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SECTOR REFLECTIONS IN THE PLANE

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Abstract. We give the optimal bi-Lipschitz reflection of a Euclidean sector in the plane. This solves a problem posed by F.W. Gehring and K. Hag.

Suppose that (X, d_X) and (Y, d_Y) are metric spaces and $f\colon X \to Y$ is a bijection. Then f is said to be *bi-Lipschitz* if there exists a constant $L \geq 1$ such that

$$\frac{1}{L} d_X(x_1, x_2) \leq d_Y\bigl(f(x_1), f(x_2)\bigr) \leq L d_X(x_1, x_2) \tag{1}$$

for all $x_1, x_2 \in X$. The *bi-Lipschitz constant* of a bi-Lipschitz map f is the smallest constant $L \geq 1$ such that (1) is satisfied. If such a map exists, then X and Y are said to be bi-Lipschitz equivalent. Moreover, f is *optimal* if its bi-Lipschitz constant L is the infimal constant over all bi-Lipschitz maps from X to Y. If X and Y are suitable spaces, then one can easily show that such a map exists.

In general, it seems that the problem of finding an optimal bi-Lipschitz map can be exceedingly difficult. There is neither a theoretical foundation for constructing such a map, nor a methodology for proving that one is optimal. Few papers in the literature consider problems of this class and of these, even fewer present solutions. Furthermore, these papers are restricted in their consideration to particular spaces and make use of specially tailored geometric arguments not necessarily applicable elsewhere. For example, in [MI], J. Milnor finds the optimal bi-Lipschitz planar image of a spherical cap in $\mathbf{R}^3$.

In this note we are interested in finding optimal bi-Lipschitz reflections with respect to the Euclidean metric. As defined in [GH], a region $D \subseteq \mathbf{R}^2$ admits a *bi-Lipschitz reflection* if there exists a bi-Lipschitz map $f\colon \overline{D} \to \mathbf{R}^2 \setminus D$ that satisfies

(i) $f(\overline{D}) = \mathbf{R}^2 \setminus D$ and
(ii) $f(x) = x$ for all $x \in \partial D$

We will denote the image $\mathbf{R}^2 \setminus D$ of $\overline{D}$ under f by D^*.

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For the remainder of this note, assume that $\theta \in [0, 2\pi]$ denotes a polar angle, $r \in \mathbf{R}^+$ a radial distance and that a coordinate pair of the form (r, θ) represents the point with Cartesian coordinates

$$(r\cos\theta, r\sin\theta),$$

unless explicitly stated otherwise. Also, denote by

$$R(\alpha) = \{(r, \alpha) : r \in \mathbf{R}^+\}$$

the ray eminating from the origin with angle α from the polar axis.

In [GH], F. W. Gehring and K. Hag consider bi-Lipschitz reflections of a closed Euclidean sector $S(\gamma)$ of angle $0 < \gamma < 2\pi$, defined in polar coordinates as the set $\{(r, \theta) : 0 \le r,\ 0 \le \theta \le \gamma\}$. At first, one would suspect that an angular adjustment map $\psi\colon S(\alpha) \to S(\beta)$ where

$$(r, \theta) \mapsto \big(r, (\beta/\alpha)\theta\big)$$

for $\alpha = \gamma$ and $\beta = 2\pi - \gamma$ would yield optimal results. Note that ψ satisfies

$$A|x_1 - x_2| \le |\psi(x_1) - \psi(x_2)| \le B|x_1 - x_2| \tag{2}$$

for all $x_1, x_2 \in S(\alpha)$, where $A = \min\big(\beta/\alpha, 1\big)$ and $B = \max\big(\beta/\alpha, 1\big)$. We observe that under this map the sector experiences either expansion or contraction, but never both.

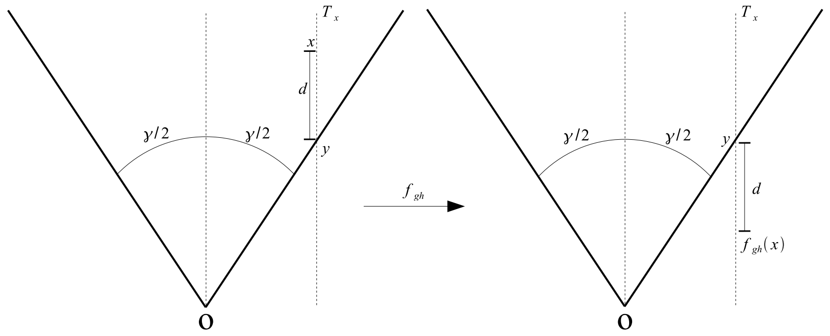

Figure 1. The Gehring–Hag bi-Lipschitz reflection of $S(\gamma)$.

