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LENGTHS OF GEODESICS ON RIEMANN SURFACES WITH BOUNDARY

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Abstract. This article concerns lengths of simple closed geodesics on hyperbolic Riemann surfaces. In particular, for a surface with boundary, it is shown that boundary length can be increased such that all simple closed geodesics are lengthened.

1. Introduction

Consider a Riemann surface S endowed with a hyperbolic metric. The surface's simple length spectrum $\Delta_0(S)$ is the ordered set of lengths $\{l_1 \leq l_2 \leq \cdots\}$ of its (interior) simple closed geodesics. The length l_1 is its systole length (the shortest non-trivial closed curve) and l_1 is known to be bounded for any given genus if S is a closed surface (without boundary). Surfaces that satisfy this bound have been a subject of active research (i.e. [6], [9]). This is perhaps the best known example of the larger study of extremal surfaces for the length of a certain simple closed geodesic. The idea of this article is to prove the following theorem, which is a natural tool for describing extremal surfaces.

Theorem 1.1. *Let S be a compact hyperbolic Riemann surface with non-empty boundary and let $\varepsilon > 0$. There exists a surface $\widetilde{S}$ of same signature with total boundary length ε greater than that of S with $\Delta_0(S) < \Delta_0(\widetilde{S})$.*

In other words, boundary length can be increased so that a simple closed geodesic on S is strictly shorter that its corresponding geodesic on $\widetilde{S}$. The surfaces S and $\widetilde{S}$ are of same topological type endowed with two different hyperbolic metrics. The reciprocal to the theorem, namely that boundary length can be decreased in order to decrease the length of all simple closed geodesics, is also true. It must noted that a similar proposition limited to the case where S is of signature $(g,1)$, was proved by Schmutz–Schaller in [9] and was used to find properties for surfaces with maximum size systoles.

The theorem, along with the convexity of geodesic length functions along earthquake paths, is used to prove the following corollaries.

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Corollary 1.2. *Let S be a maximal surface of genus g for Bers' constant. Then all simple closed geodesics intersect at least two distinct simple closed geodesics of length L_g that are the longest geodesics of distinct partitions.*

Corollary 1.3. *Let S be a surface of genus g with largest possible systole among all surfaces of same genus. Let γ be a simple closed geodesic. Then γ intersects two distinct systoles (and distinct from γ).*

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2. Preliminaries

Here a *surface* will always be a compact Riemann surface equipped with a metric of constant curvature -1. Such a surface is always locally isometric to the hyperbolic plane $\mathbf{H}$. A surface will generally be represented by S and distance on S (between points, curves or other subsets) by $d_S(\cdot,\cdot)$. The signature of such a surface will be denoted (g,n) (genus g with n boundary curves). All boundary curves must be smooth closed geodesics. A surface of signature $(0,3)$ is called a Y*-piece* or a *pair of pants* and will generally be represented by $\mathscr{Y}$ or $\mathscr{Y}_i$.

A curve, unless mentioned, will always be non-oriented and primitive. The set of all free homotopy classes of closed curves of a surface S is denoted by $\pi(S)$. A *non-trivial curve* on S is a curve which is not freely homotopic to a point. A closed curve on S is called *simple* if it has no self-intersections. Closed curves (geodesic or not) will generally be represented by greek letters (α, β, γ and γ_i etc.) whereas paths (geodesic or not) will generally be represented by lower case letters (a, b etc). The intersection number between two distinct curves α and β will be denoted $\mathrm{int}(\alpha,\beta)$. Unless otherwise specified, a *geodesic* is a simple closed geodesic curve. A *non-separating closed curve* is a closed curve γ such that the set $S\setminus\gamma$ is connected. Otherwise, a closed curve is called *separating*. The function that associates to a finite path or curve its length will be represented by $l(\cdot)$, although generally a path or a curve's name and its length will not be distinguished.

