_K](r)$ exists, it follows from the proof of [9, Lemma 2.1] that

$$\lim_{r\to 1^-} \frac{1-\mathrm{P}\,[\Psi_K](r)}{1-r} = 2\lim_{r\to 1^-} \partial\mathrm{P}\,[\Psi_K](r).$$

Hence by (1.25) we obtain

$$\lim_{r\to 1^-} \frac{1-\mathrm{P}\,[\Psi_K](r)}{1-r} = L_K. \tag{1.31}$$

From Lemma 1.1 it follows that the estimate (1.4) holds. Hence for all $z \in \mathbf{T}$ and $r \in (0;1)$,

$$\frac{|F^*(z)-F(rz)|}{1-r} \geq \frac{|F^*(z)|-|F(rz)|}{1-r} \geq \frac{1-\mathrm{P}\,[\Psi_K](|rz|)}{1-r} = \frac{1-\mathrm{P}\,[\Psi_K](r)}{1-r}.$$

This combined with (1.31) leads to (1.15), and the lemma follows. ▫

Lemma 1.4. *For every $K \geq 1$,*

$$\begin{aligned} L_K &= \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{\mathrm{d}\Phi_{1/K}(s)^2}{s\sqrt{1-s^2}} \\ &= \frac{4}{\pi}\Phi_{1/K}\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{1-2s^2}{s^2(1-s^2)^{3/2}}\Phi_{1/K}(s)^2\,\mathrm{d}s. \end{aligned} \tag{1.32}$$

Moreover, L_K is a strictly decreasing function of $K \geq 1$ such that

$$\lim_{K\to 1} L_K = L_1 = 1 \quad \textit{and} \quad \lim_{K\to+\infty} L_K = 0 \tag{1.33}$$

as well as

$$|L_{K_2} - L_{K_1}| \le L|K_2 - K_1|, \quad K_1, K_2 \ge 1, \tag{1.34}$$

where

$$L := \frac{4}{\pi}(1 + 65 \ln 2). \tag{1.35}$$

Proof. Fix $K \ge 1$. By (1.1) we get

$$\Phi_K\big(\Phi_{1/K}(s)\big) = \Phi_1(s) = s, \quad 0 \le s \le 1, \tag{1.36}$$

and hence using (1.11) we have

$$\Phi_{1/K}\Big(\sqrt{1 - \Phi_{1/K}(s)^2}\,\Big) = \sqrt{1 - \Phi_K\big(\Phi_{1/K}(s)\big)^2} = \sqrt{1 - s^2}\,, \quad 0 \le s \le 1.$$

Since the function $[0;1] \ni s \mapsto \Phi_{1/K}(s)^2$ is absolutely continuous, we thus see, integrating by substitution, that

$$L_K = \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{1}{s\sqrt{1-s^2}} \frac{\mathrm{d}\Phi_{1/K}(s)^2}{\mathrm{d}s}\,\mathrm{d}s = \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{\mathrm{d}\Phi_{1/K}(s)^2}{s\sqrt{1-s^2}}\,. \tag{1.37}$$

Note that (1.22) and (1.36) lead to

$$4^{1-K}x^K \le \Phi_{1/K}(x) \le x^K, \quad 0 \le x \le 1,\ K \ge 1. \tag{1.38}$$

Then, integrating by parts, we obtain

$$\begin{aligned} L_K &= \lim_{r \to 0^-} \frac{2}{\pi}\int_r^{1/\sqrt{2}} \frac{\mathrm{d}\Phi_{1/K}(s)^2}{s\sqrt{1-s^2}} \\ &= \lim_{r\to 0^-} \frac{2}{\pi}\frac{\Phi_{1/K}(s)^2}{s\sqrt{1-s^2}}\,\Big|_{s=r}^{s=1/\sqrt{2}} - \lim_{r\to 0^-}\frac{2}{\pi}\int_r^{1/\sqrt{2}} \frac{2s^2-1}{s^2(1-s^2)^{3/2}}\Phi_{1/K}(s)^2\,\mathrm{d}s \\ &= \frac{4}{\pi}\Phi_{1/K}\Big(\frac{1}{\sqrt{2}}\Big)^2 + \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{1-2s^2}{s^2(1-s^2)^{3/2}}\Phi_{1/K}(s)^2\,\mathrm{d}s. \end{aligned}$$

