Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 30, 2005, 3–48

CYLINDER AND HOROBALL PACKING IN HYPERBOLIC SPACE

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Abstract. A cylinder of radius r in hyperbolic space is the closed set of points within distance r of a given geodesic. We define the density of a packing of cylinders of radius r in n dimensions and prove that, when $n=3$, this density cannot exceed $(1+23e^{-r})\varrho_\infty$. Here $\varrho_\infty = 0.853276\ldots$ is the greatest possible density of a horoball packing in space, from which the above bound is obtained by applying a continuity argument.

Applications of this result are found in volume estimates for hyperbolic 3-manifolds and orbifolds where estimates on the density of cylinder packings are an essential part of identifying hyperbolic 3-manifolds with maximal automorphism groups or with high order symmetries.

We further give a generalization of Blichfeld's inequality and construct packings of horoballs in n-space with density at least 2^{1-n}.

Cylinder packings associated with the fundamental group of the orbifold obtained by performing $(m,0)$ Dehn filling on the figure of eight knot complement provide examples of dense packings for a spectrum of radii when $n=3$. We explicitly calculate the densities of these packings.

1. Introduction

A *cylinder* in hyperbolic space of radius r is the set of points within distance r of a given geodesic. The subject of this paper is packings of cylinders of a given radius in hyperbolic space. In particular we evaluate lower bounds for the density of such a packing in three dimensions. Such bounds are of interest because they can be used to improve estimates of volumes of hyperbolic manifolds in much the same way that Böröczky's bounds [Bö1], [Bö2] for the optimal packing density of balls in hyperbolic space have been used in the past [GM1]. Virtually all known bounds for the volume of hyperbolic 3-manifolds are obtained from the study of hyperbolic cylinder packings obtained as the lift of a tubular neighbourhood

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2000 Mathematics Subject Classification: Primary 51M09, 52C17, 57M50, 57N10; Secondary 51M04.

Research supported in part by grants from the N. Z. Marsden Fund and the N. Z. Royal Society (James Cook Fellowship).

of a simple closed geodesic, [GM1], [GM2], [MM1], [GMM], [P1], [P2], [P3]. In particular, applications of our results are concerning estimates on the density of cylinder packings are an essential part of identifying hyperbolic 3-manifolds with maximal automorphism groups and high order symmetries [MM3].

If a hyperbolic manifold $M = \mathbf{H}^3/\Gamma$ contains a geodesic with an embedded tubular neighbourhood of radius r, called a *collar*, then the volume of the collar provides a trivial lower bound for the volume of M. The lifts of this collar to $\mathbf{H}^3$ constitute a cylinder packing and if the density of any such packing is known not to exceed ϱ, then the volume estimate for M can be increased by a factor of ϱ^{-1}. In this way our results can be used to improve many known bounds.

In contrast to the Euclidean case, it is a non-trivial problem even to *define* what is meant by the density of a packing in hyperbolic space. Generally this is possible only in a "local" sense rather than for the packing as a whole. We make these ideas precise in Sections 2 and 4. Given these definitions, we prove that the local packing density of a cylinder packing cannot exceed

$$(1 + 23e^{-r})\varrho_\infty \tag{1}$$

in 3-dimensional space. Here $\varrho_\infty = 0.853276\ldots$ is the optimal horoball packing density in 3-space [F].

This bound is not sharp. We prove a somewhat more elaborate bound, of which (1) is a simplification, and this in turn can be slightly improved by refining the proof given here, but there seems to be no chance of obtaining a sharp bound without some essentially new methods. On the other hand, (1) is asymptotically sharp, in the sense that there is a spectrum of radii $r_n \to \infty$, for which there are packings of cylinders of radius r_n whose densities are asymptotically equal to ϱ_∞, and thus to the bound in (1). These packings are invariant under the Kleinian groups associated with $(n, 0)$ Dehn filling on the figure of eight knot complement. We consider them in more detail in Section 6.

Roughly, as $r \to \infty$ the shape of a cylinder of radius r approaches that of a horoball which we thus consider to be a degenerate cylinder of infinite radius. The bound at (1) is closely related to known density results about horoball packings. It is obtained by using the fact that, for large r, a cylinder packing is well approximated locally by a horoball packing, together with the known horoball packing of greatest density.

Horoball packings in hyperbolic n-space are in turn closely related to ball packings in Euclidean $(n-1)$-space (horoballs in the halfspace model project to balls in the boundary). In particular the best known bounds for horoball packings in space depends on the known best disk packing in the Euclidean plane. The fact that densest ball packings are still unknown in Euclidean spaces of dimension greater than two is the main difficulty in generalizing our results to dimension four or more. (We note that Hales recent solution of the Kepler conjecture, though an important result, does not help here since no uniqueness has been established).

We do however obtain some results about horoball packings in n dimensions, including a generalization of Blichfeld's inequality [Ro], and a construction of an n-dimensional horoball packing, with local density at least 2^{1-n}.

While for large r cylinder packings are approximated by horoball packings, for small r they are (locally) approximated by Euclidean cylinder packings. The densest such packing has been determined by A. Bezdek and W. Kuperberg [BK] to be that in which the cylinders are all parallel and which meet any plane perpendicular to them all in an optimal packing of disks. This packing consequently has the same density $\left(\pi/\sqrt{12}\right)$ as the optimal Euclidean disk packing. More surprisingly cylinder packings with positive density are known in which no two cylinders are parallel and indeed the possibility has not been excluded that the density of such packings may be made arbitrarily close to $\pi/\sqrt{12}$ [K].

The methods of [BK] can be adapted in a manner not too different to that of the present paper (that is by approximation) to give upper density bounds for packings by "thin" cylinders in hyperbolic space asymptotic to those of the Euclidean case. Przeworski [P2], [P3] has used this approach to find nontrivial density bounds for all r. For small r (roughly $r \leq 7.1$) these estimates are better than ours.

Obtaining sharp bounds appears to be much more difficult, as the densest Euclidean cylinder packing has no analogue in hyperbolic space. Indeed it seems possible that the optimal density of a lattice packing of $\mathbf{H}^3$ by cylinders of radius r tends to 0 with r, as in the analogous case in two dimensions [MM2].

Finally we would like to thank Mike Hilden for a very helpful correspondence.

2. Local density

Let B be a subset of a metric space X. A *packing* of X by copies of B is a set $\mathscr{P}$ of isometric copies of B in X whose interiors are disjoint. We will assume that B is closed.

We will assume that X is either the unit n-sphere $\mathbf{S}^n$, n-dimensional Euclidean space $\mathbf{R}^n$ or n-dimensional hyperbolic space $\mathbf{H}^n$. Let P denote the union of sets in a packing $\mathscr{P}$ in one of these spaces. For a packing in $\mathbf{S}^n$ the *density* ϱ of the packing $\mathscr{P}$ is then defined by

$$\varrho = \frac{\operatorname{vol}(P)}{\operatorname{vol}(\mathbf{S}^n)}.$$

For a packing in $\mathbf{R}^n$ we define

$$\varrho = \lim_{R\to\infty} \frac{\operatorname{vol}\big(P \cap B(a,R)\big)}{\operatorname{vol}\big(B(a,R)\big)},$$

where this limit exists. It is easily shown that this definition is independent of the choice of a; (see e.g. [F, pp. 161–162]).

The problem with hyperbolic packings is that it is not clear that the above limit is independent of a in this case. (For further discussion of this point see [F, Section 40]).

To avoid this problem we can define various types of "local" density. These definitions all involve partitioning the space into finite-volume regions $\{R_i\}$ with disjoint interiors, and then defining, for each i, a local density

$$\varrho_i = \frac{\mathrm{vol}\,(P \cap R_i)}{\mathrm{vol}\,(R_i)}.$$

In general, these densities will differ from one region to another, but it is often possible to find an upper bound for the ϱ_i, which can then in some sense be considered as an upper bound for the packing as a whole. If B is a set in a packing, then the *Dirichlet cell* $D(B)$ is defined by

$$\big\{z \in \mathbf{H}^n \mid \forall B' \in \mathscr{P},\ B' \neq B,\ \varrho(z,B) \leq \varrho(z,B')\big\},$$

where $\varrho(\,\cdot\,,\cdot\,)$ is the hyperbolic metric. Thus $D(B)$ is the set of all points which are at least as close to B as to any other set in the packing. Clearly the set $\{D(B) \mid B \in \mathscr{P}\}$ tessellate $\mathbf{H}^n$, and, provided these cells are of finite volume, we obtain a definition of local density. For cylinder packings the Dirichlet cells have infinite volume so that the definition needs to be modified. We do this in Section 4.

The nicest situation arises when some co-finite volume discrete group Γ acts transitively on $\mathscr{P}$. Moreover, this is also where most applications are to be found. We refer to such a packing as a *lattice packing*. It is natural in this case to let $\{R_i\}$ be the translates of a fundamental domain for Γ. In this case, the values of the local density ϱ_i are clearly all the same, and independent of the fundamental domain chosen. Moreover, as the following corollary shows, this definition is also independent of the choice of group. We refer to the local density defined in this way as the *group density* of the packing.

We define the *symmetry group* of a packing $\mathscr{P}$ to be the group of isometries which permute the members of $\mathscr{P}$.

Lemma 2.1. *Let $\mathscr{P}$ be a packing of $\mathbf{H}^n$ by cylinders or horoballs. If the endpoints of the cylinders* (*respectively tangency points of the horoballs*) *fail to lie on any codimension 2 sphere or hyperplane in $\partial\mathbf{H}^n$, then the symmetry group of $\mathscr{P}$ is discrete.*

Proof. We prove the lemma for cylinders. The proof for horoballs is similar. Let Γ be the symmetry group of $\mathscr{P}$ and $\{g_k\}$ a sequence of isometries in Γ which converges to the identity. There is a finite set of cylinders whose endpoints fail to lie on any codimension 2 sphere or hyperplane in $\partial\mathbf{H}^n$. For sufficiently large k, g_k leaves these cylinders invariant and fixes their endpoints. Therefore g_k is either a reflection or the identity. But, for sufficiently large k, g_k must be the identity and so Γ is discrete. □

The necessity of the condition that the endpoints of the cylinders fail to lie on any codimension 2 sphere or hyperplane is clear.

The lemma implies in particular that any packing in $\mathbf{H}^3$ of two or more cylinders or three or more horoballs has discrete symmetry group.

We next show the density of a lattice packing does not depend on the group.

Corollary 2.1. *Let $\mathscr{P}$ be a packing of $\mathbf{H}^n$ by horoballs, or cylinders, which is invariant under the actions of two cofinite volume discrete groups, Γ_1 and Γ_2 which act transitively on $\mathscr{P}$ and have respective fundamental domains D_1 and D_2, then*

$$\frac{\mathrm{vol}\,(P \cap D_1)}{\mathrm{vol}\,(D_1)} = \frac{\mathrm{vol}\,(P \cap D_2)}{\mathrm{vol}\,(D_2)}.$$

Proof. From the above lemma, the symmetry group, Γ, of $\mathscr{P}$ is discrete and, since it contains Γ_1 and Γ_2, it is also cofinite volume. We may therefore assume that $\Gamma_2 = \Gamma$, so that Γ_1 is a subgroup of Γ_2, of finite index, say, k. We may also assume that D_1 is the union of k disjoint translates of D_2, whence

$$\frac{\mathrm{vol}\,(P \cap D_1)}{\mathrm{vol}\,(D_1)} = \frac{k\,\mathrm{vol}\,(P \cap D_2)}{k\,\mathrm{vol}\,(D_2)} = \frac{\mathrm{vol}\,(P \cap D_2)}{\mathrm{vol}\,(D_2)}$$

which proves the corollary. □

3. Notation and definitions

Throughout this paper we use exclusively the halfspace models $\mathbf{H}^n$ of hyperbolic n-space. The boundary of this model is $\mathbf{R}^{n-1} \cup \{\infty\}$, and when $n = 3$ we tacitly identify this with the extended complex plane $\overline{\mathbf{C}}$. In the following definitions there is a dependence on the dimension n which is not made explicit.

We let $\nu(x)$ denote the vertical projection of $x \in \mathbf{H}^n$ to the boundary. We let $\varrho(x, y)$ denote the (hyperbolic) distance in these spaces, where x and y may denote either points or sets (or one of each).

For $a < b$, let $A(a, b)$ be the closed annulus in $\partial\mathbf{H}^n$ lying between the circles of radius a and b, $C(a, b) = \nu^{-1}\big(A(a, b)\big)$ and $H(a, b)$ the region of $\mathbf{H}^n$ lying on or between the two hemispheres in $\mathbf{H}^n$ centred at the origin with radii a and b.

If B is a cylinder, then $\mathrm{ax}(B)$ denotes its axis, that is the geodesic joining its endpoints. If B is a horoball, then $\mathrm{tg}(B)$ denotes its point of tangency with the boundary. We adopt the convention that a horoball B is a cylinder of infinite radius both of whose endpoints coincide at $\mathrm{tg}(B)$. We let I denote the geodesic with endpoints 0 and ∞, $B(r)$ the cylinder of radius r with axis I, when $r < \infty$, and $B(\infty)$ the horoball with boundary $x_n = 1$.