Gehring and Hag make the natural conjecture that the optimal reflection for sectors $S(\gamma)$ with $0 < \gamma \le \pi$ is a piecewise linear map defined as follows. Let $x \in S(\gamma)$, T_x be the unique line containing x parallel to the bisector of $S(\gamma)$ and y the point where T_x intersects $\partial S(\gamma)$. Then $f_{gh}(x)$ is defined to be the unique point in $S(\gamma)^*$ on the line T_x with Euclidean distance $d = |x - y|$ from y. See Figure 1 for an illustration of the map. One may readily verify that their map has bi-Lipschitz constant $\cot\big(\frac{1}{4}\gamma\big)$. There is very compelling evidence in support of Gehring and Hag's conjecture. Indeed, if we divide the sector into two parts by means of a ray through the origin, and further assume linearity in each of the parts, then the optimality of this map can be proved with elementary geometry. Furthermore, under the far less restrictive assumption of piecewise differentiability at the origin, we can use a rescaling argument to reduce the problem to the case of piecewise linearity, again implying the optimality of the Gehring–Hag map.

In light of these considerations, it is quite striking that in general their conjecture fails to hold. In this note we give the optimal reflection of Euclidean sectors $S(\gamma)$ for $0 < \gamma < 2\pi$.

In order to define the map, we first need some preparation.

Lemma 1. *For given $0 < \gamma \le \pi$ there exists unique α with $0 \le \alpha < \frac{1}{2}\gamma$ and $\alpha \le \frac{1}{4}\pi$ such that the map φ defined by*

$$t \mapsto \frac{\pi + 2t - \gamma}{\gamma - 2t} \tan^2 t$$

attains the value $\varphi(\alpha) = 1$.

Proof. It is clear that $\varphi(0) = 0$. Now, we consider two cases. If $0 < \gamma \le \frac{1}{2}\pi$, it is clear that $\varphi(t) \to \infty$ as $t \to \frac{1}{2}\gamma$. Alternatively, if $\frac{1}{2}\pi < \gamma \le \pi$, then $\gamma > \frac{1}{4}\pi$. In particular,

$$\varphi\left(\frac{\pi}{4}\right) = \frac{\frac{3}{2}\pi - \gamma}{\gamma - \frac{1}{2}\pi} \ge 1.$$

So in either case, the Intermediate Value Theorem and the strict monotonicity of φ implies that there exists a unique α in the required range such that $\varphi(\alpha) = 1$. □

By a similar consideration as was given in Lemma 1, the inequality

$$\frac{\pi + 2\alpha - \gamma}{\gamma - 2\alpha} > 1 \tag{3}$$

holds for $0 < \gamma \le \pi$ and $0 \le \alpha < \frac{1}{2}\gamma$. This fact will play an important role in the determination of the bi-Lipschitz constant of the optimal map reflection of $S(\gamma)$.

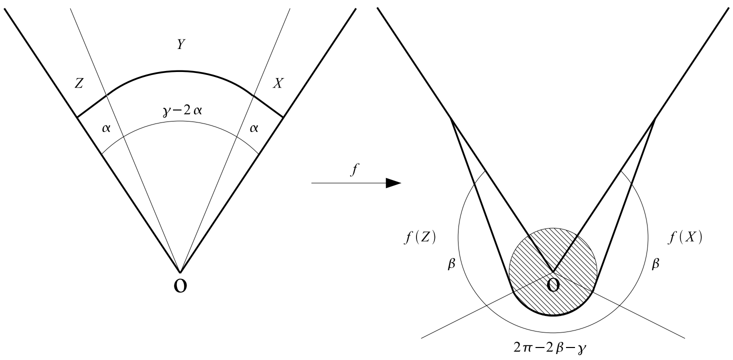

Figure 2. Optimal bi-Lipschitz reflection of $S(\gamma)$.

Let $\gamma \in (0, \pi]$ be fixed, α be as in Lemma 1, and $\beta = \frac{1}{2}\pi - \alpha$. Define $f\colon S(\gamma) \to S(\gamma)^*$ piecewise as follows:

(i) If $0 \le \theta \le \alpha$, then f is the unique linear reflection map such that $f(r,0) = (r,0)$ and $f(r,\alpha) = (r\tan\alpha, -\beta)$.

(ii) If $\alpha \le \theta \le \gamma - \alpha$, then f is the angular adjustment followed by a rescaling and reflection,

$$(r,\theta) \mapsto \left(r\tan\alpha, -\frac{2\pi - 2\beta - \gamma}{\gamma - 2\alpha}(\theta - \alpha) - \beta\right).$$

(iii) If $\gamma - \alpha \le \theta \le \gamma$, then f is the unique linear reflection map such that $f(r,\gamma) = (r,\gamma)$ and $f(r, \gamma - \alpha) = (r\tan\alpha, \gamma + \beta)$.