The simple length spectrum of a surface S is the ordered set of lengths of simple closed geodesics and will be denoted

$$\Delta_0(S)=\{l_1\leq l_2\leq\cdots\}.$$

By convention, for S with boundary only interior the lengths of interior simple closed geodesics will be allowed to appear in $\Delta_0(S)$. Consider two surfaces S and $\widetilde{S}$ with simple length spectrums $\Delta_0(S)=\{l_1\leq l_2\leq\cdots\}$ and $\Delta_0(\widetilde{S})=\{\tilde{l}_1\leq\tilde{l}_2\leq\cdots\}$. The notation $\Delta_0(S)<\Delta_0(\widetilde{S})$ is an abbreviation for $l_i<\tilde{l}_i$ for all $i\in\mathbf{N}^*$.

A *geodesic length function* is an application that associates length to geodesics according to homotopy class under the action of a continuous transformation of a surface. In [8], S. Kerckhoff proves the following theorem.

Theorem 2.1. *The geodesic length function of a simple closed curve δ is convex along earthquake paths. It is strictly convex if δ intersects γ, the simple closed geodesic along which the earthquake was performed.*

A surface of genus g can always be cut along $3g-3$ disjoint simple closed geodesics $\{\gamma_1,\dots,\gamma_{3g-3}\}$ which separate S into $2g-2$ Y-pieces. The reunion of these geodesics $\mathscr{P}=\{\gamma_1,\dots,\gamma_{3g-3}\}$ is called a partition. (If S is of signature (g,n) then a partition contains $3g-3+n$ simple closed geodesics.) If $\mathscr{P}$ is chosen such that $\max_{i=1}^{3g-3} l(\gamma_i)=l(\mathscr{P})$ is minimal among all possible partitions of S, then $l(\mathscr{P})\le L_g$, where L_g is constant depending uniquely on the genus ([1], [2]). The optimal constant L_g is often called the Bers' partition constant and the best known bound is $L_g\le 21(g-1)$ ([3, Remark 5.2.5, p. 129]). A surface of genus g will be called maximal for Bers' partition constant if it is a surface on which a minimal partition $\mathscr{P}$ satisfies $l(\mathscr{P})=L_g$.

A Y-piece can be separated into two symmetric isometric right angled hyperbolic hexagons. To do this, one can cut along the three disjoint simple geodesic paths that go between the boundary geodesics. The following list of well-known propositions concerning hyperbolic polygons (polygons in the hyperbolic plane $\mathbf{H}$) are of particular use in the proofs, and are included for sake of completeness. They can be found in either [3] or [5]. All polygons, unless specially mentioned, are considered right-angled.

Proposition 2.2 ([3, Theorem 2.4.1, p. 40]). *Let H be a hexagon with a, b and c non-adjacent sides. Let α be the remaining edge adjacent to b and c, β the remaining edge adjacent to a and c, and γ the remaining edge adjacent to a and b. Then*

$$\cosh c=\sinh a\sinh b\cosh\gamma-\cosh a\cosh b.$$

Proposition 2.3 ([3, Theorem 2.4.4, p. 42]). *Let H be a non-convex hexagon with a, b and c non-adjacent sides. Let α be the remaining edge adjacent to b and c, β the remaining edge adjacent to a and c, and γ the remaining edge adjacent to a and b. Let H be such that γ and c intersect. Then*

$$\cosh c=\sinh a\sinh b\cosh\gamma+\cosh a\cosh b.$$

A trirectangle is a quadrilateral with three right angles.

Proposition 2.4 ([3, Theorem 2.3.1, p. 37]). *Let R be a trirectangle with interior angle φ being the only non-right angle situated between sides α and β. Let a and b be the remaining sides with a adjacent to β and b adjacent to α. Then the following formulas hold*:

$$\cos\varphi=\sinh a\sinh b\quad\textit{and}\quad\sinh\alpha=\sinh a\cosh\beta.$$

The following proposition deals with quadrilaterals with only two right angles.