If A is a geodesic *ultraparallel* (that is disjoint and not meeting at ∞) to I, then the *bisecting plane* of A, which we denote by $\mathrm{bp}(A)$, is the hyperplane which perpendicularly bisects the shortest geodesic arc joining A and I. If B is a cylinder of finite radius with axis ultraparallel to I, then we define its bisecting plane, $\mathrm{bp}(B)$, to be that of its axis. If B is a horoball with $\mathrm{tg}(B) \neq \infty$, and Euclidean diameter ≤ 1, then we define $\mathrm{bp}(B)$ as the perpendicular bisector of the shortest geodesic arc joining B to $B(\infty)$.

4. Local density of cylinder packings

We now use Dirichlet regions to define the local density of a cylinder packing at a specified cylinder. A slightly modified version of the definition also applies to horoball packings.

Let B be a cylinder in a packing $\mathscr{P}$ of $\mathbf{H}^n$, and $d(B)$ its associated Dirichlet region. Since both B and $d(B)$ have infinite volume, we define density as a limit. If $r < \infty$, fix a point a on $\mathrm{ax}(B)$ and let P and Q be the hyperplanes which cut $\mathrm{ax}(B)$ perpendicularly at the two points on $\mathrm{ax}(B)$, which are on either side of, and distance d_1 and d_2, respectively, from a. Let $S(a, d_1, d_2)$ be the "slice" of space lying between P and Q. We now define

$$\varrho(B) = \lim_{d_1, d_2 \to \infty} \frac{\mathrm{Vol}\,(B) \cap S(a, d_1, d_2)}{\mathrm{Vol}\,\big(d(B) \cap S(a, d_1, d_2)\big)}, \tag{2}$$

where this limit exists. Clearly, when it does, its value is independent of the choice of a. The upper and lower densities at B are defined in the same way, with limsup and liminf respectively being used above. We denote these by $\overline{\varrho}(B)$ and $\underline{\varrho}(B)$ respectively. It is these quantities which are most important in applications.

It is natural to normalize by the assumption that $\mathrm{ax}(B) = I$. In this case a is, say, the point $(0, 0, \ldots, 0, A)$ and P and Q are the hemispheres of radius Ae^{d_1} and Ae^{-d_2} centred at the origin.

We might attempt to apply this definition when $r = \infty$, but, in this case, the "slice" $S(a, d_1, d_2)$ degenerates into a region lying between two parallel hyperplanes, which both contain $\mathrm{tg}(B)$ in their boundaries. Since (for $n > 2$) this still meets B in a region of infinite volume, we modify this region slightly. Let A_i $(1 \leq i \leq n-1)$ be mutually perpendicular hyperplanes, all touching the boundary at $\mathrm{tg}(B)$ and at one other chosen point $a \in \partial\mathbf{H}^n$. Let P_i, Q_i be hyperplanes, parallel to and either side of A_i, whose intersections with the horosphere ∂B are distance d_{i1} and d_{i2}, respectively, from $A_i \cap \partial B$. Let S_i be the region lying between P_i and Q_i and S the intersection of these regions. Now define the *local density* $\varrho(B)$ of $\mathscr{P}$ *at* B, by

$$\varrho(B) = \lim \frac{\mathrm{vol}\,(B \cap S)}{\mathrm{vol}\,\big(d(B) \cap S\big)}, \tag{3}$$

the limit being taken as each $d_{i1}, d_{i2} \to \infty$ for each i. Clearly this limit, if it exists, is independent of the choice of the A_i and a. As before, the upper and lower densities at B, $\overline{\varrho}(B)$ and $\underline{\varrho}(B)$, are defined using the obvious modification. It is natural to normalize by the assumption that $\mathrm{tg}(B) = \infty$. In this case the P_i, Q_i are parallel pairs of vertical Euclidean hyperplanes, and S is the inverse projection of a box in $\mathbf{R}^{n-1} = \partial\mathbf{H}^n$.

It is easily shown that, if some cofinite volume Kleinian group Γ leaves $\mathscr{P}$ invariant and acts transitively on it, then $\varrho(B)$ coincides with the group density of $\mathscr{P}$ and so, in particular, that $\varrho(B)$ exists and is independent of B. To see this, in the case $r < \infty$, let Γ_B be the stabilizer in Γ of a cylinder B, which we may assume to have axis I. Since Γ is cofinite volume, Γ_B contains (the restriction to $\mathbf{H}^n$ of) a map of the form $\mathbf{x} \to kA\mathbf{x}$, where A is an orthogonal matrix and $k > 1$. Let k_0 be the smallest such $k > 1$, and let p be the maximum order of an elliptic in Γ_B (setting $p = 1$ if there are none). Since Γ acts transitively on $\mathscr{P}$, it also does so on the Dirichlet regions and these tessellate $\mathbf{H}^n$. Therefore $d(B) \cap H(1, d_0)$ is the union of p fundamental domains for Γ and, in particular, is of finite volume. It readily follows that $\varrho(B)$ coincides with the group density of $\mathscr{P}$.

In the case $r = \infty$ we may assume that $\mathrm{tg}(B) = \infty$. In this case the fact that Γ is cofinite volume implies that the stabilizer of B in Γ is the Poincaré extension of a cofinite volume Euclidean group. Let this group have compact fundamental domain E, then $\nu^{-1}(E) \cap d(B)$ is a fundamental domain for Γ_B and so again it is evident that $\varrho(B)$ and the group density of $\mathscr{P}$ coincide.

We have shown in particular, for a packing with a transitive symmetry group Γ, that if Γ is cofinite volume, then the local density at each cylinder is finite. It remains an open problem to determine whether or not the converse of this is true.

In three or more dimensions, the set of points equidistant from two geodesics is generally not a hyperplane. (This is so only when the geodesics span a two dimensional space.) Consequently the boundary of the Dirichlet region of a cylinder is very complicated in general. For this reason we define a more manageable region as follows. Let B be a cylinder of radius $r \le \infty$ in a packing $\mathscr{P}$. For each $C \in \mathscr{P}$ ($C \ne B$), let D be the hyperplane which perpendicularly bisects the shortest geodesic arc joining B and C (when $r < \infty$ we could equivalently use the shortest geodesic arc joining the axes of B and C), then D determines a halfspace containing B. Define the *polyhedral region* of B, $p(B)$, to be the intersection of these halfspaces, taken over all $C \ne B$. In order to justify the terminology we must show that $p(B)$ is indeed a polyhedron. The proof of this is deferred to the next section (Corollary 5.1).

Clearly $p(B)$ contains B and the $p(B)$ ($B \in \mathscr{P}$) are disjoint, though they will not in general tessellate space. In two dimensions, and for horoballs in all dimensions, $p(B)$ is simply $d(B)$. For large r, $p(B)$ is a good approximation to $d(B)$ (Lemma 8.1 below).

The next lemma gives a more convenient way of expressing $\varrho(B)$. Some further definitions will be useful.

For a measurable set $A \subseteq \mathbf{C}$ with $0 \notin A$, define

$$\mu(A) = \int_A \frac{\mathrm{d}x\,\mathrm{d}y}{x^2+y^2} = \text{Area}\big(\text{Log}\,(A)\big).$$

Observe that this measure is invariant under complex multiplication.

Lemma 4.1. *Let $\mathscr{P}$ be a packing of cylinders of radius $r<\infty$ in $\mathbf{H}^3$, and suppose $B = B(r) \in \mathscr{P}$, then*

$$\varrho(B) = \lim_{a\to 0,\, b\to\infty} \frac{\text{vol}\,\big(B\cap C(a,b)\big)}{\text{vol}\,\big(d(B)\cap C(a,b)\big)}, \tag{4}$$

in the sense that either neither limit exists, or both limits exist and are equal. Corresponding results hold for upper and lower densities.

Proof. Let $\theta(s)$ be the angle between the boundary of $B(s)$ and $\partial\mathbf{H}^3$. We have

$$\sin\theta(s) = 1/\cosh s, \quad \tan\theta(s) = 1/\sinh s$$

(see e.g. [Be, Section 7.20]).

For all $x<y$ we have

$$\text{Vol}\,\big(B(s)\cap C(x,y)\big) = \tfrac{1}{2}\mu\big(A(x,y)\big)/\tan^2\theta(s) = \pi\log(y/x)\sinh^2 s \tag{5}$$

and

$$\text{Vol}\,\big(B(s)\cap H(x,y)\big) = \pi\log(y/x)\sinh^2 s.$$

Let $s<r$ be chosen. Since $B(s)\subseteq B(r) = B\subseteq d(B)$, we have

$$\begin{aligned}\frac{\text{vol}\,\big(\big(C(x,y)\big)\cap d(B)\big)\backslash B(s)\big)}{\text{vol}\,\big(\big(C(x,y)\big)\cap d(B)\big)\big)} &\geq \frac{\text{vol}\,\big(\big(C(x,y)\big)\cap B(r)\big)\backslash B(s)\big)}{\text{vol}\,\big(\big(C(x,y)\big)\cap B(r)\big)\big)} \\ &= \frac{\sinh^2 r - \sinh^2 s}{\sinh^2 r},\end{aligned}$$

whenever the fraction on the left is defined (that is not ∞/∞). Choose a, b so that $a<b' = b\sin\theta(s)$. Then

$$\big(C(a,b')\cap d(B)\big)\backslash B(s) \subseteq H(a,b)\cap d(B)$$

see Figure 1.

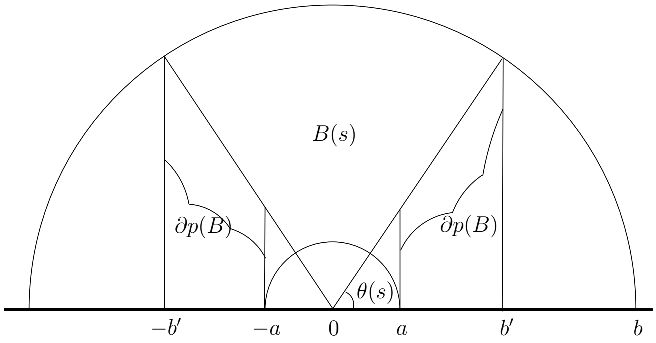

Figure 1.

From these inequalities it follows that

$$
\begin{aligned}
\frac{\operatorname{vol}\big(B\cap H(a,b)\big)}{\operatorname{vol}\big(d(B)\cap H(a,b)\big)} &\le \frac{\pi\log(b'/a)\sinh^2 r}{\operatorname{vol}\big(\big(C(a,b')\cap d(B)\big)\backslash B(s)\big)}\left(\frac{\log(b/a)}{\log(b'/a)}\right)\\
&= \frac{\operatorname{vol}\big(B\cap C\big(a,b')\big)}{\operatorname{vol}\big(\big(C(a,b')\cap d(B)\big)\backslash B(s)\big)}\left(\frac{\log(b/a)}{\log(b/a)+\log\sin\theta(s)}\right) \qquad (6)\\
&\le \frac{\operatorname{vol}\big(B\cap C(a,b')\big)}{\operatorname{vol}\big(\big(C(a,b')\cap d(B)\big)\big)}\left(\frac{\log(b/a)}{\log(b/a)+\log\sin\theta(s)}\right)\\
&\quad\times\left(\frac{\sinh^2 r}{\sinh^2 r-\sinh^2 s}\right).
\end{aligned}
$$

In the limit as $a\to\infty$, $b\to 0$, the middle term goes to 1, and, since s can be chosen arbitrarily small, one half of the theorem follows. To prove the reverse inequality, let $\eta\in(0,1)$ be chosen arbitrarily. We have

$$d(B)\cap H(a,b)\subseteq\big(d(B)\cap C(\eta a,b)\big)\cup\big(H(a,\infty)\cap\nu^{-1}\big(B(0,\eta a)\big)\big).$$

The volume of the second set in union is finite and depends only on η. Abbreviating it to C, we have

$$
\begin{aligned}
\lim_{a\to 0,\,b\to\infty}\frac{\operatorname{vol}\big(B\cap H(a,b)\big)}{\operatorname{vol}\big(d(B)\cap H(a,b)\big)} &\ge \frac{\operatorname{vol}\big(B\cap C(a,b)\big)}{\operatorname{vol}\big(d(B)\cap C(\eta a,b)\big)+C}\\
&= \frac{\operatorname{vol}\big(B\cap C(\eta a,b)\big)}{\operatorname{vol}\big(d(B)\cap C(\eta a,b)\big)+C}\left(\frac{\log(b/a)}{\log(b/a)-\log\eta}\right).
\end{aligned}
$$

The required inequality follows by letting $b/a\to\infty$.

The same arguments give the corresponding results for upper and lower densities. ◻

Let $h(x,y)$ be the vertical distance from the point $(x,y) \in \partial \mathbf{H}^3$ to $p(B)$. In view of (5) the above theorem can be expressed by saying that $\varrho(B)^{-1}$ is $1/\sinh^2 r$ of the limiting mean of $(x^2+y^2)h(x,y)^{-2}$ with respect to the measure μ.