The domain of definition of f in (i), (ii), and (iii) above corresponds to X, Y, and Z, respectively, in Figure 2. Note that on the rays $R(\alpha)$ and $R(\gamma - \alpha)$ the definition of f is consistent.

Lemma 2. *Suppose $S(\gamma)$ is a Euclidean sector of angle $0 < \gamma \le \pi$, and the map $f\colon S(\gamma) \to S(\gamma)^*$ is defined as above. Then the bi-Lipschitz constant L of f is given by $\cot\alpha$, where α is as specified in Lemma* 1.

Proof. Consider first the closed sector with $0 \le \theta \le \alpha$, corresponding to X in Figure 2. Let T_1 be the line segment connecting the points with polar coordinates $(1,0)$ and $(\cos\alpha, \alpha)$, and T_2 be the line segment connecting $(1,0)$ and $(\cos\beta, -\beta)$. These are labeled in Figure 3. Then $T_2 = f(T_1)$. Moreover, T_1 is orthogonal to the ray $R(\alpha)$ and T_2 to the ray

$$f\big(R(\alpha)\big) = \{(r\tan\alpha, -\beta) : r \in \mathbf{R}^+\}.$$

Thus in this sector, f can be thought of as the composition of two maps. First, it is a rescaling by a factor of $\tan\alpha$ and $\cot\alpha$ in the orthogonal directions $R(\alpha)$ and $R(-\beta)$, respectively, and then a reflection. It is obvious that the reflection is an isometry. Hence, f satisfies

$$\tan\alpha|x_1 - x_2| \le |f(x_1) - f(x_2)| \le \cot\alpha|x_1 - x_2| \tag{4}$$

for all $x_1, x_2 \in \{(r,\theta) : 0 \le r,\ 0 \le \theta \le \alpha\}$. A similar consideration shows that f satisfies inequality (4) for all $x_1, x_2 \in \{(r,\theta) : 0 \le r,\ \gamma - \alpha \le \theta \le \gamma\}$, which corresponds to Z in Figure 2. Note that the bounds in (4) are the best possible.

Now suppose that $x_1, x_2 \in \{(r,\theta) : 0 \le r,\ \alpha \le \theta \le \gamma - \alpha\}$, which corresponds to Y in Figure 2. In this sector, f is a sector expansion followed by a rescaling. Hence by inequality (2), we see that f satisfies

$$\tan\alpha|x_1 - x_2| \le |f(x_1) - f(x_2)| \le \frac{2\pi - 2\beta - \gamma}{\gamma - 2\alpha}\tan\alpha|x_1 - x_2|. \tag{5}$$

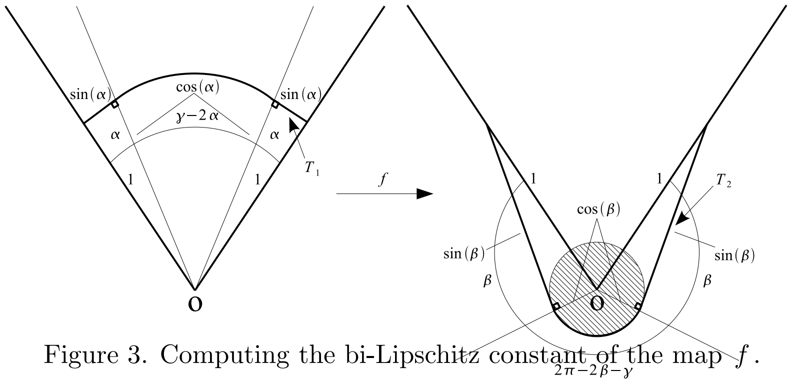

Figure 3. Computing the bi-Lipschitz constant of the map f.

Note that by (3), the multiplicative constant on the right-hand side of (5) is in fact larger than the constant on the left-hand side. Again, it is clear that the bounds in (5) are the best possible. Combining (4) and (5),

$$L = \max\left(\cot\alpha, \frac{2\pi - 2\beta - \gamma}{\gamma - 2\alpha}\tan\alpha\right).$$

It follows from Lemma 1 that

$$\cot\alpha = \frac{2\pi - 2\beta - \gamma}{\gamma - 2\alpha}\tan\alpha$$

and so $L = \cot\alpha$ is the bi-Lipschitz constant of f. ▫

Theorem 3. *Let $S(\gamma)$ be a Euclidean sector of angle $0 < \gamma \le \pi$. Then the unique optimal bi-Lipschitz reflection is given by the map f as defined above.*