Proposition 2.5 ([3, Theorem 2.3.1, p. 38]). *Let R be a convex quadrilateral with two right interior angles. Let γ be the side of R between the two right angles. Let c be the side opposite γ and a and b the remaining sides. Then*

$$\cosh c = \cosh a \cosh b \cosh \gamma - \sinh a \sinh b.$$

And in the non-convex case the following proposition holds.

Proposition 2.6 ([3, Theorem 2.3.1, p. 38]). *Let R be a non-convex quadrilateral with two right interior angles. Let γ be the side of R between the two right angles and c be the side that intersects γ. Let a and b be the remaining sides. Then*

$$\cosh c = \cosh a \cosh b \cosh \gamma + \sinh a \sinh b.$$

Finally:

Proposition 2.7 ([3, Example 2.2.7, p. 36]). *Let P be a pentagon with four right angles. Let φ be the (only non-right) interior angle between two sides, a and b. Let α be the other side adjacent to b and β the other side adjacent to a. Let c be the remaining edge. Then the following formulas hold*:

$$\cosh c = -\cosh a \cosh b \cos \varphi + \sinh a \sinh b \quad \textit{and} \quad \frac{\cosh a}{\cosh \alpha} = \frac{\cosh b}{\cosh \beta} = \frac{\cosh c}{\cosh \varphi}.$$

3. Main theorem and corollaries

The proof of the main theorem is primarily based on the following technical lemma. The lemma essentially states that by decreasing a boundary length on a Y-piece, one decreases the length of all simple geodesic paths between the other boundary geodesics.

Lemma 3.1. *Let $\mathscr{Y} = (\alpha, \beta, \gamma)$ be a Y-piece. Let $d_{\alpha\beta}$, $d_{\alpha\gamma}$ and $d_{\beta\gamma}$ be $\mathscr{Y}$'s three perpendiculars. Let c be a border to border simple geodesic path from α to β (from α to α, respectively). Let p and q be the initial and end points of c. For a positive $\varepsilon < l(\gamma)$ let $\mathscr{Y}' = (\alpha, \beta, \gamma')$ with $l(\gamma') = l(\gamma) - \varepsilon$. Let p' be the point on $\alpha \subset \mathscr{Y}'$ corresponding to p (such that $d_{\mathscr{Y}'}(p', d_{\alpha\gamma'}) = d_{\mathscr{Y}}(p, d_{\alpha\gamma})$ and $d_{\mathscr{Y}'}(p', d_{\alpha\beta}) = d_{\mathscr{Y}}(p, d_{\alpha\beta})$) and q' the point on $\beta \subset \mathscr{Y}'$ corresponding to q. Let c' be the simple geodesic path on $\mathscr{Y}'$ from p' to q'. Then $c' < c$.*

Proof. The three arcs $d_{\alpha\beta}$, $d_{\alpha\gamma}$ and $d_{\beta\gamma}$ separate $\mathscr{Y}$ into two right-angled hexagons. The geodesic path c can either cross one or both hexagons. This explains the different configurations of c in what follows. For what follows also notice that if the length of γ decreases, then so does $d_{\alpha\beta}$ (Proposition 2.2).

Let c be a simple arc on $\mathscr{Y}$ from α to β. Then one the two following figures hold.

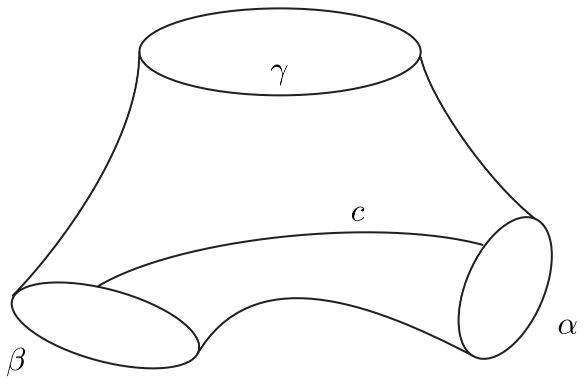

Figure 1. A Y-piece with a geodesic arc from α to β.