Our main result is

Theorem 4.1. *If $\mathscr{P}$ is a packing of $\mathbf{H}^3$ by cylinders of radius r, and $B \in \mathscr{P}$, then*

$$\overline{\varrho}(B) \leq (1 + 23e^{-r})\varrho_\infty, \tag{7}$$

where $\varrho_\infty = 0.853276\ldots$ is the density of the optimal horoball packing.

Of course this theorem only has content when the right-hand side of (7) is less than 1, which occurs for $r > 4.896\ldots$.

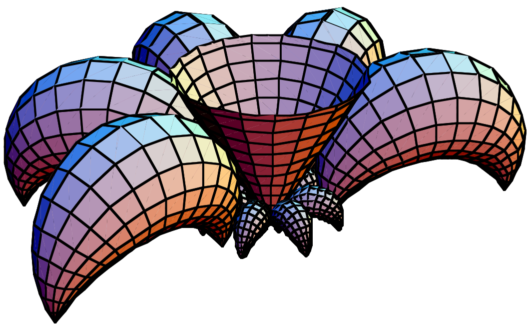

Figure 2. A cylinder packing in $\mathbf{H}^3$.

We define the "optimal packing" function $\lambda(r)$ by

$$\lambda(r) = \sup_{\mathscr{P}} \sup_{B \in \cap P} \overline{\varrho}(B) \tag{8}$$

where the supremum is taken over all packings by cylinders of radius r. Our main result shows $\lambda(r) \leq (1+23e^{-r})\varrho_\infty$. The Figure of 8 packings discussed below give conjectural values for $\lambda(r)$ for specific values of r. We next sketch a proof that $\lambda(r)$ is upper semi-continuous, however it would be very useful to have more generic information about $\lambda(r)$. For instance, is $\lambda(r)$ continuous, or even monotone concave as the values for the Figure of 8 packings might suggest? Notice that Böröczky's bound [Bö1], [Bö2], [BF] for the optimal packing of spheres of radius r in hyperbolic space exhibits these features, though of course the exact values remain unknown for any value of r.

Theorem 4.2. *The function $\lambda(r)$ is upper semi-continuous.*

Proof. It is clear that λ is a well defined function, $0 \le \lambda(r) \le 1$. Also

$$\liminf_{r \nearrow r_0} \lambda(r) \ge \lambda(r_0) \tag{9}$$

as we may slightly decrease the radii of cylinders (keeping their axes fixed) of any nearly optimal packing of radius r_0 effecting a continuous decrease in the density.

Let $\varepsilon > 0$ and set $\limsup_{r_1 \nearrow r_0} \lambda(r_i) = \alpha$. The desired conclusion will follow as soon as we exhibit a packing containing a cylinder B_0 of radius r_0 and $\overline{\varrho}(B_0) > \alpha - \varepsilon$. Choose packings $\mathscr{P}_i$ of cylinders of radius r_i such that

$$\lambda(r_i) < \sup_{B \in \mathscr{P}_i} \overline{\varrho}(B) - \tfrac{1}{2}\varepsilon.$$

For each i choose $B_i \in \mathscr{P}_i$ so that

$$\sup_{B \in \mathscr{P}_i} \overline{\varrho}(B) < \overline{\varrho}(B_i) + \tfrac{1}{2}\varepsilon.$$

Whence

$$\lambda(r_i) > \overline{\varrho}(B_i) - \varepsilon.$$

We normalise the packings so that each B_i has the same axis. Fix j and let K_j denote the closed hyperbolic ball of radius j centered at the origin. We suppose that $\frac{1}{2}r_0 < r_i < 2r_0$ for all i. Then K_j meets a finite number (independent of i) of cylinders of any of the packings $\mathscr{P}_i$. Thus we can select a subsequence r_{i_j} such that the packings $\mathscr{P}_{i_j}$ converge uniformly on K_j to a packing of cylinders about B_0, a cylinder of radius r_0 with the same axis as the B_i. The notion of convergence is clear here, we require the convergence of the endpoints on the boundary of hyperbolic space, a compact set. We inductively construct such subsequences for all j. The usual Cantor diagonal process provides us with a limit packing about B_0, the convergence being uniform on compact subsets. It is a simple matter to observe from the definition that the density of the packing about B_0 is at least $\alpha - \varepsilon$. $\square$

5. Basic lemmas

It is possible to define unambiguously the rotation angle between two ultraparallel *oriented* geodesics g_1 and g_2 in $\mathbf{H}^3$ (modulo 2π) in the following way. Let g be the common perpendicular to g_1 and g_2 and for $i = 1, 2$, let p_i be the point of intersection of g and g_i, and r_i the ray along g_i emanating from p_i in the direction of the orientation of g_i. Orient g in the direction from g_1 to g_2. Now define the rotation angle θ between g_1 and g_2 to be the angle obtained by going from r_1 to r_2 in the anticlockwise direction determined by the orientation of g and the right-hand rule. Clearly this definition is independent of the ordering of g_1 and g_2. It does not apply however when g_1 and g_2 intersect, in which case the angle is defined only up to sign. Thus the angle between two ultraparallel non-oriented geodesics is well defined modulo π.

Lemma 5.1. *Let g_1, g_2, g, p_1, p_2, r_1, r_2 and θ be as above. For $i = 1, 2$ let a_i be a point on g_i displaced α_i from p_i in the direction of g_i. Let l denote the distance from p_1 to p_2, then*

(1) $\cosh\bigl(\varrho(a_1, a_2)\bigr) = \cosh\alpha_1 \cosh\alpha_2 \cosh l - \sinh\alpha_1 \sinh\alpha_2 \cos\theta$.

(2) $\cosh^2\bigl(\varrho(g_1, a_2)\bigr) = \cosh^2\alpha_2 \cosh^2 l - \sinh^2\alpha_2 \cos^2\theta$.

(3) *If $l_1 + l_2 = l$ and $\cosh^2 l_2 \geq \cosh^2 l_1 + 1$, then every point on the plane Π which meets g perpendicularly at the point distance l_1 from p_1, is closer to g_1 than to g_2.*

Proof. Let t and u be the rays emanating from p_2 and passing through the points p_1 and a_1 respectively (t is thus half of the geodesic g).

These two rays, along with the ray r_2 form the edges of a cone P (degenerate when $\theta = 0$) which has a vertex at p_2. See Figure 3.

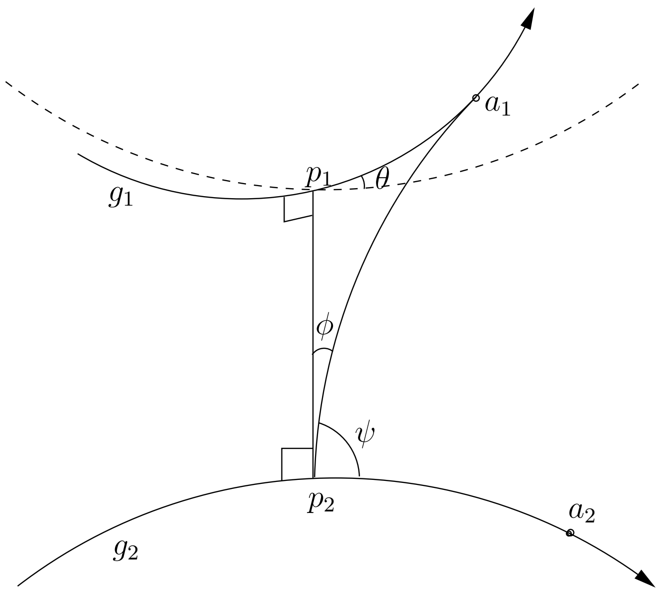

Figure 3.

Let ϕ and ψ be the angles between u and t and between u and r_2 respectively. The angle between t and r_2 is $\frac{1}{2}\pi$ and the angle between the faces of P which meet along t is θ. Spherical cosine rule applied to the link of P at p_2, then gives

$$\cos\psi = \cos\theta \sin\phi. \tag{10}$$

Let $y = \varrho(a_1, p_2)$. Since the triangle $a_1 p_1 p_2$ is right-angled we have (see e.g. [Be, Theorem 7.11.2]),

$$\sinh \alpha_1 = \sinh y \sin \phi. \tag{11}$$

Combining (10) and (11) gives

$$\cos \psi = \cos \theta \, \frac{\sinh \alpha_1}{\sinh y}. \tag{12}$$

Hyperbolic cosine rule applied to the triangle $a_1 p_2 a_2$ gives

$$\cosh \varrho(a_1, a_2) = \cosh y \cosh \alpha_2 - \cos \psi \sinh y \sinh \alpha_2,$$

which combined with Pythagoras theorem and (12) gives the first part of the lemma. The second part follows by minimizing this distance with respect to α_1 and is a straightforward exercise in calculus.

If l_1 and l_2 satisfy the conditions of (3) and if $x \in \Pi$ is distance β from g, then by (2)

$$\begin{aligned} \cosh^2\bigl(\varrho(x, g_1)\bigr) &\le \cosh^2 \beta \cosh^2 l_1 \le \cosh^2 \beta (\cosh^2 l_2 - 1) \\ &< \cosh^2 \beta \cosh^2 l_2 - \sinh^2 \beta \le \cosh^2\bigl(\varrho(x, g_2)\bigr). \end{aligned}$$

This proves (3) of the lemma. □

Corollary 5.1. *Let $\mathscr{P}$ be a packing of $\mathbf{H}^n$ by cylinders of radius r. Let B and C be distinct cylinders in $\mathscr{P}$, $u \in \mathrm{ax}(B)$. Let $l(B, C)$ be the shortest geodesic arc from $\mathrm{ax}(B)$ to $\mathrm{ax}(C)$ and let $\mathrm{bp}(B, C)$ be the hyperplane perpendicular to and bisecting it.*

Let $R > r$. If $\mathrm{bp}(B, C)$ meets the open ball $B(u, R)$, then $\mathrm{ax}(C)$ meets $B(u, R + \alpha)$, where α is defined by

$$\cosh \alpha = (\cosh R)/(\sinh r). \tag{13}$$

The set of hyperplanes $\mathrm{bp}(B, C)$ ($B, C \in \mathscr{P}$, $B \ne C$) is locally finite, and so $p(B)$ is a polyhedron.

Proof. We prove the lemma for 3 dimensions. The general case follows by restricting to the subspace of $\mathbf{H}^n$ spanned by $\mathrm{ax}(B)$ and $\mathrm{ax}(C)$.

Let x be the point in $l(B, C) \cap ax(B)$. If $\mathrm{bp}(B, C)$ meets $B(u, R)$, then there is a geodesic g in $\mathrm{bp}(B, C)$, which passes through $B(u, R)$ and $\mathrm{bp}(B, C) \cap l(B, C)$. Let $\beta = \varrho(u, x)$. We have

$$\cosh^2 \beta \, \sinh^2(r) < \cosh^2 \beta \, \cosh^2(r) - \sinh^2 \beta \le \cosh^2 R,$$

using Lemma 5.1(2), whence $\beta < \alpha$, and so

$$\varrho\bigl(u, ax(C)\bigr) \le \varrho(u, x) + \varrho\bigl(x, ax(C)\bigr) < \alpha + 2r.$$

Since the axes of the cylinders of $\mathscr{P}$ are distance at least $2r$ apart from each other, only finitely many of them can meet any ball. It follows from the above that this must also be true of the set of hyperplanes $\mathrm{bp}(B, C)$ ($C \ne B$), and, by definition, it follows that $p(B)$ is a polyhedron. □

The next lemma gives the angle between two geodesics in terms of the cross ratio of their four end points.

Lemma 5.2. *Let L_1 and L_2 be ultraparallel geodesics in $\mathbf{H}^3$ oriented from end points z_1 to z_2 and w_1 to w_2 respectively, and let κ be the cross ratio*

$$[z_1, w_1, w_2, z_2] = (z_1 - w_2)(w_1 - z_2)/(z_1 - w_1)(w_2 - z_2).$$

If δ is the distance, and θ the rotation angle, between L_1 and L_2, then

$$\cosh(\delta + i\theta) = \frac{\kappa + 1}{\kappa - 1}, \tag{14}$$

$$\kappa = \coth^2\big(\tfrac{1}{2}(\delta + i\theta)\big) \tag{15}$$

$$\frac{1}{1-\kappa} = [z_1, z_2, w_1, w_2] = \frac{(z_1 - w_1)(z_2 - w_2)}{(z_1 - z_2)(w_1 - w_2)} = -\sinh^2\big(\tfrac{1}{2}(\delta + i\theta)\big) \tag{16}$$

$$\frac{\kappa}{\kappa - 1} = \cosh^2\big(\tfrac{1}{2}(\delta + i\theta)\big) \tag{17}$$

and

$$\cosh\delta = \frac{1 + |\kappa|}{|1 - \kappa|}. \tag{18}$$

Proof. Since δ, θ and the cross ratio κ are all invariant under orientation preserving isometries, we may assume that the common perpendicular of L_1 and L_2 is I. By applying a further orientation preserving isometry if necessary we may assume that L_1 has end points -1 and 1 and is oriented from -1 to 1 and that L_2 is oriented from $-ke^{i\theta}$ to $ke^{i\theta}$, where $k = e^{\delta}$. A simple calculation gives (15) whence (14), (16) and (17) all follow.