This and (1.37) yield (1.32). Fix $K_2, K_1 \ge 1$. Then by (1.32),

$$\begin{aligned} L_{K_2} - L_{K_1} &= \frac{4}{\pi}\big(\Phi_{1/K_2}\big(1/\sqrt{2}\,\big)^2 - \Phi_{1/K_1}\big(1/\sqrt{2}\,\big)^2\big) \\ &\quad + \frac{2}{\pi}\int_0^{1/\sqrt{2}} \frac{1-2s^2}{s^2(1-s^2)^{3/2}}\big(\Phi_{1/K_2}(s)^2 - \Phi_{1/K_1}(s)^2\big)\,\mathrm{d}s. \end{aligned} \tag{1.39}$$

If $K_1 < K_2$ then $\Phi_{1/K_2}(s) < \Phi_{1/K_1}(s)$ for $0 < s < 1$, and by (1.39) we have $L_{K_2} < L_{K_1}$. Thus L_K is a strictly decreasing function of K.

Assume now that $K_1 \le K_2 \le 2K_1$ and set $R := K_1/K_2$. By (1.1) and (1.22) we have for every $s \in [0;1]$,

$$\begin{aligned}0 &\le \Phi_{1/K_1}(s)^2 - \Phi_{1/K_2}(s)^2 = \Phi_{1/R}\big(\Phi_{1/K_2}(s)\big)^2 - \Phi_{1/K_2}(s)^2 \\ &\le 16^{1-R}\big(\Phi_{1/K_2}(s)^2\big)^R - \Phi_{1/K_2}(s)^2.\end{aligned} \tag{1.40}$$

Since $\frac{1}{2} \le R \le 1$ it follows that the function $[0;1] \ni t \mapsto 16^{1-R}t^R - t$ is increasing, and hence (1.40) and (1.38) yield

$$0 \le \Phi_{1/K_1}(s)^2 - \Phi_{1/K_2}(s)^2 \le 16^{1-R}s^{2K_1} - s^{2K_2}, \quad 0 \le s \le 1. \tag{1.41}$$

Since

$$\frac{1-2s^2}{(1-s^2)^{3/2}} \le \frac{1}{\sqrt{1-s^2}} \le \sqrt{2}\,, \quad 0 \le s \le \frac{1}{\sqrt{2}}\,, \tag{1.42}$$

we conclude from (1.41) that

$$\begin{aligned}&\int_0^{1/\sqrt{2}} \frac{1-2s^2}{s^2(1-s^2)^{3/2}}\big(\Phi_{1/K_1}(s)^2 - \Phi_{1/K_2}(s)^2\big)\,\mathrm{d}s \\ &\quad\le \sqrt{2}\int_0^{1/\sqrt{2}} \big(16^{1-R}s^{2K_1-2} - s^{2K_2-2}\big)\,\mathrm{d}s \\ &\quad= 2\left[16^{1-R}\frac{2^{-K_1}}{2K_1-1} - \frac{2^{-K_2}}{2K_2-1}\right] \\ &\quad= \frac{2^{1-K_1}}{2K_1-1}[16^{1-R}-1] + 2\left[\frac{2^{-K_1}}{2K_1-1} - \frac{2^{-K_2}}{2K_2-1}\right].\end{aligned} \tag{1.43}$$

From (1.41) we also see that

$$\begin{aligned}\Phi_{1/K_1}\big(1/\sqrt{2}\big)^2 - \Phi_{1/K_2}\big(1/\sqrt{2}\big)^2 &\le 16^{1-R}2^{-K_1} - 2^{-K_2} \\ &= 2^{-K_1}(16^{1-R}-1) + 2^{-K_1} - 2^{-K_2}.\end{aligned} \tag{1.44}$$

Applying Lagrange's mean-value theorem we have

$$16^{1-R} - 1 \le \frac{K_2-K_1}{K_2}16\ln 16 \le (K_2-K_1)16\ln 16 \tag{1.45}$$

and

$$\begin{aligned}\frac{2^{-K_1}}{2K_1-1} - \frac{2^{-K_2}}{2K_2-1} &\le \frac{K_2-K_1}{2^{K_1}}\left[\frac{\ln 2}{2K_1-1} + \frac{2}{(2K_1-1)^2}\right] \\ &\le \frac{2+\ln 2}{2}(K_2-K_1)\end{aligned} \tag{1.46}$$

as well as

$$2^{-K_1} - 2^{-K_2} \leq (K_2 - K_1)2^{-K_1} \ln 2 \leq (K_2 - K_1)\tfrac{1}{2}\ln 2. \tag{1.47}$$

Combining (1.44), (1.45), (1.47) and (1.43), (1.45), (1.46) with (1.39) we obtain the following estimate

$$0 \leq L_{K_1} - L_{K_2} \leq L(K_2 - K_1), \quad 1 \leq K_1 \leq K_2 \leq 2K_1, \tag{1.48}$$

with the constant L given by (1.35).