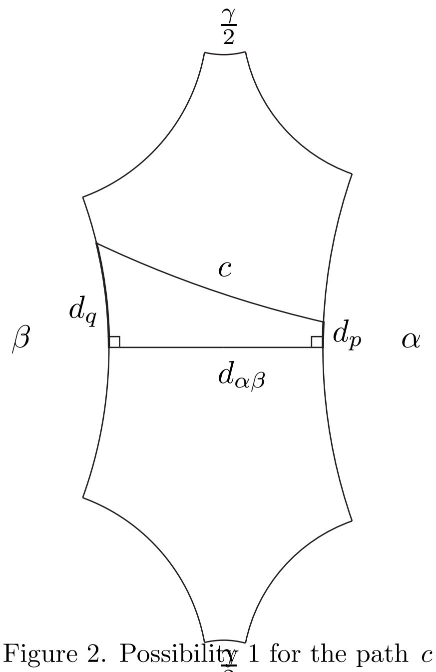

Figure 2. Possibility 1 for the path c.

Consider the quadrilaterals containing c and $d_{\alpha\beta}$ (convex in Figure 2 and non-convex in Figure 3). The paths (and their lengths) d_p and d_q are defined as the two remaining sides of these quadrilaterals, as in the figures. (The path d_p is between p and $d_{\alpha\beta}$, and the path d_q is between q and $d_{\alpha\beta}$.) Notice that according to the hypotheses these lengths do not change under the influence of the modification to $\mathscr{Y}$. Using the formula for quadrilaterals (Proposition 2.5), in Figure 2 we have the following expression for the length of c:

$$\cosh c = \cosh d_p \cosh d_q \cosh d_{\alpha\beta} - \sinh d_p \sinh d_q .$$

In Figure 3, the quadrilateral with sides d_p, d_q, c and $d_{\alpha\beta}$ is the situation described in Proposition 2.6. Thus:

$$\cosh c = \cosh d_p \cosh d_q \cosh d_{\alpha\beta} + \sinh d_p \sinh d_q .$$

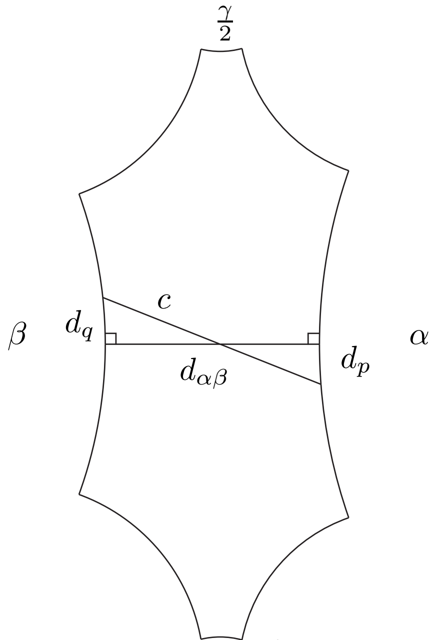

Figure 3. Possibility 2 for the path c.

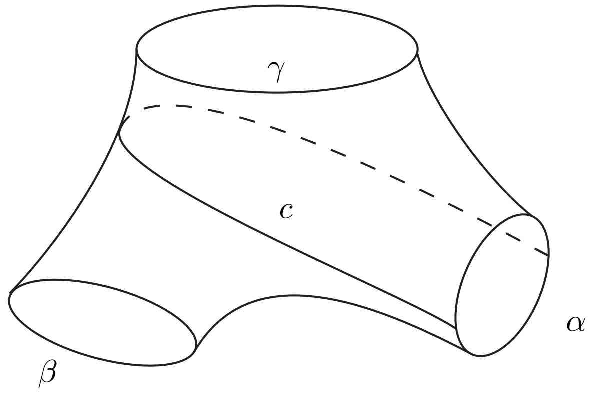

Figure 4. A Y-piece with a geodesic arc from α to α.

In both cases it is easy to see that the length of c is strictly reduced by a reduction of length γ.

Let c be a simple arc on $\mathscr{Y}$ from α to α. The paths d_p and d_q are defined as previously and verify the configuration of either Figures 5 or 6. In both cases we can extract the hexagon depicted in Figure 7.