As in [GM1] we have

$$\begin{aligned} 2\cosh^2\big(\tfrac{1}{2}\delta\big) &= |\cosh^2 \tfrac{1}{2}(\delta + i\theta)| + |\sinh^2 \tfrac{1}{2}(\delta + i\theta)| + 1 \\ &= \left|\frac{1}{1-\kappa}\right| + \left|\frac{\kappa}{1-\kappa}\right| + 1 \end{aligned} \tag{19}$$

and (18) follows. □

Lemma 5.3. *Let C be a cylinder with endpoints 1 and w. If the rotation angle between* $\mathrm{ax}(C)$ *and* I*, oriented from 1 to w and from 0 to ∞ respectively, is θ and the distance from* $\mathrm{ax}(C)$ *to I is r, then* $\mathrm{bp}(C)$ *is the hemisphere with centre*

$$\sqrt{w}\left(\frac{\left|1+\sqrt{w}\,\right|+\left|1-\sqrt{w}\,\right|}{\left|1+\sqrt{w}\,\right|-\left|1-\sqrt{w}\,\right|}\right)=\coth\bigl(r+\tfrac{1}{2}i\theta\bigr)\coth r$$

and radius

$$2\sqrt{|w|}\left(\frac{\left|\sqrt{1-w}\,\right|}{\left|1+\sqrt{w}\,\right|-\left|1-\sqrt{w}\,\right|}\right)=\left|\coth\bigl(r+\tfrac{1}{2}i\theta\bigr)\right|\operatorname{cosech} r.$$

[*Note: The choice of square root is immaterial provided it is made consistently throughout.*]

Sketch of proof. Applying the Möbius transform

$$\phi(z)=\frac{z+\sqrt{w}}{z-\sqrt{w}} \tag{20}$$

maps the geodesics with endpoints $\{0,\infty\}$ and $\{1,w\}$ to those with endpoints $\{-1,1\}$ and $\{\alpha,-\alpha\}$ respectively, where $\alpha=\bigl(1+\sqrt{w}\,\bigr)/\bigl(1-\sqrt{w}\,\bigr)$. The perpendicular bisector of the shortest geodesic arc joining these geodesics is the hemisphere S centred at the origin with radius $\left|\sqrt{\alpha}\,\right|$. We have

$$\phi^{-1}(z)=\sqrt{w}\left(\frac{z+1}{z-1}\right), \tag{21}$$

which takes the line $\mathbf{R}$ to $\sqrt{w}\,\mathbf{R}$, so that it takes S to a hemisphere perpendicular to $\sqrt{w}\,\mathbf{R}$ which meets it at the points $\phi^{-1}\bigl(\pm\left|\sqrt{\alpha}\,\right|\bigr)$. These are the points

$$\sqrt{w}\left(\frac{\left|\sqrt{1-w}\,\right|+\left|1-\sqrt{w}\,\right|}{\left|\sqrt{1-w}\,\right|-\left|1-\sqrt{w}\,\right|}\right),\qquad \sqrt{w}\left(\frac{\left|\sqrt{1-w}\,\right|-\left|1-\sqrt{w}\,\right|}{\left|\sqrt{1-w}\,\right|+\left|1-\sqrt{w}\,\right|}\right),$$

and the equations in w now follow. To obtain the equations in r we use (15) and the identity $|\cosh z\pm\sinh z|=\exp(\pm\operatorname{Re}z)$. ◻

Lemma 5.4. *Let A be a geodesic in $\mathbf{H}^3$ at distance $2r$ from, and with rotation angle θ to I. Let $B=e^{2i\phi}A$. If $2s$ is the distance, and ψ the rotation angle, between A and B, then we have, after substituting $\psi+\pi$ for ψ if necessary,*

$$\sin^2\phi\sinh^2(2r+i\theta)=\sinh^2\bigl(s+\tfrac{1}{2}i\psi\bigr). \tag{22}$$

Proof. By applying a scale change if necessary, we may assume that the endpoints of A are 1 and w. Using (16) and (17)

$$\sinh^2\left(s+\tfrac{1}{2}i\psi\right) = -[1, w, e^{2i\phi}, e^{2i\phi}w] = -\frac{\left(1-e^{2i\phi}\right)^2 w}{(1-w)^2 e^{2i\phi}}$$

$$= 4w\sin^2\phi/(1-w)^2 \tag{23}$$

$$= 4\sin^2\phi\,\sinh^2\left(r+\tfrac{1}{2}i\theta\right)\,\cosh^2\left(r+\tfrac{1}{2}i\theta\right) \tag{24}$$

$$= \sin^2\phi\,\sinh^2(2r+i\theta), \tag{25}$$

which is what we wanted to prove. □

6. Figure of eight packings

Let Γ_n denote the fundamental group of the orbifold obtained by performing $(n,0)$ Dehn surgery on the figure of eight knot complement. From [HLM, Propositions 6.2 and 6.3], [1] the volume of the orbifold $\mathbf{H}^3/\Gamma_n$ is

$$V_n = \int_{2\pi/n}^{2\pi/3} \operatorname{arccosh}\left(2+\cos t - 2\cos^2 t\right) \mathrm{d}t,$$

and length of its singular set is

$$\tau_n = 2\operatorname{arccosh}\left(2+\cos(2\pi/n) - 2\cos^2(2\pi/n)\right).$$

From [HLM] we also calculate that the minimum distance between axes of elliptics of order n (which is the distance between the midpoints of the top and bottom diagonals in Figure 11 of [HLM]) is $2r_n$, where

$$\sinh^2 r_n = \frac{-1+2\cos(2\pi/n)+\sqrt{3-2\cos(2\pi/n)}}{4\left(1-\cos(2\pi/n)\right)}.$$

We thus have a formula for the packing density ϱ_n of the cylinders of radius r_n around the elliptic axes, which is

$$\varrho_n = \frac{\pi\tau_n\sinh^2 r_n}{nV_n}.$$

In the limit as $r_n \to \infty$ this density is $\sqrt{3}/(2V) = 0.853276\ldots$, where $V = 1.0149\ldots$ is the volume of a regular ideal tetrahedron. This is the density ϱ_∞ of the optimal horoball packing [F].

Some values for V_n, τ_n, r_n and ϱ_n are given in Table 1.

---

[1] the formulae in [HLM] are given in terms of $x = \sqrt{3-2\cos(2\pi/n)}/(2+\sqrt{3-2\cos(2\pi/n)})$.

n	V_n	τ_n	r_n	ϱ_n
7	1.4118	2.4462	0.8964	0.8112
8	1.5439	2.2568	1.0129	0.8197
9	1.6386	2.0817	1.1175	0.8263
10	1.7086	1.9248	1.2125	0.8310
11	1.7616	1.7858	1.3000	0.8346
12	1.8026	1.6629	1.3802	0.8373
∞	2.02988	0	∞	0.8533

Table 1. Data for Γ_n.

In order to investigate the local structure of the packings associated with these groups, we obtain an explicit matrix representation. The figure of eight knot complement has fundamental group Γ with presentation,

$$\langle a, b \mid aba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}\rangle,$$

in which both a and b are parabolic. Adding the relations $a^n = b^n = 1$ gives a presentation for Γ_n. The mappings

$$a \to \begin{pmatrix} \alpha + i\sqrt{(\gamma+\beta^2)/2} & \sqrt{(\gamma-\beta^2)/2} \\ \sqrt{(\gamma-\beta^2)/2} & \alpha - i\sqrt{(\gamma+\beta^2)/2} \end{pmatrix}, \tag{26}$$

$$b \to \begin{pmatrix} \alpha - i\sqrt{(\gamma+\beta^2)/2} & \sqrt{(\gamma-\beta^2)/2} \\ \sqrt{(\gamma-\beta^2)/2} & \alpha + i\sqrt{(\gamma+\beta^2)/2} \end{pmatrix}, \tag{27}$$

where $\alpha = \cos(\pi/n)$, $\beta = \sin(\pi/n)$ and $\gamma = \frac{1}{4}\big(1 + \sqrt{(1-4\alpha^2)(5-4\alpha^2)}\big)$, give a representation of this group in $\mathrm{SL}(2, \mathbf{C})$ [BH].

Interpreted as a Möbius transformation, the matrix for a has fixed points

$$i\frac{\sqrt{2(\gamma+\beta^2)} \pm 2\beta}{\sqrt{2(\gamma-\beta^2)}}.$$

By conjugation we may shift the fixed points of a to 0 and ∞, to get matrix representatives for a and b respectively

$$A = \begin{pmatrix} e^{i\pi/n} & 0 \\ 0 & e^{-i\pi/n} \end{pmatrix}, \tag{28}$$

$$B = \begin{pmatrix} \alpha - i\gamma/\beta & -y + i\beta + i\gamma/\beta \\ -y - i\beta + i\gamma/\beta & \alpha + i\gamma/\beta \end{pmatrix}, \tag{29}$$

where $y = \sqrt{2(\gamma+\beta^2)}$. A calculation shows that

$$L = BA^{-1}B^{-1}A^2B^{-1}A^{-1}B$$

has the same axis as A, and is hyperbolic with translation length τ_n.

Let f and g be the Möbius transformations corresponding to the matrices A and B respectively. From the above, f and $gf^{-1}g^{-1}f^2g^{-1}f^{-1}g$ generate the stabilizer of I. Let $C = B(r_n)$, the cylinder with axis I and radius r_n. A series of tedious but routine calculations establish the following facts. The geodesic $g(I)$ is distance $2r_n$ and rotation angle θ_n to I, where

$$\sin^2\theta_n = \frac{\cosh^2 r_n\bigl(1-4\sin^2(\pi/n)\cosh^2 r_n\bigr)}{\sin^2(\pi/n)(1-4\cosh^2 r_n)}. \tag{30}$$

Since $\sin(\pi/n)\sinh(2r_n+i\theta_n)$ lies on the ellipse parameterized by $\sinh(r_n+it)$ ($t\in\mathbf{R}$), Lemma 5.4 shows that $\varrho\bigl(fg(I),g(I)\bigr) = 2r_n$, whence the cylinders $f^kg(C)$ ($0\le k<n$) form a tier around C, with $f^kg(C)$ touching $f^{k-1}g(C)$, $f^{k+1}g(C)$ and C. Next, $gfg^{-1}(\infty)/gfg^{-1}(0)$, and $g(\infty)/g(0)$ are conjugate, so that $gfg^{-1}(I)$ also has distance $2r_n$ and rotation angle θ_n to I, but is rotated in the opposite direction to I. The cylinders with axes $f^kgfg^{-1}(C)$ form a second tier around C and, since $\varrho\bigl(gfg^{-1}(I),g(I)\bigr) = \varrho\bigl(fgfg^{-1}(I),g(I)\bigr) = 2r_n$, each cylinder in this second tier touches two in the first (and *vice versa*).

Since also $f^2g^{-1}f(I) = \phi\bigl(g(I)\bigr)$ and $fg^{-1}fg(I) = \phi\bigl(gfg^{-1}(I)\bigr)$, where $\phi(z) = e^{-\tau/2}z$, the cylinders $g^{-1}f(C)$ and $g^{-1}fg(C)$ generate a third and fourth tier, which touch in the same way.

Finally, since $gfg^{-1}(\infty)/fg^{-1}f(0)$ and $g(\infty)/gfg^{-1}(0)$ are conjugate, this is also true of the second and third tiers.

Consequently the set $\mathscr{S}$ of cylinders touching C contains at least four orbits under the stabilizer of I, and the cylinders with axes $g(I)$, gfg^{-1}, $g^{-1}f(I)$ and $g^{-1}fg(I)$ are representatives of each of these orbits. Presumably these are the only orbits in $\mathscr{S}$. For large n this is readily proved. First we show that the number of orbits must in any case be even.

Lemma 6.1. *Let $\mathscr{P}$ be a cylinder packing of $\mathbf{H}^3$, upon which a finite-covolume Kleinian group without elliptics of order two acts transitively. Let $B\in\mathscr{P}$, Γ_B be the stabilizer of B in Γ and $\mathscr{S}$ be the set of cylinders in $\mathscr{P}$ which touch B. There are an even number of Γ_B-orbits in $\mathscr{S}$.*

Proof. For $C\in\mathscr{S}$ let $[C]$ denote the Γ_B-orbit containing C. Let ϕ mapping the set of Γ_B-orbits to itself be defined by $\phi([C]) = [g^2(C)]$, where $g\in\Gamma$ maps C to B. We show that ϕ is well defined and has no fixed points. Since ϕ is then clearly involutive as well, it effects a pairing of the orbits and the lemma follows.

