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Complex dynamics, value distributions, and potential theory

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Abstract. Value distributions and complex dynamics are intertwined in that they involve the study of preimages and forward-images under iteration, respectively. In this article, we first show the equivalence of the dynamical exceptional set of a rational map $f\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ and both the dynamical Nevanlinna and Valiron exceptional sets. As a consequence, we establish several convergence theorems of the potentials of the averaged value distributions, which are stronger than what can be obtained from the general axiomatic potential theory.

1. Introduction

A rational map f is a holomorphic endomorphism of the Riemann sphere $\widehat{\mathbf{C}}$.

Notation 1.1. Rat denotes the set of all rational endomorphisms of $\widehat{\mathbf{C}}$. $\widehat{\mathbf{C}}$ is identified with the set of all constant maps of $\widehat{\mathbf{C}}$.

In the case that f is non-invertible, the Fatou and Julia strategy for studying the complex dynamics $(\widehat{\mathbf{C}}, f)$, which treats *forward-images* under iterations, is the separation of $\widehat{\mathbf{C}}$ into two completely invariant complementary subsets, one of which is the *Fatou set* $F(f)$, the region of normality of $\{f^k := f^{\circ k}\}$, and the other the *Julia set* $J(f)$. In other words, the restricted dynamical systems $\bigl(F(f), f\bigr)$ and $\bigl(J(f), f\bigr)$ are tame and chaotic, respectively. Consequently, the dynamical system *around* $J(f)$ has an *almost covering* feature, that is,

$$E(f) := \widehat{\mathbf{C}} - \bigcap_{U\colon \text{an open neighborhood of a point of } J(f)} \left(\bigcup_{k\in\mathbf{N}} f^k(U) \right)$$

consists of at most two points.

Definition 1.1 (dynamical exceptional set). $E(f)$ is called the *dynamical exceptional set* of f.

From this almost covering feature, a Nevanlinna theoretical study, which treats *preimages* under iterations, arises naturally.

For the Nevanlinna theory, see, for example, [3] and [5].

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Notation 1.2. The Dirac measure at $w \in \widehat{\mathbf{C}}$ is written as δ_w.

Definition 1.2 (averaged value distribution). For distinct $f, g \in \mathrm{Rat}$, put $\mu(f,g) := \sum_{f(w)=g(w)} \delta_w$, where the summation takes into account the multiplicity of each root of the equation $f = g$. The *averaged value distribution* of f for g is the probability measure $\mu(f,g)/(\deg f + \deg g)$.

The spherical area measure and the chordal distance on $\widehat{\mathbf{C}}$ are given by

$$\sigma(w) = \frac{\mathrm{d}x\,\mathrm{d}y}{\pi(1+|w|^2)^2} \quad (w = x+iy) \qquad \text{and} \quad [z,w] = \frac{|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}},$$

respectively. We note that they are normalized as $\sigma(\widehat{\mathbf{C}}) = 1$ and $[0,\infty] = 1$.

Definition 1.3 (dynamical Nevanlinna theory [12]). For $f, g \in \mathrm{Rat}$, the *pointwise proximity function* is defined by

$$\big(w(g,f)\big)(z) := \log \frac{1}{[g(z), f(z)]} \colon \widehat{\mathbf{C}} \to [0,\infty],$$

which is δ-subharmonic, and the *mean proximity* by

$$m(g,f) := \int_{\widehat{\mathbf{C}}} w(g,f)\,\mathrm{d}\sigma \in [0,\infty).$$

Let $\mathscr{F}$ be a *rational sequence* $\{f_k\}_{k=0}^{\infty} \subset \mathrm{Rat}$ with increasing degrees $\{d_k := \deg f_k\}$. For $g \in \mathrm{Rat}$, the dynamical *Nevanlinna* and *Valiron exceptionalities* are defined by

$$\mathrm{NE}(g;\mathscr{F}) := \liminf_{k\to\infty} \frac{m(g,f_k)}{d_k} \in [0,\infty],$$
$$\mathrm{VE}(g;\mathscr{F}) := \limsup_{k\to\infty} \frac{m(g,f_k)}{d_k} \in [0,\infty],$$

respectively.

From now on, we consider the *iteration* sequence $\{f^k\}_{k=1}^{\infty}$ of $f \in \mathrm{Rat}$ of degree $d \geq 2$.

Definition 1.4 (dynamical Nevanlinna and Valiron exceptional sets). The dynamical *Nevanlinna* and *Valiron* exceptional sets of f in $\widehat{\mathbf{C}}$ are defined by

$$E_N(f) := \big\{p \in \widehat{\mathbf{C}} \colon \mathrm{NE}(p;\{f^k\}) > 0\big\},$$
$$E_V(f) := \big\{p \in \widehat{\mathbf{C}} \colon \mathrm{VE}(p;\{f^k\}) > 0\big\},$$

respectively.