Notice that h, the common perpendicular between c and β is contained in the hexagon as in Figure 7. β_1 and β_2 are defined as the two parts of β separated by h. Notice that the pentagons $(Q_1 \cup Q_2)$ and $(Q_1 \cup Q_2)$ are right-angled, except in q and p, respectively. Thus, using Proposition 2.7 the following formulas hold:

$$\frac{\cosh c_1}{\sinh d_{\alpha\beta}} = \frac{\cosh d_q}{\sinh h} \quad \text{and} \quad \frac{\cosh c_2}{\sinh d_{\alpha\beta}} = \frac{\cosh d_p}{\sinh h}.$$

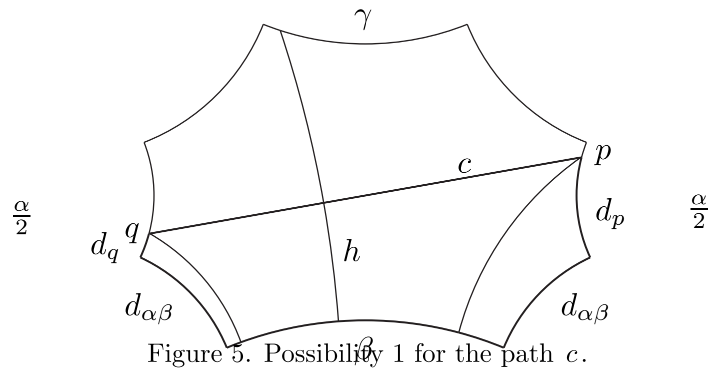

Figure 5. Possibility 1 for the path c.

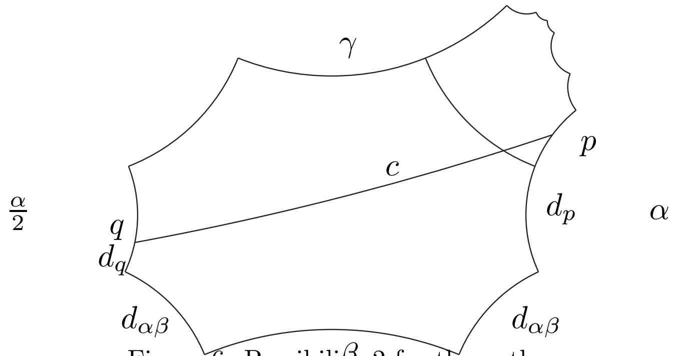

Figure 6. Possibility 2 for the path c.

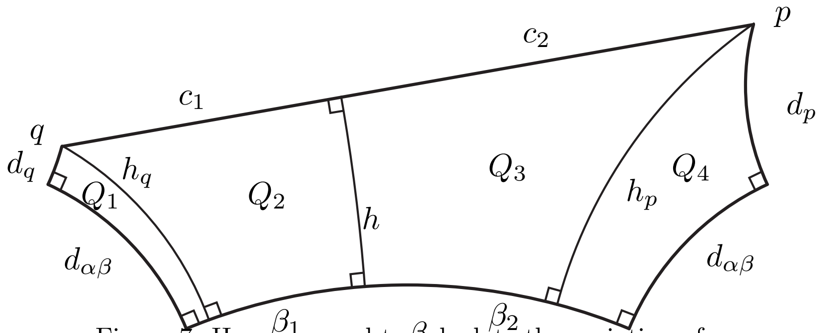

Figure 7. Hexagon used to calculate the variation of c.

Thus

$$\cosh c_1 = \frac{\cosh d_q}{\cosh d_q} \cosh c_2. \tag{1}$$

From this formula, and because $c = c_1 + c_2$, if either c_1 or c_2 decreases then so does c.

When there is a variation of γ then, because β is constant, either β_1 or β_2 is reduced (or stays constant). The situation being symmetric, suppose that β_1 is reduced or left constant.