If $C \in \mathscr{S}$ and $g(C) = B$, then clearly $g^2(C) \in \mathscr{S}$. Suppose that $[C_1] = [C]$ and $g_1(C_1) = B$. Then, for some $\tau \in \Gamma_B$, $\tau(C_1) = C$. Now $g_1\tau^{-1}g^{-1}$ fixes B so that, for some $\tau_1 \in \Gamma_B$, $g_1 = \tau_1 g\tau$. It follows that

$$g_1^2(C_1) = (\tau_1 g\tau)^2(C_1) = \tau_1 g(B) = \tau_1 g^2(C) \in [g^2(C)], \tag{31}$$

so that ϕ is well defined. Now suppose that fixes some Γ_B-orbit $\mathscr{O}$. By composing g with a member of Γ_B if necessary, we may assume that, for some $g \in \Gamma$, and $C \in \mathscr{O}$, $g^2(C) = C$. But, since g must fix the point of intersection of C and B, g^2 must be the identity, whence g is elliptic of order two, contrary to our assumption. □

Thus, in the figure of 8 case, it will follow that $\mathscr{S}$ contains four orbits provided that it has fewer than six. Suppose that $\mathscr{S}$ comprises k orbits. Each cylinder in $\mathscr{S}$ contains a ball of radius r_n, tangent to $B(r_n)$ at the same point as the cylinder. We take a set of k such balls—one representing each orbit—and apply Theorem 3.27 of [GM2], to obtain the inequality

$$k\mathscr{V}(r_n) \le \text{Vol.}\big(B(r_n)/\Gamma_n\big) \le \text{Vol.}(\mathbf{H}^3/\Gamma_n),$$

where

$$\mathscr{V}(t) = \frac{\sqrt{3}}{2}\tanh(t)\cosh(2t)\text{arcsinh}^2\left(\frac{\sinh(t)}{\cosh(2t)}\right).$$

Thus $k \le V_n/\mathscr{V}(r_n)$ and calculation shows that $k < 6$, hence $k \le 4$ for $n \ge 10$. Presumably this can be slightly improved using the bounds of [P1].

The cylinders in $\mathscr{S}$ consist of alternating nested tiers of n, each obtained from the last by reflection through a plane containing I and dilation by a factor of $e^{\tau/4}$. Moreover each cylinder in $\mathscr{S}$ touches six others in $\mathscr{S}$. We conjecture that the figure of eight packings are the only cylinder packings with this last property that are completely invariant under a finite covolume Kleinian group. In the limit as $n \to \infty$, these packings become the familiar hexagonal packing of horoballs associated with the original figure of eight group.

7. Horoball packings

When dealing with horoball packings in $\mathbf{H}^n$, it is natural to make the normalizing assumption that one of the horoballs in the packing is $B(\infty)$. The remaining horoballs in the packing must touch the boundary at points of $\mathbf{R}^{n-1}$. If B is such a horoball, then $\text{bp}(B)$ comprises exactly the points equidistant from B and $B(\infty)$ and it is easy to show that the diameter of B is the square of the radius of $\text{bp}(B)$ (Figure 4).

If every $B \ne B(\infty)$ has diameter 1, then their projections into the boundary give a packing of $\mathbf{R}^{n-1}$, by equal balls of diameter 1, and this correspondence is obviously bijective. Thus, in some sense, horoball packings can be seen as generalizations of Euclidean ball packings. Moreover, k-dimensional faces of $d\big(B(\infty)\big)$ project into k-dimensional faces of Dirichlet cells in the Euclidean packing. The main result of this section generalizes the inequality of Blichfeldt [Ro], which states that, given any $k+1$ disjoint unit balls in $\mathbf{R}^n$, any point which is equidistant from their centres (in the context of a packing, any point in a $(n-k)$-dimensional face of a Dirichlet polyhedron) is distance at least $\sqrt{2k/(k+1)}$ from each centre. In view of the above discussion, this gives the result that, for any $k+1$ unit spheres in $\mathbf{R}^{n+1}$ with centres in $\mathbf{R}^n$, which are all distance at least 1 apart from each other, the projection to $\mathbf{R}^n$ of the points of intersection of these spheres are distance at least $\sqrt{k/2(k+1)}$ from each centre (each sphere in $\mathbf{R}^{n+1}$ has a corresponding disk in $\mathbf{R}^n$ with the same centre and half the radius. These disks are mutually disjoint so that Blichfeldt's inequality can be applied). The next lemma is essentially a generalization of Blichfeldt's inequality from Euclidean packings to horoball packings.

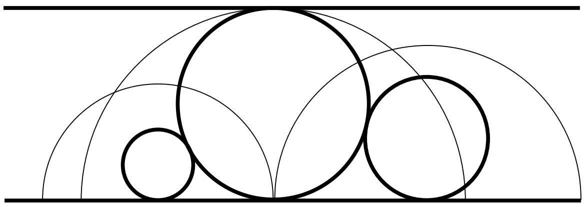

Figure 4. Horoball packing.

Let B_i $(i = 1,2)$ be horoball with (Euclidean) radius r_i, tangent to the boundary at $\mathbf{x}_i$. A simple calculation shows that B_1 and B_2 touch when $|\mathbf{x_1}-\mathbf{x_2}|^2 = 4r_1r_2$, that is when the product of the radii of their bisecting planes is equal to the distance between their centres. This motivates the constraint in Lemma 7.1.

Lemma 7.1. *Let $C_1, C_2, \ldots, C_k$ be $k \le n$ spheres in $\mathbf{R}^n$. Let C_i have centre $\mathbf{c_i}$ and radius r_i and suppose that*

$$r_i \le 1 \quad (\forall i) \tag{32}$$

and

$$|\mathbf{c_i}-\mathbf{c_j}| \ge r_i r_j \quad (\forall i \ne j). \tag{33}$$

Let P denote the space spanned by the $\mathbf{c_i}$. If $\bigcap_{i=1}^n C_i$ is nonempty, then let $\mathbf{z_1}$ and $\mathbf{z_2}$ (with possibly $\mathbf{z_1} = \mathbf{z_2}$) denote the points of this intersection most distant from P. (Since the intersection of spheres is either a sphere, a point or empty, *it is clear that there are at most two such points and that these have the same projection to P.)*

For given $r_1, \dots, r_k$, satisfying (32), let $h = h(r_1, r_2, \dots, r_k)$ be the supremum of dist $(\mathbf{z_1}, P)$ *taken over all values of $\mathbf{c_i}$ subject to (33).*

(1) *h is a monotone increasing function of each r_i.*
(2) *If equality holds in (33) for each $i \neq j$ and $\bigcap_{i=1}^{n} C_i \neq \emptyset$, then*

$$h = \operatorname{dist}(\mathbf{z_1}, P)$$

if and only if the projection of $\mathbf{z_1}$ to P lies in the convex hull of the $\mathbf{c_i}$.

Proof. Let π denote projection onto P.

If $k < n$, then, by restricting everything to intersections with any k-dimensional plane containing the $\mathbf{c_i}$, we are reduced to the case $k = n$, which we assume henceforth.

We also assume that $P = \mathbf{R}^{n-1}$ and that $\mathbf{z_1}$, $\mathbf{z_2}$ are the points $(0, \dots, 0, \pm z)$, $(z \geq 0)$.

Let $x_i = |\mathbf{c_i}|$. We thus have

$$r_i^2 = x_i^2 + z^2. \tag{34}$$

When $\mathbf{c_i} \neq \mathbf{0}$ let $\hat{\mathbf{c}}_\mathbf{i}$ be the unit vector $\mathbf{c_i}/x_i$. If $\mathbf{c_i} = \mathbf{0}$ let $\hat{\mathbf{c}}_\mathbf{i}$ be an arbitrary unit vector. We may assume that these $\hat{\mathbf{c}}_\mathbf{i}$ are chosen to be at angle at least $\frac{1}{2}\pi$ to the other $\hat{\mathbf{c}}_\mathbf{j}$.

Let θ_{ij} be the angle between the vectors $\hat{\mathbf{c}}_\mathbf{i}$ and $\hat{\mathbf{c}}_\mathbf{j}$ (Figure 5).

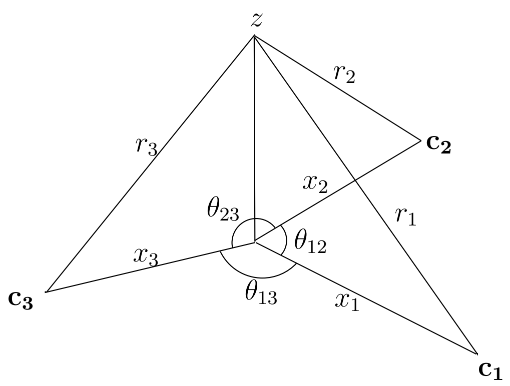

Figure 5.

The following is equivalent to (33).

$$r_i^2 r_j^2 \leq x_i^2 + x_j^2 - 2x_i x_j \cos\theta_{ij} \quad (\forall i \neq j). \tag{35}$$

To prove the first part of the theorem we show that, if $r_j < 1$, it is possible to change the vectors $\mathbf{c_i}$, while fixing $\mathbf{z_1}$, $\mathbf{z_2}$ and all the r_i $(i \neq j)$ and incrementally increasing r_j (in view of (34), this means that the x_i $(i \neq j)$ are also fixed, while x_j increases), in such away that the inequalities (32) and (35) still hold. We may assume that $j = 1$.

If the angle between $\mathbf{\hat{c}_1}$ and each other $\mathbf{\hat{c}_i}$ is at least $\frac{1}{2}\pi$, then we may increase the length of $\mathbf{c_1}$, while leaving $\mathbf{z_1}$ and $\mathbf{z_2}$ fixed, without violating (35).

Otherwise $\mathbf{c_1} \neq \mathbf{0}$ and some θ_{1i} is acute. The vectors $\mathbf{\hat{c}_i}$ lie on the sphere S^{n-2}. We let $\alpha(\mathbf{\hat{c}_i}) \in \left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$,—the azimuthal angle of $\mathbf{\hat{c}_i}$—be defined by the condition that $\sin\bigl(\alpha(\mathbf{\hat{c}_i})\bigr)$ is the $(n-1)$th component of $\mathbf{\hat{c}_i}$.

We may assume that $\mathbf{\hat{c}_2}, \mathbf{\hat{c}_3}, \ldots, \mathbf{\hat{c}_n}$ lie on the sphere $\{\mathbf{x} \in S^{n-2} \mid \alpha(\mathbf{x}) = \nu\}$ for some $\nu \geq 0$. If $\alpha(\mathbf{\hat{c}_1}) \geq \nu$, then all the $\mathbf{\hat{c}_i}$ $(i \geq 2)$ can be projected onto the "equator"($\{\mathbf{x} \in S^{n-2} \mid \alpha(\mathbf{x}) = 0\}$), without reducing any of the θ_{ij}. Hence we may assume that $\alpha(\mathbf{\hat{c}_1}) = \mu \leq \nu$. Thus there are unit vectors $\mathbf{u_i}$ in S^{n-3} for which $\mathbf{\hat{c}_1} = \bigl((\cos\mu)\mathbf{u_1}, (\sin\mu)\bigr)$ and $\mathbf{\hat{c}_i} = \bigl((\cos\nu)\mathbf{u_i}, (\sin\nu)\bigr)$ $(i \geq 2)$. Since the angle between $\mathbf{\hat{c}_1}$ and some other $\mathbf{\hat{c}_i}$ is acute, we have $\nu - \mu < \frac{1}{2}\pi$, whence, for all $i \neq 1$

$$\cos\theta_{i1} = \mathbf{\hat{c}_i}\cdot\mathbf{\hat{c}_1} = a_i \cos\mu\cos\nu + \sin\mu\sin\nu$$

where $a_i \leq 1$. Now continuously decrease μ, holding $\mathbf{u_1}$ fixed and letting $x_1 = A\sec(\nu - \mu)$, where A is determined by the initial values of μ and x_1.

Since z is fixed we have $\mathrm{d}x_1^2 = \mathrm{d}r_1^2 > 0$, so that, to ensure that (35) is preserved, it suffices to show that

$$\frac{\mathrm{d}(x_1\cos\theta_{i1})}{\mathrm{d}\mu} \geq 0. \tag{36}$$

We have

$$\begin{aligned}\mathrm{d}(x_1\cos\theta_{i1})/\mathrm{d}\mu &= -A\sec(\nu-\mu)\bigl[a_i\sin\mu\cos\nu - \sin\nu\cos\mu \\ &\qquad + \tan(\nu-\mu)(a_i\cos\mu\cos\nu + \sin\mu\sin\nu)\bigr] \\ &= A\sec(\nu-\mu)(1-a_i)\cos\nu\bigl[\tan(\nu-\mu)\cos\mu + \sin\mu\bigr].\end{aligned}$$

Since $\nu - \mu < \frac{1}{2}\pi$, we have $\tan(\nu - \mu) \geq -\tan\mu$, so that the expression above is non-negative. This proves the first part of the theorem.

To prove the second part of the theorem, we now assume that equality holds in (33) for each $i \neq j$ and $\bigcap_{i=1}^n C_i \neq \emptyset$. If $\pi(\mathbf{z_1}) = \mathbf{0}$ does not lie in the convex hull of the $\mathbf{c_i}$, then all the $\mathbf{\hat{c}_i}$ lie in some open hemisphere of S^{n-2} (Figure 5), which we assume to be the hemisphere $\{\mathbf{x} \in S^{n-2} \mid \alpha(\mathbf{x}) > 0\}$. It is easily seen that, by reducing each $\alpha(\mathbf{\hat{c}_i})$ by the same positive decrement, and a further small perturbation in the case where two or more of the $\mathbf{c_i}$ project to the same point on the equator, all the θ_{ij} can be strictly increased. Consequently each (35) holds strictly, and it is thus possible to increase z, leaving the r_i fixed and reducing each x_i, while preserving (35). We have thus shown that, when $\pi(\mathbf{z_1})$ does not lie in the convex hull of the $\mathbf{c_i}$, $h > \operatorname{dist}(\mathbf{z_1}, P)$.