We shall use several notions from the geometric measure theory and the potential theory. For the details, see, for example, [1], [10], and [4].

Convention. We consider only weak convergences for measures.

Theorem 1 (All exceptional sets are same). *For* $f \in \mathrm{Rat}$ *of degree* ≥ 2,

$$E_N(f) = E_V(f) = E(f).$$

Remark 1.1. In [11], Theorem 1 has been already implicitly applied to the Siegel–Cremer linearizability problem of rational maps.

The proof relies on a fundamental theorem in complex dynamics proved by Lyubich [6] and independently by Freire–Lopes–Mañé [2]:

Theorem 1.1 ([6] and [2]). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$. *For every* $p \in \widehat{\mathbf{C}} - E(f)$,

$$\lim_{k\to\infty} \frac{\mu(f^k, p)}{d^k} = \lim_{k\to\infty} \frac{(f^k)^*\sigma}{d^k} =: \mu_f. \tag{1}$$

An important conclusion from Theorem 1 is a convergence theorem of the *potentials* of the averaged value distributions.

Definition 1.5 (spherical potential). For a regular measure μ on $\widehat{\mathbf{C}}$, the *potential* is defined by

$$V_\mu := \int_{\widehat{\mathbf{C}}} \log \frac{1}{[\,\cdot\,, w]} \, \mathrm{d}\mu(w) : \widehat{\mathbf{C}} \to [-\infty, \infty],$$

which is a δ-subharmonic function on $\widehat{\mathbf{C}}$.

Remark 1.2. In the potential theory, the potential is sometimes defined as $-V_\mu$, but the definition will be more convenient in our study.

The (axiomatic) potential theory implies that when positive regular measures $\{\mu_k\}$ converges to μ, then

$$\liminf_{k\to\infty} V_{\mu_k} = V_\mu \tag{2}$$

quasieverywhere on $\widehat{\mathbf{C}}$.

Theorem 2 (convergence theorem of potentials). *Let* $f \in \mathrm{Rat}$ *be of degree* $d \geq 2$ *and* $p \in \widehat{\mathbf{C}} - E(f)$. *If* p *is not a fixed point of* f, *then*

$$\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k} = V_{\mu_f} \tag{3}$$

on $\widehat{\mathbf{C}}$. *If* p *is a fixed point of* f, (3) *holds on* $\widehat{\mathbf{C}} - \bigcup_{k\geq 0} f^{-k}(p)$.

We also characterize such points that the potentials actually *converge* there.

Theorem 3 (convergence of potentials and pointwise behavior). *Let $f \in \mathrm{Rat}$ be of degree $d \geq 2$. For $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$,*

$$\lim_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) = V_{\mu_f}(q) \tag{4}$$

if and only if

$$\lim_{k\to\infty} \frac{1}{d^k}\log\frac{1}{[p, f^k(q)]} = 0. \tag{5}$$

Hence it follows from the classification of the Fatou components that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$, if (4) (or (5)) does not hold, then either

(i) p is periodic and $q \in \bigcup_{k\geq 0} f^{-k}(p)$,
(ii) $p, q \in J(f)$, or
(iii) p is not periodic but contained in a rotation domain and

$$q \in \mathrm{Fol}_f(p), \tag{6}$$

which is called the *foliated equivalence class* of p (cf. [8]) defined by

$$\mathrm{Fol}_f(p) := \left\{q \in \widehat{\mathbf{C}} : \overline{\mathrm{GO}_f(p)} = \overline{\mathrm{GO}_f(q)}\right\},$$

where $\mathrm{GO}_f(p)$ is the *grand orbit* of p defined by

$$\mathrm{GO}_f(p) := \{q \in \widehat{\mathbf{C}} : \text{for some } k, l \in \mathbf{N},\ f^k(p) = f^l(q)\}.$$

Conversely, in the case (i), (5) never holds. Concerning (ii), it is possible to find such an f that for some $p, q \in J(f)$ not satisfying (i), (5) does not hold: for example, let $f(z) = z^d$, choose such a sequence $\{a_l\}_{l=1}^{\infty} \subset \mathbf{N}$ that $\limsup_{l\to\infty}(a_{l+1} - a_l)/d^{a_l} \geq 1$, and set $p = 1$ and $q = \exp\bigl(2i\pi\sum_{l=1}^{\infty} 1/d^{a_l}\bigr)$. Then $p, q \in J(f) = \{|z| = 1\}$, $q \in \widehat{\mathbf{C}} - \bigcup_{k\geq 0} f^{-k}(p)$ and, by a direct calculation, $\limsup_{m\to\infty}(1/d^{a_m})\log\bigl(1/[p, f^{a_m}(q)]\bigr) \geq \log d > 0$.