We can now use the trirectangle Q_1 and Proposition 2.4. In Figure 7 the following geodesic paths need to be defined: h_q is the minimal path from q to β,

h_p is the minimal path from p to β, and β_{11} and β_{12} are the two parts of β_1 separated by h_q. The following formula holds:

$$\sinh h_q = \sinh d_{\alpha\beta} \cosh d_q.$$

Thus h_q is reduced by a reduction of γ. In the same trirectangle we can see that

$$\sinh \beta_{11} = \frac{\sinh d_q}{\cosh h_q}.$$

This proves that β_{11} increases, and this implies that $\beta_{12} = \beta_1 - \beta_{11}$ decreases. Now in the trirectangle Q_2 the following formula holds:

$$\sinh c_1 = \sinh \beta_{12} \cosh h_q.$$

Both β_{11} and h_q decrease and this implies that c_1 decreases as well. We can now conclude that c decreases as well. □

Theorem 3.2. *Let S be a surface of signature (g,n) with $n > 0$. Let $\gamma_1, \dots, \gamma_n$ be the boundary geodesics of S. For $(\varepsilon_1, \dots, \varepsilon_n) \in (\mathbf{R}^+)^n$ with at least one $\varepsilon_i \neq 0$, there exists a surface $\widetilde{S}$ with boundary geodesics of length $\gamma_1 + \varepsilon_1, \dots, \gamma_n + \varepsilon_n$ such that all corresponding simple closed geodesics in $\widetilde{S}$ are of length strictly greater than those of S $(\Delta_0(S) < \Delta_0(\widetilde{S}))$.*

Proof. For a given partition of S, a simple closed geodesic is either an element of the partition, or transversally intersects the partition. In the latter case, it can be seen as the union of simple geodesic arcs, each arc joining geodesics in the partition. The previous lemma will thus be our main tool for comparing lengths of simple geodesics between surfaces.

Let $\mathscr{P}$ be a partition of S. We shall replace S by the following surface $\widetilde{S}$. Let γ_i be a boundary geodesic and $\mathscr{Y}_{\gamma_i}(\alpha, \beta, \gamma_i)$ be the element of $\mathscr{P}$ with γ_i a boundary geodesic. For $\varepsilon > 0$, replace $\mathscr{Y}_{\gamma_i}$ with $\mathscr{Y} = (\alpha, \beta, \gamma_i + \varepsilon)$ without modifying the twist parameters.

Let us denote $\widetilde{\mathscr{P}}$ the partition on $\widetilde{S}$ corresponding to $\mathscr{P}$. Now let $\tilde{\delta}$ be a simple closed geodesic on $\widetilde{S}$. We shall now find a corresponding geodesic δ on S. To find δ, proceed as follows. The geodesic $\tilde{\delta}$ is either an element of $\widetilde{\mathscr{P}}$, or intersects the partition. If we are in the latter case, then $\tilde{\delta} = \bigcup_{i=1}^{m} \tilde{c}_i$ where each $\tilde{c}_i$ is a simple geodesic path from an element of $\widetilde{\mathscr{P}}$ to another. To each $\tilde{c}_i$ one can find a corresponding c_i on S as in the previous lemma. Now let $\delta = \mathscr{G}\big(\bigcup_{i=1}^{m} c_i\big)$. Clearly, by the previous lemma, if δ intersects $\mathscr{Y}_{\gamma_i}$, then $l(\delta) < l(\tilde{\delta})$.

The same process can be performed by increasing the length of multiple boundary geodesics simultaneously. Notice that all simple closed geodesics on S are in one-to-one correspondence with those of $\widetilde{S}$. For a given partition $\mathscr{P}$ on S, all simple closed geodesics that have a transversal intersection with a Y-piece whose boundary length has been increased correspond to a simple closed geodesic on $\widetilde{S}$ whose length is strictly greater. Conversely, all other simple closed geodesics on S correspond to simple closed geodesics on $\widetilde{S}$ with identical length.