For the converse, let the $\mathbf{c_i}$ vary subject to (33), with the r_i remaining fixed. Let $d = \operatorname{dist}(\mathbf{z_1}, P)$. Suppose equality holds in (33) with $\mathbf{c_i} = \mathbf{c_i'}$ and that $\pi(\mathbf{z_1}) = \mathbf{0}$ lies in the convex hull of the $\mathbf{c_i'}$, then there are non-negative numbers β_i whose sum is 1, such that $\mathbf{0} = \pi(\mathbf{z_1}) = \sum_{i=1}^{n} \beta_i \mathbf{c_i'}$. We have

$$0 \le \left| \sum_{i=1}^{n} \beta_i \mathbf{c_i} \right|^2 = \sum_{i=1}^{n} {\beta_i}^2 (r_i^2 - d^2) + 2 \sum_{1 \le i < j} \beta_i \beta_j \mathbf{c_i} \cdot \mathbf{c_j}. \tag{37}$$

Since, by (33),

$$|\mathbf{c_i}|^2 + |\mathbf{c_j}|^2 - 2\mathbf{c_i} \cdot \mathbf{c_j} \ge r_i r_j \quad (\forall i \ne j),$$

we have

$$\mathbf{c_i} \cdot \mathbf{c_j} \le \tfrac{1}{2}(r_i^2 + r_j^2 - 2d^2 - r_i r_j). \tag{38}$$

Hence

$$0 \le \sum_{i=1}^{n} {\beta_i}^2 (r_i^2 - d^2) + \sum_{1 \le i < j} \beta_i \beta_j (r_i^2 + r_j^2 - 2d^2 - r_i r_j), \tag{39}$$

and so

$$d^2 \le \sum_{i=1}^{n} {\beta_i}^2 r_i^2 + \sum_{1 \le i < j} \beta_i \beta_j (r_i^2 + r_j^2 - r_i r_j). \tag{40}$$

Now observe that equality holds throughout when $\mathbf{c_i} = \mathbf{c_i'}$. Hence d is maximized in this case. This completes the proof. ◻

For applications we use the following.

Corollary 7.1. *In the preceding lemma let r_1 be fixed, and the other r_i allowed to vary, subject to (32). Let d be the distance between $\mathbf{c_1}$ and $\pi(\mathbf{z_1})$. Then*

$$d \ge \frac{(2r_1^2 - 1)\sqrt{2(k-1)}}{2\sqrt{|2(k-1)r_1^2 - k + 2|}}. \tag{41}$$

Proof. We may assume that $r = r_1 > 1/\sqrt{2}$, since the bound (41) is trivial otherwise. By (1) of the preceding theorem, h is maximized, and hence $d = \sqrt{r^2 - h^2}$ is minimized, when $r_i = 1$ for every $i \ge 2$. Suppose that this is so.

We recall that the distance d_n between the centroid and vertex of a regular Euclidean n-dimensional simplex with edge length 1 is $d_n = \sqrt{n/\big(2(n+1)\big)} < 1/\sqrt{2}$, so that, since we are assuming $r > 1/\sqrt{2}$, it is possible to place the $\mathbf{c_i}$ so that equality holds in each of (33). In this case the $\mathbf{c_i}$ are the vertices of a simplex Σ whose "base" B, spanned by $\mathbf{c_2}, \dots, \mathbf{c_k}$, is regular with edge lengths of 1, and $\mathbf{c_1}$ is joined to each other $\mathbf{c_i}$ by an edge of length r. By symmetry it is clear that $\pi(\mathbf{z_1})$ must lie on the line segment joining $\mathbf{c_1}$ and the centroid of B. Let l be the length of this segment, then

$$l^2 = r^2 - d_{k-2}^2.$$

Let z be the distance from $\mathbf{z_1}$ to $\pi(\mathbf{z_1})$. Recall that $\pm z$ is the nth component of $\mathbf{z_1}$. We have

$$d^2 + z^2 = r^2 \tag{42}$$

and

$$z^2 + (l-d)^2 + d_{k-2}^2 = 1,$$

whence, eliminating z,

$$d = (2r^2 - 1)/(2l),$$

and the corollary follows. □

Blichfeldt's inequality is this result with $r_1 = 1$.

Theorem 7.1. *There exists a packing of horoballs in $\mathbf{H}^n$ with local density at least 2^{1-n} at each horoball.*

Proof. Let $\phi(n) = \big(\phi_1(n), \phi_2(n)\big)$ be any bijection from $\mathbf{N}$ to $\mathbf{N}^2 \cup \{(0,0)\}$ with the property that $\phi_1(n) < n$. Let $\mathscr{P}_0$ be the packing in $\mathbf{H}^n$ comprising the single horoball $B_{(0,0)} = B(\infty)$.

We define inductively a sequence of packings $\mathscr{P}_n$, each comprising $B_{(0,0)}$, and disjoint horoballs $B_{(j,m)}$ ($1 \le j \le n, 1 \le m \le \infty$), with the property that every horoball in $\mathscr{P}_n$ other than $B_{(0,0)}$ lies within (hyperbolic) distance $\log 2$ of at least one of the horoballs $B_{\phi(j)}$ ($1 \le j \le n$), and the local density of $\mathscr{P}_n$ at each of these $B_{\phi(j)}$ is at least 2^{1-n}.

We have already defined $\mathscr{P}_0$, which satisfies these conditions vacuously. For the induction step suppose that horoballs $B_{(j,m)}$ have been defined and that $\mathscr{P}_n$, comprising this collection of horoballs together with $B_{(0,0)}$ is a packing with the required properties.

To construct $\mathscr{P}_{n+1}$, let ψ be an isometry mapping $B_{\phi(n+1)}$ to $B_{(0,0)}$. To the packing $\psi(\mathscr{P}_n)(= \{\psi(B) \mid B \in \mathscr{P}_n\})$, successively add horoballs $B'_{n+1,1}$, $B'_{n+1,2}$, $B'_{n+1,3} \dots$, each of Euclidean diameter $\frac{1}{2}$, and chosen so that the absolute value of $\operatorname{tg}(B'_{n+1,m})$ is minimized subject to the interior of $B'_{n+1,m}$ being disjoint from all the horoballs of $\psi(\mathscr{P}_n)$ and all the $B'_{n+1,i}$ with $i < m$. Let $\mathscr{P}'_n$ denote this extended packing. Each $B'_{n+1,m}$ is hyperbolic distance $\log 2$ from $\psi(B_{\phi(n)}) = B_{(0,0)}$.

For $\mathbf{x} \in \partial\mathbf{H}^n \backslash \{\infty\} = \mathbf{R}^{n-1}$ define the "height" function $h(\mathbf{x})$ to be the Euclidean length of the vertical line segment from x to the boundary of the polyhedral region $p(B_{(0,0)})$ defined relative to the packing $\mathscr{P}'_n$. We prove that $h(\mathbf{x})$ is at least $\frac{1}{2}$ on all but a compact subset of $\mathbf{R}^{n-1}$ and is always positive. For suppose some $h(\mathbf{x})$ is zero. Then, if B in $\mathscr{P}'_n$ has Euclidean diameter d, $|\mathbf{x} - \mathrm{tg}(B)| \geq \sqrt{d}$, and so a new horoball (of any Euclidean diameter up to 1) can be added to $\mathscr{P}'_n$, tangent to the boundary at $\mathbf{x}$, without overlapping any of the existing horoballs but, in view of the way $\mathscr{P}'_n$ was constructed, this is impossible.

Let K be the set of points in $\mathbf{R}^{n-1}$ within distance $\left(\frac{3}{4}\right) + \log 2$ of any $\mathrm{tg}\big(\psi(B_{\phi(m)})\big)$ $(m \leq n)$, then K is compact. By the induction hypothesis, if B is a horoball of diameter $\frac{1}{2}$, with $\mathrm{tg}(B) \notin K$, then B is disjoint from all of the horoballs in $\psi(\mathscr{P}_n)$. If $h(\mathbf{x}) < \frac{1}{2}$, it is also disjoint from all of the $B'_{n+1,i}$, because $\mathrm{bp}(B'_{n+1,i})$ has radius $1/\sqrt{2}$, and so we must have $|\,\mathrm{tg}(B'_{n+1,i} - \mathbf{x})|^2 \geq \left(1/\sqrt{2}\,\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Again this contradicts the minimality assumption in the construction of $\mathscr{P}'_n$, so that we must have $h(\mathbf{x}) \geq \frac{1}{2}$ for all $\mathbf{x} \notin K$.

The local density of $\mathscr{P}'_n$ at $B_{(0,0)}$ is thus at least 2^{1-n}.

Now define $B_{(n+1,i)} = \psi^{-1}(B'_{n+1,i})$ and $\mathscr{P}_{n+1} = \psi^{-1}(\mathscr{P}'_n)$. Clearly $\mathscr{P}_{n+1}$ satisfies the induction hypotheses, and $\mathscr{P}_n \subseteq \mathscr{P}_{n+1}$.

Finally let $\mathscr{P} = \cup \mathscr{P}_n$. Since $\phi(n)$ is a bijection, $\mathscr{P}$ has local density at least 2^{1-n} at each horoball, as required. □

8. Packing cylinders of large radius

In this section we prove Theorem 7. We first prove that, for large r, the polyhedral region is a good approximation to the Dirichlet region, and then we work with the polyhedral region for the remainder of the proof.

The following lemma shows that after slightly shrinking $p(B_\infty)$ it lies inside $d(B_\infty)$.

Lemma 8.1. *Let $\mathscr{P}$ be a packing of $\mathbf{H}^3$ by cylinders of radius*

$$r > \operatorname{arcsinh}\left(\frac{1}{\sqrt{2}}\right),$$

including the cylinder $B(r)$. Suppose that the polyhedral region $p\big(B(r)\big)$ of $B(r)$ lies in $B(r_0)$ for some $r_0 > r$. Let

$$\tilde{p}(B_\infty) = \{(x, y, kz) \mid (x, y, z) \in p(B_\infty)\}$$

where $k = 1 + (1 + \operatorname{sech} r)\sinh^2 r_0 / \big(2(2\sinh^2 r - 1)(\cosh r - 1)^2\big)$. Then $\tilde{p}\big(B(r)\big) \subseteq d\big(B(r)\big)$.

Proof. Let $B \neq B(r)$ be in $\mathscr{P}$. We have $\varrho\big(\mathrm{ax}(B), I\big) = 2s \geq 2r$. By scaling we may assume that the centre of $\mathrm{bp}(B)$ is 1, whence, by Lemma 5.3, its radius is $\operatorname{sech} s$. By Pythagoras, it follows that the geodesic g perpendicular to $\mathrm{bp}(B)$ and to I has endpoints $\pm \tanh s$.

Let $t = s'$ be the (unique) solution of

$$\cosh^2(2s - t) - \cosh^2(t) = 1.$$

Since $\cosh^2$ is convex we also have

$$\cosh^2 s - \cosh^2 s' \leq \tfrac{1}{2}. \tag{43}$$

Let $\widetilde{P}(B)$ be the hemisphere which is perpendicular to g at a distance s' from I. By Lemma 5.1 all points on or above $\widetilde{P}(B)$ are closer to I than to $\mathrm{ax}(B)$. Let C and P denote respectively the centre of $\widetilde{P}(B)$ and its point of intersection with g. The (Euclidean) right triangle OPC has angle θ at the origin where $\cos\theta = \tanh s'$, whence $C = \big(\tanh(s)\big)/\big(\tanh(s')\big)$ and the radius of $\widetilde{P}(B)$ is $\big(\tanh(s)\big)/\big(\sinh(s')\big)$. Since $p\big(B(r)\big) \subseteq B(r_0)$, every point of $\mathrm{bp}(B) \cap \partial p\big(B(r)\big)$ has vertical component at least

$$h_0 = (1 - \operatorname{sech}\, s)\operatorname{cosech} r_0. \tag{44}$$

If $q \in \partial \mathbf{H}^3$ lies underneath $\mathrm{bp}(B)$ let $\nu^{-1}(\{q\})$ meet $\mathrm{bp}(B)$ and $\widetilde{P}(B)$ in points y and z respectively, then z is maximized for given y by choosing q to be real and greater than 1.