In the third case (iii), we have the following.

Theorem 4 (convergence of potentials and rotation domains). *Let $f \in \mathrm{Rat}$ be of degree $d \geq 2$. For $p, q \in \widehat{\mathbf{C}}$ such that p is not periodic but contained in a rotation domain and $q \in \mathrm{GO}_f(p)$, (5) (and (4)) holds.*

Hence in (iii), we have a stronger assertion that

$$q \in \mathrm{Fol}_f(p) - \mathrm{GO}_f(p) \tag{7}$$

than (6). The following is open.

Problem (a new problem on rotation domains). For some $p \in \widehat{\mathbf{C}}$ not periodic but contained in a rotation domain and $q \in \mathrm{Fol}_f(p) - \mathrm{GO}_f(p)$, is it possible that (5) does not hold?

If the answer to this problem is No, we can conclude that $\{V_{\mu(f^k,p)/d^k}\}$ always converges to V_{μ_f} on $F(f)$ except for the trivial case (i).

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2. The continuity of the potential of μ_f

In the rest of this paper, we fix $f \in \mathrm{Rat}$ of degree $d \geq 2$.

Notation 2.1. (i) $\|\cdot\|$ denotes the Euclidean norm on $\mathbf{C}^2$.
(ii) $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \big(i/(2\pi)\big)(\overline{\partial} - \partial)$, hence $\mathrm{dd}^c = (i/\pi)\partial\overline{\partial}$.
(iii) $\pi\colon \mathbf{C}^2 - O \to \widehat{\mathbf{C}}$ is the canonical projection which maps (z_0, z_1) to z_1/z_0 when $z_0 \neq 0$. Here O is the origin in $\mathbf{C}^2$.
(iv) An integration $\int_{\widehat{\mathbf{C}}} \phi \, \mathrm{d}\mu$ is also written as $\langle \phi, \mu \rangle$.

Lemma 2.1 (continuity of the potential [7]). *V_{μ_f} is continuous on $\widehat{\mathbf{C}}$.*

This lemma was proved by Mañé [7], whose proof was based on a quite technical lemma. We shall give a short proof of Lemma 2.1 by the pluripotential theory rather than his technical lemma.

Proof. There exists a homogeneous polynomial map $F\colon \mathbf{C}^2 \to \mathbf{C}^2$ of degree d such that $\pi \circ F = f \circ \pi$ on $\mathbf{C}^2 - O$. The *escaping rate function* $G^F\colon \mathbf{C}^2 \to [-\infty, +\infty)$ is the limit of $\{\log \|F^k\|/d^k\}$, which is uniform on $\mathbf{C}^2 - O$ (cf. [13, Theorem 1.5]). Hence G^F is continuous on $\mathbf{C}^2 - O$ and plurisubharmonic on $\mathbf{C}^2$.

Since $\pi^*\sigma = \mathrm{dd}^c \log \|\cdot\|$, it follows from Theorem 1.1 that $\pi^*\mu_f = \mathrm{dd}^c G^F$.

Lemma 2.2 (a Stokes formula). *For $p \in \widehat{\mathbf{C}}$,*

$$\mathrm{dd}^c \log \frac{1}{[\,\cdot\,, p]} = \sigma - \delta_p, \tag{8}$$

and hence for every regular probability measure μ on $\widehat{\mathbf{C}}$,

$$\mathrm{dd}^c V_\mu = \sigma - \mu. \tag{9}$$

Hence $dd^c(\pi^* V_{\mu_f} - \log\|\cdot\| + G^F) = 0$, which concludes that $\pi^* V_{\mu_f}$ is continuous on $\mathbf{C}^2 - O$ and hence V_{μ_f} is continuous on $\widehat{\mathbf{C}}$. □

3. Proof of Theorem 1

Definition 3.1 (accumulation and convergence loci). The *accumulation* and *convergence* loci of the *averaged* value distributions of f in $\widehat{\mathbf{C}}$ are defined by

$$A(f) := \left\{ p \in \widehat{\mathbf{C}} : \text{there exists } \{k_j\}_{j=0}^{\infty} \subset \mathbf{N} \text{ such that } \lim_{j\to\infty} \frac{\mu(f^{k_j}, p)}{d^{k_j}} = \mu_f \right\},$$

$$\mathrm{Conv}(f) := \left\{ p \in \widehat{\mathbf{C}} : \lim_{k\to\infty} \frac{\mu(f^k, p)}{d^k} = \mu_f \right\},$$

respectively.