To construct $\widetilde{S}$, proceed as follows. Let $\big(\tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_1)\big) = (\varepsilon_1/k, \ldots, \varepsilon_n/k)$ where k is a positive integer to be determined. Take a partition $\mathscr{P}$ on S, and construct a surface S_1 as described above with boundary geodesics of lengths $\gamma_1 + \tilde{\varepsilon}_1, \ldots, \gamma_n + \tilde{\varepsilon}_n$. On S_1, choose a different partition $\mathscr{P}_1$ and repeat the process to obtain a surface S_2 etc. For each step, as the correspondence between geodesics is one-to-one, there continues to be a one-to-one correspondence between geodesics on S and geodesics on, say, S_m. Clearly, by choosing the partitions $\mathscr{P}_m$ correctly, it takes a finite number of steps N so that the length of *all* simple closed geodesics on S are strictly inferior to their corresponding geodesics on S_N. The number N depends on the genus and the number of boundary geodesics γ_i whose length is increased in the process. Now replace k by N and the corresponding construction proves the theorem.

Notice that it suffices to have one boundary geodesic whose length increases. □

The idea of studying maximal surfaces has induced us to focus on how to increase the length of simple closed geodesics. However, the above proof also applies to the reduction of lengths, and the following theorem holds as well.

Theorem 3.3. *Let S be a surface of signature (g, n) with $n > 0$. Let $\gamma_1, \ldots, \gamma_n$ be the boundary geodesics of S. For $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbf{R}^-)^n$ with at least one $\varepsilon_i \neq 0$, there exists a surface $\widetilde{S}$ with boundary geodesics of length $\gamma_1 + \varepsilon_1, \ldots, \gamma_n + \varepsilon_n$ such that all corresponding simple closed geodesics in $\widetilde{S}$ are of length strictly less than those of S $\big(\Delta_0(S) > \Delta_0(\widetilde{S})\big)$.*

As a first corollary to Theorem 3.2, let us first look at properties of maximal surfaces for Bers' partition constant.

Corollary 3.4. *Let S be a maximal surface of genus g for Bers' constant. Then all simple closed geodesics intersect at least two distinct simple closed geodesics of length L_g that are the longest geodesics of distinct partitions.*

Proof. Let γ be a non-separating simple closed geodesic on S. Open S along γ to obtain a surface $\widetilde{S}$ of signature $(g-1, 2)$. By the previous proposition, $\widetilde{S}$ can be replaced by a new surface S' with all lengths of simple closed geodesics longer, including γ. If S was maximal for genus g, then the surface obtained by gluing S' back together again along the image of γ must have a shorter partition. The only geodesics that could have been shortened by this operation were those that intersected γ, thus γ must have intersected a geodesic δ that was of length L_g and that completed a partition as the longest geodesic. Now twist γ to obtain a new surface S'' such that the image of δ is longer. Re-perform the operation described above and S can only have been maximal if there was a second distinct geodesic like δ, which proves the result. □

Remark 3.5. An upper bound on L_g is known ([4], [3]). On a maximal surface the shortest simple closed geodesic σ intersects a geodesic of length L_g. With this in mind, using the collar theorem [7], we can give a lower bound on σ (which is not sharp) for a maximal surface in the following manner. The collar theorem states that there is a lower bound $2\operatorname{arcsinh}(2/\sigma)$ for the length of any closed geodesic that intersects σ. Thus for σ too small, we have

$$2\operatorname{arcsinh}(2/\sigma) > L_g,$$

and thus the surface in question cannot be maximal. Applied differently, the same argument shows that surfaces with small systoles admit a partition containing the systoles and with all geodesics of length inferior to L_g.

A second corollary, similar to the first one, concerns surfaces with maximum size systoles. The problem of maximum size systoles is exposed in [9], and a sharp bound is known for surfaces of genus 2 ([6]). The proof of this corollary is almost identical to the previous corollary and is left to the reader.

Corollary 3.6. *Let S be a surface of genus g with largest possible systole among all surfaces of same genus. Let γ be a simple closed geodesic. Then γ intersects two distinct systoles (and distinct from γ).*

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