In this case we have,

$$\begin{aligned} q &= 1 + \sqrt{\operatorname{sech}^2 s - y^2}\,, \\ z^2 &= \frac{\tanh^2 s}{\sinh^2 s'} - \left(1 + \sqrt{\operatorname{sech}^2 s - y^2} - \frac{\tanh s}{\tanh s'}\right)^2. \end{aligned}$$

By (43)

$$\begin{aligned} \frac{\tanh^2 s}{\tanh^2 s'} - 1 &= \tanh^2 s\left(1 + \frac{1}{\sinh^2 s'}\right) - 1 \\ &\leq \tanh^2 s\left(1 + \frac{2}{2\sinh^2 s - 1}\right) - 1 \\ &= \frac{1}{\cosh^2 s(2\sinh^2 s - 1)}, \end{aligned}$$

whence

$$\frac{\tanh s}{\tanh s'} - 1 \leq \frac{1}{2\cosh^2 s(2\sinh^2 s - 1)}. \tag{45}$$

We therefore have

$$\begin{aligned} z^2 &\le \frac{2\tanh^2 s}{2\sinh^2 s - 1} - (\operatorname{sech}^2 s - y^2) + 2\left(\frac{\tanh s}{\tanh s'} - 1\right)\sqrt{\operatorname{sech}^2 s - y^2} \\ &\le \frac{2\tanh^2 s}{2\sinh^2 s - 1} - (\operatorname{sech}^2 s - y^2) + \frac{\sqrt{\operatorname{sech}^2 s - y^2}}{\cosh^2 s(2\sinh^2 s - 1)} \\ &= y^2 + \frac{1 + \sqrt{\operatorname{sech}^2 s - y^2}}{\cosh^2 s(2\sinh^2 s - 1)} \\ &\le y^2 + \frac{1 + \operatorname{sech} s}{\cosh^2 s(2\sinh^2 s - 1)} \\ &= y^2\left(1 + \frac{1 + \operatorname{sech} s}{y^2\cosh^2 s(2\sinh^2 s - 1)}\right). \end{aligned}$$

By (44), if $q \in \nu^{-1}\big(\mathrm{bp}(B) \cap \partial p\big(B(r)\big)\big)$, then

$$\frac{z}{y} \le 1 + \frac{(1 + \operatorname{sech} s)\sinh^2 r_0}{2(2\sinh^2 s - 1)(\cosh s - 1)^2},$$

and since the right-hand side above is decreasing in s, the theorem follows. □

Lemma 8.2. *Let H_1 and H_2 be the bisecting planes of axes A_1 and A_2 respectively, which are distance $2s$ and $2t$, respectively from I, with $s, t \ge r$. Let the centres of H_1 and H_2 be z and w respectively.*

If $\varrho(A_1, A_2) \ge 2r$ and

$$2e^{-2r} + 4e^{2(|s-t|-r)} \le 1, \tag{46}$$

then

$$|z - w| \ge 2|z|e^{r-s-t}\big[[(1 - 2e^{-2r})(1 - 2e^{-2r} - 4e^{2(|s-t|-r)})]^{1/2} - e^{|s-t|-r} - e^{-r}\big]. \tag{47}$$

If $r \ge \log 20$ and

$$|z - w| \ge 2|z|e^{r-s-t},$$

$$\big[1 + e^{-r} + 2.62e^{-2r} + (3 + 3.04e^{-2r})e^{|s-t|-r} + 4(1 - e^{-4r})^{-2}e^{2|s-t|-r-s-t}\big], \tag{48}$$

then $\varrho(A_1, A_2) \ge 2r$. The right-hand side above does not exceed the simpler bound

$$|z|(\operatorname{sech} u)[1.06e^{r-v} + 0.16] \tag{49}$$

where u and v are respectively the smaller and greater of s and t.

If $r \ge \log 20$ and $|s - t| \ge \log 2.68$, then either one of the bisecting planes H_1, H_2 lies underneath the other or else $\varrho(A_1, A_2) \ge 2r$.

Proof. We assume $s \le t$. Let A_1 and A_2 have endpoints z_1, z_2 and w_1, w_2 respectively. If I, A_1 and A_2 are oriented from 0 to ∞, z_1 to z_2 and w_1 to w_2 respectively, let θ and ϕ denote the rotation angles of A_1 and A_2 respectively to I.

By Lemma 5.3,

$$\begin{aligned} z_1 &= z\tanh\bigl(s+\tfrac12 i\theta\bigr)\tanh s,\\ z_2 &= z\coth\bigl(s+\tfrac12 i\theta\bigr)\tanh s,\\ w_1 &= w\tanh\bigl(t+\tfrac12 i\phi\bigr)\tanh t,\\ w_2 &= w\coth\bigl(t+\tfrac12 i\phi\bigr)\tanh t, \end{aligned}$$

so that

$$\begin{aligned} (z_1-w_1)(z_2-w_2) &= (z\tanh s - w\tanh t)^2\\ &\quad + zw\tanh s\tanh t\left[2-\coth\left(t+\tfrac12 i\phi\right)\tanh\bigl(s+\tfrac12 i\theta\bigr)\right.\\ &\quad \left.-\tanh\bigl(t+\tfrac12 i\phi\bigr)\coth\left(s+\tfrac12 i\theta\right)\right]\\ &= (z\tanh s - w\tanh t)^2\\ &\quad - \frac{4zw\sinh^2\bigl((s-t)+\frac12 i(\theta-\phi)\bigr)}{\sinh(2s+i\theta)\sinh(2t+i\phi)}\tanh s\tanh t, \end{aligned}$$

whence

$$\begin{aligned} \frac{(z_1-w_1)(z_2-w_2)}{(z_1-z_2)(w_1-w_2)} &= \frac{(z\tanh s - w\tanh t)^2\sinh(2t+i\phi)\sinh(2s+i\theta)}{4zw\tanh s\tanh t}\\ &\quad + \sinh^2\bigl((s-t)+\tfrac12 i(\theta-\phi)\bigr). \end{aligned}$$

Setting $2x = \varrho(A_1, A_2)$, (16) now gives

$$\begin{aligned} \sinh^2 x - \cosh^2(s-t) &\le \left|\frac{(z\tanh s - w\tanh t)^2\sinh(2t+i\theta)\sinh(2s+i\theta)}{4zw\tanh s\tanh t}\right|\\ &\le \cosh^2 x + \cosh^2(s-t). \end{aligned}$$

By a change of scale, we now assume that $z = 1$.

Setting $d = |z-w|$ and $a = (\tanh t - \tanh s)$ we have

$$M|w| \le |(z-w)\tanh t - a|^2 \le N(1+d), \tag{50}$$

and so

$$M(1-d) - 2ad\tanh t \le d^2\tanh^2 t + a^2 \le N(1+d) + 2ad\tanh t, \tag{51}$$

where

$$M = \frac{4\big(\sinh^2 x - \cosh^2(s-t)\big)\tanh s \tanh t}{\cosh 2s \cosh 2t},$$

$$N = \frac{4\big(\cosh^2 x + \cosh^2(s-t)\big)\tanh s \tanh t}{\sinh 2s \sinh 2t} = \frac{\cosh^2 x + \cosh^2(s-t)}{\cosh^2 s\, \cosh^2 t}.$$

Observe that the condition (46) ensures that $M > 0$.
Solving (51) gives

$$\begin{aligned} &- a\coth t - \tfrac{1}{2}M(\coth^2 t) + M^{1/2}\coth t\big(1 + a\coth t + \tfrac{1}{4}M\coth^2 t\big)^{1/2} \\ &\quad \le d \le a\coth t + \tfrac{1}{2}N(\coth^2 t) + N^{1/2}\coth t\big(1 + a\coth t + \tfrac{1}{4}N\coth^2 t\big)^{1/2}. \end{aligned}$$

Now suppose that $x \ge r$, then the first inequality in (50), and hence the lower bound for d above remains true if we set $x = r$. With this substitution we have

$$\begin{aligned} M\coth^2 t &\ge \frac{4\tanh r\big(e^{2r}(1-e^{-2r})^2 - e^{2|s-t|}(1+e^{-2|s-t|})^2\big)}{e^{2s}e^{2t}(1+e^{-4r})^2} \\ &\ge 4(1-2e^{-2r})e^{2(r-s-t)}\big[1 - 2e^{-2r} - 4e^{2(|s-t|-r)}\big], \\ M^{1/2}\coth t &\ge 2(1-2e^{-2r})^{1/2}e^{r-s-t}\big[1 - 2e^{-2r} - 4e^{2(|s-t|-r)}\big]^{1/2}. \end{aligned}$$

Similarly

$$M\coth^2 t \le 4e^{2(r-s-t)}.$$

Also, since $s \le t$, we have

$$0 \le a < a\coth t = 1 - \frac{\tanh s}{\tanh t} \le 1 - \tanh s \le 2e^{-2s} = 2e^{r-s-t}(e^{|s-t|-r}).$$

Combining these inequalities gives

$$\begin{aligned} d &\ge -2e^{r-s-t}e^{|s-t|-r} - 2e^{r-s-t}e^{-r} \\ &\quad + 2(1-2e^{-2r})^{1/2}e^{r-s-t}\big[1 - 2e^{-2r} - 4e^{2(|s-t|-r)}\big]^{1/2} \\ &\ge 2e^{r-s-t}\big[[(1-2e^{-2r})(1-2e^{-2r} - 4e^{2(|s-t|-r)})]^{1/2} - e^{|s-t|-r} - e^{-r}\big], \end{aligned}$$

which gives (47).

On the other hand, if $x \le r$, then

$$\begin{aligned} N\coth^2 t &\le \left[\frac{4(1+e^{-2t})^2}{(1-e^{-2t})^2}\right]\left[\frac{e^{2r}(1+e^{-2r})^2 + e^{2|s-t|}(1+e^{-2|s-t|})^2}{e^{2s}e^{2t}(1+e^{-2t})^2(1+e^{-2s})^2}\right] \\ &\le 4(1-e^{-4t})^{-2}e^{2(r-s-t)}\big[(1+e^{-2r})^2 + 4e^{2(|s-t|-r)}\big] \\ &\le 2e^{r-s-t}(1-e^{-4t})^{-2}(2e^{-r})\big[(1+e^{-2r})^2 + 4e^{2(|s-t|-r)}\big] \end{aligned}$$

whence, from the second inequality.

$$N^{1/2}\coth t \le 2(1-e^{-4t})^{-1}e^{r-s-t}\big[1+e^{-2r}+2e^{|s-t|-r}\big]$$

and

$$\begin{aligned} N\coth^2 t &\le 4(1-e^{-4t})^{-2}\big[(1+e^{-2r})^2e^{-2r}+4e^{-4r}\big] \\ &= 4e^{-2r}(1-e^{-4t})^{-2}(1+6e^{-2r}+e^{-4r}). \end{aligned}$$

Using these bounds we have, for $x\le r$,

$$\begin{aligned} d &\le 2e^{r-s-t}e^{|s-t|-r} \\ &\quad + 2e^{r-s-t}(1-e^{-4t})^{-2}e^{-r}\big[(1+e^{-2r})^2+4e^{2|s-t|-s-t}\big] \\ &\quad + 2e^{r-s-t}(1-e^{-4t})^{-1}\big[1+e^{-2r}+2e^{|s-t|-r}\big] \\ &\quad \times\big(1+2e^{-2r}+e^{-2r}(1-e^{-4t})^{-2}(1+6e^{-2r}+e^{-4r})\big)^{1/2} \\ &\le 2e^{r-s-t}(1-e^{-4t})^{-2}\big[e^{|s-t|-r}+e^{-r}[(1+e^{-2r})^2+4e^{2|s-t|-s-t}] \\ &\quad +(1+e^{-2r}+2e^{|s-t|-r})\big(1+2e^{-2r}+e^{-2r}(1+6e^{-2r}+e^{-4r})\big)^{1/2}\big]. \end{aligned}$$

The assumption that $r\ge\log 20$ now gives

$$\begin{aligned} d &\le 2e^{r-s-t}(1-e^{-4t})^{-2}\big[e^{|s-t|-r}+e^{-r}+0.100125e^{-2r}+4e^{-r}e^{2|s-t|-s-t} \\ &\quad +\big(1+e^{-2r}+2e^{|s-t|-r}\big)(1+(3.01500625/2)e^{-2r})\big] \\ &\le 2e^{r-s-t}(1-e^{-4t})^{-2}\big[1+e^{-r}+2.612e^{-2r}+(3+3.02e^{-2r})e^{|s-t|-r} \\ &\quad +4e^{-r}e^{2|s-t|-s-t}\big] \\ &\le 2e^{r-s-t}\big[1+e^{-r}+2.62e^{-2r}+(3+3.04e^{-2r})e^{|s-t|-r} \\ &\quad +4(1-e^{-4t})^{-2}e^{2|s-t|-r-s-t}\big], \end{aligned}$$

which gives (48).

Recalling that $s\le t$ we also get,

$$\begin{aligned} d &\le 2e^{r-s-t}\big[1+e^{-r}+2.62e^{-2r}+(3+3.04e^{-2r})e^{t-s-r} \\ &\quad +4(1-e^{-4t})^{-2}e^{t-3s-r}\big] \\ &\le 2e^{-s}\big[1.057e^{r-t}+3.01e^{-r}+4.00006e^{-3r}\big] \\ &\le (\operatorname{sech} s)[1.06e^{r-t}+0.16], \end{aligned}$$

which gives (49).