Assume now that $2K_1 < K_2$. Then $2^m \geq K_2/K_1$ for some natural number $m \geq 2$, and so $R := (K_2/K_1)^{(1/m)} \leq 2$. Replacing K_1 by $R^{n-1}K_1$ and K_2 by $R^n K_1$ in (1.48) we get

$$0 \leq L_{R^{n-1}K_1} - L_{R^n K_1} \leq L(R^n K_1 - R^{n-1}K_1), \quad n = 1, 2, \ldots, m,$$

and hence

$$\begin{aligned} 0 \leq L_{K_1} - L_{K_2} &= \sum_{n=1}^{m} (L_{R^{n-1}K_1} - L_{R^n K_1}) \\ &\leq \sum_{n=1}^{m} L(R^n K_1 - R^{n-1}K_1) = L(K_2 - K_1). \end{aligned}$$

Thus the inequalities in (1.48) hold for all $K_1, K_2 \geq 1$ satisfying $K_1 \leq K_2$, which yields (1.34).

From (1.34) and (1.32) it follows that

$$\lim_{K \to 1} L_K = L_1 = \frac{2}{\pi} \int_0^{1/\sqrt{2}} \frac{2\mathrm{d}s}{\sqrt{1 - s^2}} = 1,$$

which implies the first part in (1.33). The second part in (1.33) follows from (1.32) combined with (1.38) and (1.42), which completes the proof. □

Remark 1.5. It is worth noting that Lemma 1.1 still holds provided the assumption "$F \in \mathrm{QCH}(\mathbf{D}; K)$ and $F(0) = 0$" is replaced by the weaker one: $F = \mathrm{P}[f]$ for some $f \in \mathrm{Hom}^+(\mathbf{T})$ which admits a K-quasiconformal extension G to $\mathbf{D}$ satisfying $G(0) = 0$. This may be achieved after simple modification of the proof of Lemma 1.1. As a matter of fact the inequalities (1.9) are still valid under the new assumption, and the remaining part of the proof of Lemma 1.1 runs unchanged. Thus from the proof of Lemma 1.3, it follows that the left-hand side in (1.15) is still not less than L_K under the new assumption.

2. Main estimations for QCH-maps

Lemmas 1.1 and 1.3 enable us to modify easily the proofs of [10, Lemma 0.5] and [10, Theorem 0.6] to obtain, due to Lemma 1.4, their asymptotically sharp improvements. As a result we derive the following two theorems. For the convenience of the reader we give however their proofs.

Theorem 2.1. *Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbf{D}$ satisfying $F(0) = 0$. If f is the boundary limiting valued function of F, then*

$$d_f \geq \frac{1}{K} \max\left\{\frac{2}{\pi}, L_K\right\}. \tag{2.1}$$

Moreover, the right-hand side in (2.1) *is a decreasing and continuous function of $K \geq 1$ with values in* $(0;1]$.

Proof. From [9, Lemma 2.1] it follows that for a.e. $z \in \mathbf{T}$ both the functions $\partial \mathrm{P}\,[f]$ and $\bar{\partial} \mathrm{P}\,[f]$ have radial limiting values at z and the following equalities hold:

$$\begin{aligned} 2z \lim_{r\to 1^-} \partial \mathrm{P}\,[f](rz) &= \lim_{r\to 1^-} \left[\frac{f(z) - \mathrm{P}\,[f](rz)}{1-r} + zf'(z)\right], \\ 2\bar{z} \lim_{r\to 1^-} \bar{\partial} \mathrm{P}\,[f](rz) &= \lim_{r\to 1^-} \left[\frac{f(z) - \mathrm{P}\,[f](rz)}{1-r} - zf'(z)\right]. \end{aligned} \tag{2.2}$$

Hence for a.e. $z \in \mathbf{T}$,

$$\begin{aligned} &\lim_{r\to 1^-} [z\partial F(rz) + \bar{z}\bar{\partial} F(rz)] = \lim_{r\to 1^-} \frac{f(z) - F(rz)}{1-r}, \\ &\lim_{r\to 1^-} [z\partial F(rz) - \bar{z}\bar{\partial} F(rz)] = zf'(z). \end{aligned} \tag{2.3}$$