Theorem 1.1 means $\widehat{\mathbf{C}} - E(f) = \mathrm{Conv}(f)$. We show the following, which contains Theorem 1:

Theorem 5 (characterizations of exceptional sets). *We have*

$$E_N(f) = E_V(f) = E(f) = \widehat{\mathbf{C}} - \mathrm{Conv}(f) = \widehat{\mathbf{C}} - A(f).$$

Proof. For $p \in \widehat{\mathbf{C}}$ and $k \in \mathbf{N}$, it follows from (8) that

$$dd^c \frac{w(f^k, p)}{d^k} = \frac{(f^k)^*\sigma - \mu(f^k, p)}{d^k}.$$

Hence for every C^∞ function ϕ on $\widehat{\mathbf{C}}$,

$$\left| \left\langle \phi, \frac{(f^k)^*\sigma - \mu(f^k, p)}{d^k} \right\rangle \right| = \left| \left\langle \frac{w(f^k, p)}{d^k}, dd^c\phi \right\rangle \right| \le \left(\max_{\widehat{\mathbf{C}}} \left| \frac{dd^c\phi}{\sigma} \right| \right) \frac{m(p, f^k)}{d^k}.$$

The $\liminf$ and $\limsup$ of the right-hand side are 0 if $p \in \widehat{\mathbf{C}} - E_N(f)$ and $p \in \widehat{\mathbf{C}} - E_V(f)$, respectively, which in fact implies that $A(f) \supset \widehat{\mathbf{C}} - E_N(f)$ and $\mathrm{Conv}(f) \supset \widehat{\mathbf{C}} - E_V(f)$ since $\{w(f^k, p)/d^k\}$ is a sequence of δ-subharmonic functions on $\widehat{\mathbf{C}}$.

For the proof of the following, see [12] or [11].

Lemma 3.1 (Riesz decomposition). *For $p, q \in \widehat{\mathbf{C}}$,*

$$w(f, p)(q) = V_{\mu(f,p) - f^*\sigma}(q) + m(f, p).$$

By Lemma 3.1, for $p, q \in \widehat{\mathbf{C}}$ and $k \in \mathbf{N}$,

$$\frac{w(f^k, p)(q)}{d^k} = V_{(\mu(f^k, p) - (f^k)^*\sigma)/d^k}(q) + \frac{m(f^k, p)}{d^k}. \tag{10}$$

We integrate both sides of (10) by $\mathrm{d}\mu_f(q)$. Then by the Fubini theorem and $f_*\mu_f = \mu_f$ from (1), we have

$$\frac{1}{d^k} V_{\mu_f}(p) = \left\langle V_{\mu_f}, \frac{\mu(f^k, p) - (f^k)^*\sigma}{d^k} \right\rangle + \frac{m(f^k, p)}{d^k}. \tag{11}$$

Since V_{μ_f} is continuous on $\widehat{\mathbf{C}}$ by Lemma 2.1, (11) implies that $A(f) \subset \widehat{\mathbf{C}} - E_N(f)$ and $\mathrm{Conv}(f) \subset \widehat{\mathbf{C}} - E_V(f)$.

Finally we show that $E(f) \subset E_N(f)$. We recall the following (cf. [9]).

Lemma 3.2 (algebraic characterization of the exceptional set).

$$E(f) = \{p \in \widehat{\mathbf{C}} : \textit{periodic of period} \le 2 \textit{ and critical of order } d-1\}.$$

Let $p \in E(f)$. When p is of period one, by the Böttcher theorem, there exists a neighborhood U'' of p such that f is conformally conjugate to $z \mapsto z^d$ there. When p is of period two, there exist neighborhoods U of p and U' of $f(p)$ such that f^2 is conformally conjugate to $z \mapsto z^{d^2}$ on U and $f(U') \subset U$, and we put $U'' := U \cup U'$. For such U'', there exists $C > 0$ such that $\int_{U''} w(f^k, p)\,\mathrm{d}\sigma > Cd^k$ for every $k \in \mathbf{N}$. Hence $\mathrm{NE}(p; \{f^k\}) \ge C$, which implies $p \in E_N(f)$. Now the proof is completed. □

4. Proof of Theorems 2 and 3

Lemma 4.1 (convergence of potentials).

$$\lim_{k\to\infty} V_{(f^k)^*\sigma/d^k} = V_{\mu_f} \tag{12}$$

on $\widehat{\mathbf{C}}$.