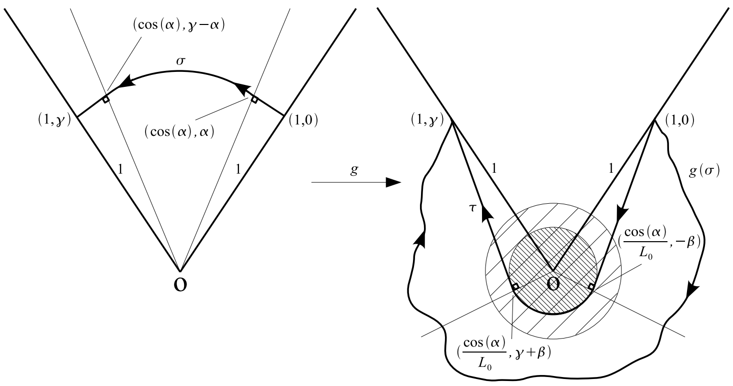

Figure 4. The path σ in $S(\gamma)$ and the paths τ and $g(\sigma)$ in $S(\gamma)^*$. The path $g(\sigma)$ avoids the lightly shaded disk centered at the origin and the path τ must avoid the more darkly shaded disk.

Proof. Suppose that $\sigma\colon [0,1] \to S(\gamma)$ is the path of constant speed that runs from $(1,0)$ to $(\cos\alpha, \alpha)$ along a line segment, then to $(\cos\alpha, \gamma-\alpha)$ along a circular arc of radius $\cos\alpha$ centered at the origin and finally to $(1,\gamma)$ along a line segment. Similarly, define $\tau\colon[0,1]\to S(\gamma)^*$ to be a path of constant speed that runs from $(1,0)$ to $(\cos\beta,-\beta)$ along a line segment, then to $(\cos\beta,\gamma+\beta)$ along a circular arc of radius $\cos\beta$ centered at the origin and then to $(1,\gamma)$ along a line segment.

Note that σ is orthogonal to the line segments emanating from the origin to the points $(\cos\alpha,\alpha)$ and $(\cos\alpha,\gamma-\alpha)$. Similarly, τ is orthogonal to the line segments with endpoints $(0,0),(\cos\beta,-\beta)$ and $(0,0),(\cos\beta,\gamma+\beta)$. Also, $f(\sigma)=\tau$.

Let $L_0=\cot\alpha$, the bi-Lipschitz constant of f, and suppose that $g\colon S(\gamma)\to S(\gamma)^*$ is any arbitrary L-bi-Lipschitz reflection of $S(\gamma)$. Furthermore, suppose that $L\le L_0$. Then by the bi-Lipschitz inequality, $g(\sigma)$ must avoid an open disk of radius $\cos\alpha/L$ centered at the origin and so also an open disk of radius $\cos\alpha/L_0$ centered at the origin. These disks are sketched in Figure 4. Obviously, τ is the unique shortest path connecting the points with polar coordinates $(1,0)$ and $(1,\gamma)$ in $S(\gamma)^*$, avoiding the latter disk. Hence,

$$L\ge\frac{l\big(g(\sigma)\big)}{l(\sigma)}\ge\frac{l(\tau)}{l(\sigma)}=L_0, \tag{6}$$

where l signifies the length of a curve. The equality in (6) follows from the consideration given in the computation of the bi-Lipschitz constant of f from the proof of Lemma 2. In particular, f attains its bi-Lipschitz constant along the path σ. Therefore f is optimal.

Lastly, we show the uniqueness of the optimal map. Suppose that g is an arbitrary L_0-bi-Lipschitz reflection of $S(\gamma)$. From the argument above it follows that $g(\sigma)$ and τ are both parameterizations of the same curve. We combine inequality (6) with the fact that the bi-Lipschitz constant of g is L_0 to obtain

$$l\big(g(\sigma)\big)=L_0 l(\sigma)=l(\tau). \tag{7}$$

Moreover, for all $t\in[0,1]$ we have both

$$l\big(g(\sigma\mid[0,t])\big)\le L_0 l(\sigma\mid[0,t]) \tag{8}$$

and

$$l\big(g(\sigma\mid[t,1])\big)\le L_0 l(\sigma\mid[t,1]). \tag{9}$$

It follows that we have equality in (8) and (9) by (7) and so $g(\sigma)$ is of constant speed. Therefore $g\big(\sigma(t)\big)=\tau(t)=f(\sigma(t))$ for all $t\in[0,1]$ and so the maps g and f agree along the path σ. Applying this argument to rescalings of σ, we see that the map g agrees with the map f everywhere and so the uniqueness is proven. $\square$

Corollary 4. *Suppose $S(\gamma)$ is a Euclidean sector of angle $\pi \le \gamma < 2\pi$. Then the unique optimal bi-Lipschitz reflection is given by the map f, where f^{-1} is the optimal bi-Lipschitz reflection of $S(2\pi - \gamma)$.*

Proof. This is obvious since the inverse of an L-bi-Lipschitz map is also L-bi-Lipschitz. This fact also implies that the inverse of a unique optimal bi-Lipschitz map is also unique and optimal. □

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