Since F is a K-quasiconformal mapping, we see from (2.3) that for a.e. $z \in \mathbf{T}$,

$$\begin{aligned} |f'(z)| &= \lim_{r\to 1^-} |z\partial F(rz) - \bar{z}\bar{\partial} F(rz)| \geq \lim_{r\to 1^-} \big(|\partial F(rz)| - |\bar{\partial} F(rz)|\big) \\ &\geq \frac{1}{K} \lim_{r\to 1^-} \big(|\partial F(rz)| + |\bar{\partial} F(rz)|\big) \\ &\geq \frac{1}{K} \lim_{r\to 1^-} \big(|z\partial F(rz) + \bar{z}\bar{\partial} F(rz)|\big) \\ &= \frac{1}{K} \lim_{r\to 1^-} \left|\frac{f(z) - F(rz)}{1-r}\right|. \end{aligned}$$

Thus (2.1) follows immediately from Lemma 1.3. The remaining part of the theorem is a simple conclusion from Lemma 1.4. □

Theorem 2.2. *Given $K \geq 1$ let F be a K-quasiconformal and harmonic self-mapping of $\mathbf{D}$ satisfying $F(0)=0$. Then the inequalities*

$$|\partial F(z)| \geq \frac{K+1}{2K} \max\left\{\frac{2}{\pi}, L_K\right\} \tag{2.4}$$

and

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq \frac{1}{2}\left(1+\frac{1}{K}\right)^2 \max\left\{\frac{4}{\pi^2}, L_K^2\right\} \tag{2.5}$$

hold for every $z \in \mathbf{D}$. Moreover, the right-hand sides in (2.4) *and* (2.5) *are decreasing and continuous functions of $K \geq 1$ with values in* $(1/\pi; 1]$ *and* $(2/\pi^2; 2]$, *respectively.*

Proof. Since F is a K-quasiconformal mapping, we have

$$(K+1)|\bar{\partial} F(w)| \leq (K-1)|\partial F(w)|, \quad w \in \mathbf{D},$$

and hence

$$2(K^2+1)|\partial F(w)|^2 \geq (K+1)^2(|\partial F(w)|^2 + |\bar{\partial} F(w)|^2), \quad w \in \mathbf{D}. \tag{2.6}$$

From (2.2) it follows that for a.e. $z \in \mathbf{T}$ the following limits exist and

$$2 \lim_{r\to 1^-} \left(|\partial F(rz)|^2 + |\bar{\partial} F(rz)|^2\right) = |f'(z)|^2 + \lim_{r\to 1^-} \left|\frac{f(z)-F(rz)}{1-r}\right|^2. \tag{2.7}$$

Combining (2.7) with (2.1) and (1.15) we see that for a.e. $z \in \mathbf{T}$,

$$\lim_{r\to 1^-} \left(|\partial F(rz)|^2 + |\bar{\partial} F(rz)|^2\right) \geq \frac{1}{2}\left(1+\frac{1}{K^2}\right) \max\left\{\frac{4}{\pi^2}, L_K^2\right\}. \tag{2.8}$$

From this and (2.6) it follows that for a.e. $z \in \mathbf{T}$,

$$\lim_{r\to 1^-} |\partial F(rz)| \geq \frac{K+1}{2K} \max\left\{\frac{2}{\pi}, L_K\right\}. \tag{2.9}$$

Applying now [10, Lemma 0.3] we deduce (2.4). Then (2.5) follows directly from (2.4) and the identity

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 = 2\left(|\partial F(z)|^2 + |\bar{\partial} F(z)|^2\right), \quad z \in \mathbf{D}.$$

The remaining part of the theorem is a simple conclusion from Lemma 1.4. □

Applying the Lipschitz condition (1.34) we obtain the following estimate

$$L_K \geq 1 - L(K-1), \quad K \geq 1, \tag{2.10}$$

which is fairly good for small K close to 1. Due to (2.10) we may easily derive from Theorems 2.1 and 2.2 the following corollaries that give more explicit estimates as compared to (2.1), (2.4) and (2.5) for K sufficiently close to 1.

Corollary 2.3. *Under assumptions of Theorem* 2.1, *if moreover* $1 \leq K \leq 1 + (1 - 2/\pi)/L$, *then*

$$d_f \geq \frac{1}{K} - (K-1)\frac{L}{K}. \tag{2.11}$$

Corollary 2.4. *Under assumptions of Theorem* 2.2, *if moreover* $1 \leq K \leq 1 + (1 - 2/\pi)/L$, *then the inequalities*

$$|\partial F(z)| \geq 1 - (K-1)\left(\frac{1}{2K} + L\right) \tag{2.12}$$

and

$$|\partial_x F(z)|^2 + |\partial_y F(z)|^2 \geq 2 - (K-1)\left(\frac{2}{K} + 4L\right) \tag{2.13}$$

hold for every $z \in \mathbf{D}$.

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