Proof. Let s and t be local holomorphic sections of π on $\mathbf{C}$ and $\widehat{\mathbf{C}} - \{0\}$, respectively. Since there exists a holomorphic function c on $\mathbf{C} - \{0\}$ such that $t = cs$ on $\mathbf{C} - \{0\}$, we have $t^*F = c^d s^*F$ and $t^*G^F = s^*G^F + \log|c|$ there. Hence

$$H_k(z) := \begin{cases} s^*\left(\dfrac{1}{d^k}\log\|F^k\| - G^F\right) & \text{on } \mathbf{C}, \\ t^*\left(\dfrac{1}{d^k}\log\|F^k\| - G^F\right) & \text{on } \widehat{\mathbf{C}} - \{0\} \end{cases}$$

is a well-defined function on $\widehat{\mathbf{C}}$, and tends to 0 uniformly on $\widehat{\mathbf{C}}$ as $k \to \infty$. By $\pi^*\sigma = \mathrm{dd}^c \log\|\cdot\|$ and $\pi^*\mu_f = \mathrm{dd}^c G^F$, it follows that

$$\frac{(f^k)^*\sigma}{d^k} - \mu_f = \mathrm{dd}^c H_k$$

on $\widehat{\mathbf{C}}$. Now for every $q \in \widehat{\mathbf{C}}$,

$$V_{(f^k)^*\sigma/d^k}(q) - V_{\mu_f}(q) = \left\langle \log\frac{1}{[\,\cdot\,, q]}, \mathrm{dd}^c H_k \right\rangle = \left\langle H_k, \mathrm{dd}^c \log\frac{1}{[\,\cdot\,, q]} \right\rangle = \langle H_k, \sigma - \delta_q \rangle$$

converges to 0. □

We shall finish the proof of Theorems 2 and 3.

From Theorem 1, (10) and Lemma 4.1, it follows that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$,

$$\liminf_{k\to\infty} \frac{w(f^k,p)(q)}{d^k} = \liminf_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) - V_{\mu_f}(q). \tag{13}$$

When $\liminf_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) \neq V_{\mu_f}(q)$ at $q \in \widehat{\mathbf{C}}$, it follows that there exist $N \in \mathbf{N}$ and $\delta > 0$ such that for every $k \geq N$,

$$[f^k(q), p] \leq e^{-\delta d^k}. \tag{14}$$

Hence p is a fixed point of f. When $q \in \widehat{\mathbf{C}} - \bigcup_{k>0} f^{-k}(p)$, it follows from (14) that p is a superattractive fixed point of f of order $d-1$. Hence $p \in E(f)$ from Lemma 3.2, which is a contradiction. Now we have proved Theorem 2.

Again, from Theorem 1, (10) and Lemma 4.1, it follows that for $p \in \widehat{\mathbf{C}} - E(f)$ and $q \in \widehat{\mathbf{C}}$,

$$\limsup_{k\to\infty} \frac{w(f^k,p)(q)}{d^k} = \limsup_{k\to\infty} V_{\mu(f^k,p)/d^k}(q) - V_{\mu_f}(q); \tag{15}$$

(13) and (15) conclude the proof of Theorem 3. □

5. Proof of Theorem 4

It is enough to give a proof in the case of $q = p$.

Let D denote the rotation domain containing p, and S the unique analytic f-invariant circle in D containing p. Without loss of generality, we assume that $D \subset \mathbf{C}$ and the period of D equals one, and choose an analytic *linearizing* map h of $f \mid D$ that conformally maps D onto a disc or a concentric annulus centered at the origin containing the unit circle, and as S onto it. Then on D,

$$h \circ f = \lambda \cdot h,$$

where $\lambda = e^{2i\pi\alpha}$ for some $\alpha \in \mathbf{R} - \mathbf{Q}$ is called the *rotation number* of D.

Notation 5.1. $A \asymp B$ means $A/C < B < CA$ for some constant C.

Then the following uniform estimate holds: on $S \Subset D$,

$$[f^k(z), z] \asymp |f^k(z) - z| \asymp |h \circ f^k(z) - h(z)| = |\lambda^k - 1| \cdot |h(z)| = |\lambda^k - 1|,$$

where the implicit constants are independent of $k \in \mathbf{N}$ and $z \in S$. Hence

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{[p, f^k(p)]} = \limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|}. \tag{16}$$

On the right-hand side of (16), we have already shown the following.

Theorem 5.1 (a priori bound [11]). *The rotation number λ of every rotation domain satisfies*

$$\limsup_{k\to\infty} \frac{1}{d^k} \log \frac{1}{|\lambda^k - 1|} = 0. \tag{17}$$

Combining (16) and (17) we finish the proof. □

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