We now prove the last statement of the theorem. We suppose that $|s-t|\ge 2.68$. Using Lemma 5.3, the bisecting plane H_2 lies under H_1 exactly when

$$|w|\operatorname{sech} t+|z-w|\le|z|\operatorname{sech} s,$$

which follows if

$$(|z-w|+|z|)\operatorname{sech} t+|z-w| \leq |z| \operatorname{sech} s$$

or equivalently

$$|z-w| \leq \frac{\operatorname{sech} s-\operatorname{sech} t}{1+\operatorname{sech} t}. \tag{52}$$

If, on the other hand, if (52) fails, then, since $s \leq t$, (49) shows that $\varrho(A_1, A_2) \geq 2r$ if

$$\frac{\operatorname{sech} s-\operatorname{sech} t}{1+\operatorname{sech} t} \geq \operatorname{sech} s[1.06 e^{r-t}+0.16]. \tag{53}$$

Since $2.68>(1+1.17)/0.81$, we have, in turn,

$$\begin{aligned}
e^{t-s} &\geq (1+1.17)/0.81, \\
e^{s-t} &\leq 0.81-1.17 e^{s-t} \leq 0.81-1.17 e^{r-t}, \\
e^{-t} &\leq e^{-s}(0.82-1.17 e^{r-t})/1.0025, \\
\operatorname{sech} t &\leq \operatorname{sech} s(0.82-1.17 e^{r-t}), \\
\operatorname{sech} s-\operatorname{sech} t &\geq \operatorname{sech} s(1.17 e^{r-t}+0.18), \\
\operatorname{sech} s-\operatorname{sech} t &\geq \operatorname{sech} s(1.06 e^{r-t}+0.16)(1+2/20.05).
\end{aligned}$$

Thus (53) follows and the proof of the theorem is complete. □

Lemma 8.3. *Let $\mathscr{P}$ be a packing of $\mathbf{H}^3$ by cylinders of radius $r \geq \log 20$, B a cylinder in $\mathscr{P}$ and $p(B)$ the polyhedral region of B in $\mathscr{P}$. Then there exists a packing $\mathscr{P}_1$ of $\mathbf{H}^3$ by radius r cylinders, which also contains B and such that the polyhedral region of B in $\mathscr{P}_1$ lies in the intersection of $p(B)$ and the cylinder of radius r_0 about $\operatorname{ax}(B)$, where r_0 is defined by*

$$\frac{\cosh^2 r}{\cosh^2 r_0}=g^5 E^{-2}-\left(\sqrt{2}\left(\sqrt{5}+1\right) \operatorname{sech} r+\left(\sqrt{5}+2\right) \operatorname{sech}^2 r\right), \tag{54}$$

where $g=\frac{1}{2}\left(\sqrt{5}-1\right)$ is the golden mean and

$$E=E(r)=(1+e^{-2r})(1+9.27 e^{-r}). \tag{55}$$

Proof. Suppose that we have $r_0>r_1>r$, such that for any packing $\mathscr{P}'$ of cylinders of radius r, and $B \in \mathscr{P}'$, with axis I, the following holds: if $z \in p(B)$ is distance r_0 from I, g is the geodesic through z which is perpendicular to I and x the point on g which lies between z and $g \cap I$ at distance r_1 from I, and if B_1 is a cylinder of radius r, whose axis meets g perpendicularly at distance

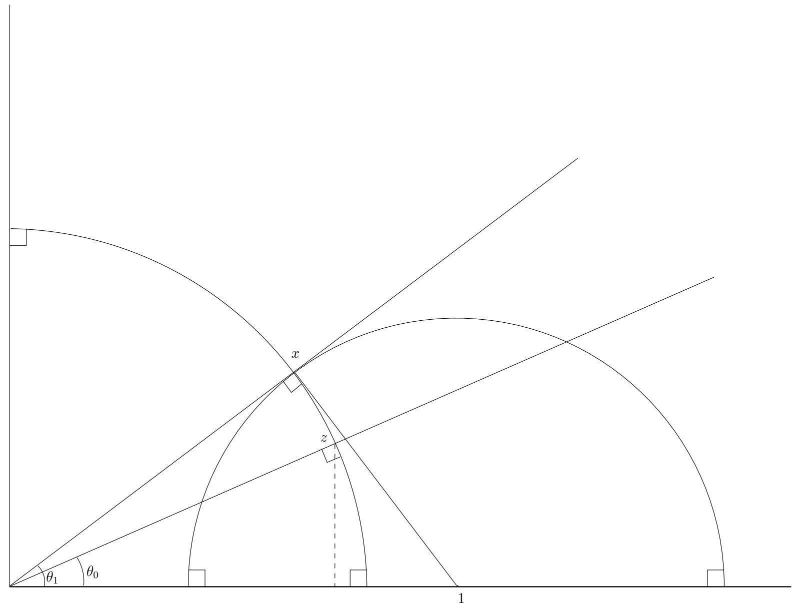

Figure 6.

$2r_1$ from I (so that $x \in \mathrm{bp}(B_1)$), then the bisecting planes of any cylinder in $\mathscr{P}'$ which intersects B_1 lies under $\mathrm{bp}(B_1)$.

Given that such r_0 and r_1 exist, we can construct a new packing by adding in the cylinder B_1 and discarding those cylinders of $\mathscr{P}$ which meet it. The polyhedral region of B in the new packing clearly lies in the corresponding region of the original packing.

By an obvious modification of the induction step in Theorem 7.1 we can, by iterating this process, construct a packing $\mathscr{P}_1$ which has the property that $p(B)$, (defined relative to $\mathscr{P}_1$) lies in the cylinder with axis I, radius r_0, as required by the theorem.

It remains to show that r_0 and r_1 with the required properties exist and that r_0 can take the value given in (54). For the moment we let any $r_0 > r_1 > r$ be chosen subject to the lower bound

$$\cosh r_1 \geq E \cosh r, \tag{56}$$

where E is defined by (55). We find under what conditions they have the required property. We define θ_i $(i = 0, 1)$ by $\cos\theta_i = \tanh r_i$ $(\sin\theta_i = \operatorname{sech} r_i)$, so that the cylinder of radius r_i about I is at angle θ_i to $\partial\mathbf{H}^3$.

Let $B_1 \in \mathscr{P}$ be chosen as above. We may suppose by scaling that the centre of $\mathrm{bp}(B_1)$ is 1, whence $z = \cos\theta_1(\cos\theta_0, 0, \sin\theta_0)$. Let $B_2 \in \mathscr{P}$ have bisecting plane with radius ϱ and centre y, and let $\varrho(\mathrm{ax}(B_2), I) = 2s \geq 2r$.

We distinguish two cases. First we show that, whatever the value of r_1, if $|r_1 - s|$ is sufficiently large, then either $\mathrm{bp}(B_2)$ lies under $\mathrm{bp}(B_1)$ or B_1 and B_2 are disjoint. For smaller s, we show that the same conclusion still holds, provided that either ϱ is less than some lower bound ϱ_0 or greater than some upper bound ϱ_1. When we choose r_0 and r_1 appropriately, we get $\varrho_0 = \varrho_1$, in which case r_0 and r_1 have the required properties.

If $|s - r_1| \geq \log 2.68$, then, by the last assertion of Lemma 8.2, either $\mathrm{bp}(B_2)$ lies under $\mathrm{bp}(B_1)$ or B_1 and B_2 are disjoint (from the definition of B_1 it is clearly impossible for $\mathrm{bp}(B_1)$ to lie under $\mathrm{bp}(B_2)$). We may therefore assume that

$$|s - r_1| \leq \log 2.68. \tag{57}$$

If

$$|y| < \frac{1 - \operatorname{sech} r}{1 + \operatorname{sech} r}, \tag{58}$$

then

$$|y| \operatorname{sech} s + \operatorname{sech} r_1 < |y - 1|, \tag{59}$$

whence $\mathrm{bp}(B_2)$ is disjoint from and not underneath $\mathrm{bp}(B_1)$, so that B_1 and B_2 are disjoint. We may thus assume that (58) fails. In this case, by Theorem 5.3,

$$\operatorname{sech} s = \frac{\varrho}{|y|} \leq \frac{(1 + \operatorname{sech} r)\varrho}{1 - \operatorname{sech} r}$$

and so, by Lemma 8.2 and (57), B_1 and B_2 are disjoint, when

$$|y - 1| \geq 2|y| e^{r - r_1 - s}(1 + 9.27 e^{-r}). \tag{60}$$

(The right-hand side in (60) is an upper bound for the right-hand side of (48) with $|s - t| = \log 2.68$.) Since z cannot lie under $\mathrm{bp}(B_2)$, we have, when $\varrho^2 \geq \cos^2 \theta_1$

$$|y - \cos \theta_0 \cos \theta_1| \geq \sqrt{\varrho^2 - \cos^2 \theta_1 \sin^2 \theta_0}\,,$$

whence

$$|y - 1| \geq \cos \theta_0 \cos \theta_1 - 1 + \sqrt{\varrho^2 - \cos^2 \theta_1 \sin^2 \theta_0}\,.$$

Therefore if

$$\varrho^2 - \cos^2 \theta_1 \sin^2 \theta_0 \geq \left[E \varrho \cosh r \sin \theta_1 + 1 - \cos \theta_0 \cos \theta_1\right]^2,$$

where $E = E(r)$ is defined in (55), then, by (60), B_1 and B_2 are disjoint.

This rearranges to the quadratic

$$(1 - A^2)\varrho^2 - 2AB\varrho - (\cos^2 \theta_1 \sin^2 \theta_0 + B^2) \geq 0,$$

where $A = E\cosh r \sin\theta_1$, $B = 1 - \cos\theta_0\cos\theta_1$. Since, by (56), $A < 1$, this has solution

$$\varrho^2 \geq \frac{2A^2B^2 + (1-A^2)(\cos^2\theta_1\sin^2\theta_0 + B^2) + 2AB\sqrt{\cos^2\theta_1\sin^2\theta_0(1-A^2) + B^2}}{(1-A^2)^2}.$$

We have $B \leq \operatorname{sech}^2 r$ so that above follows from

$$\begin{aligned}\varrho^2 &\geq \frac{2\operatorname{sech}^4 r + (1-A^2)(\sin^2\theta_0 + \operatorname{sech}^4 r) + 2\sqrt{2}\operatorname{sech}^3 r}{(1-A^2)^2} \\ &= \frac{\operatorname{sech}^2 r}{(1-A^2)}\left[\frac{\cosh^2 r}{\cosh^2 r_0} + \frac{2\sqrt{2}}{1-A^2}\operatorname{sech} r + \frac{3-A^2}{1-A^2}\operatorname{sech}^2 r\right].\end{aligned} \tag{61}$$

Suppose now that B_1 and B_2 meet, and let $d = |y-1|$ be the distance between the centres of their bisecting planes, then, by Lemma 8.2

$$d \leq E\varrho\cosh r\sin\theta_1 = A\varrho,$$

whence

$$d + \varrho \leq (A+1)\varrho.$$

The bisecting plane $\mathrm{bp}(B_2)$ lies beneath $\mathrm{bp}(B_1)$ if $d + \varrho \leq \sin\theta_1$, so for this it suffices to have

$$\varrho \leq \frac{\sin\theta_1}{A+1}. \tag{62}$$

Combining (61) and (62), r_0 and r_1 have the required property if

$$\frac{\sin^2\theta_1}{(A+1)^2} = \frac{\operatorname{sech}^2 r}{(1-A^2)}[h_0^2 + K],$$

where

$$K = \frac{2\sqrt{2}}{1-A^2}\operatorname{sech} r + \frac{3-A^2}{1-A^2}\operatorname{sech}^2 r$$

and $h_i = \cosh r/\cosh r_i$ $(i = 0, 1)$.

Simplifying gives

$$\frac{h_1^2}{1+A} = \frac{h_0^2 + K}{1-A},$$

or

$$h_0^2 = \frac{(1-Eh_1)(Eh_1)^2}{E^2(1+Eh_1)} - K. \tag{63}$$

The first and main term on the right-hand side can be maximized by setting $Eh_1 = g = \frac{1}{2}(\sqrt{5}-1)$ (which complies with the inequality (56)). This approximately maximizes h_0.

This value of Eh_1 gives

$$K = \sqrt{2}\left(\sqrt{5}+1\right)\operatorname{sech} r + \left(\sqrt{5}+2\right)\operatorname{sech}^2 r$$

and the right-hand side of (63) is $g^5E^{-2} - K$.

The lemma follows. □

Lemma 8.4. *Let $\alpha \le \beta < 1$ and let H_1 and H_2 be the upper hemispheres in $\mathbf{H}^3$ with centre 1 radius α and centre k radius $k\beta$ respectively. Let $0 < \tau < 1$. If*

$$k \ge (1-\beta^2)^{-1}\left[1 + \frac{\beta}{\sqrt{1-\tau^2}} + \frac{(2\tau+1)\beta^2}{2\tau}\right],$$

then, if $\nu(H_1 \cap H_2)$ is nonempty, it is distance at least $\tau\alpha$ from the centre of H_1.

Proof. An elementary calculation shows that $\nu(H_1 \cap H_2)$ crosses the real axis at $1+\tau\alpha$ when $d = k-1$ is the positive root of

$$(1-\beta^2)d^2 - 2(\tau\alpha+\beta^2)d + (\alpha^2-\beta^2)$$

and that if d is greater than this root, then $\nu(H_1 \cap H_2)$ crosses real axis at a number greater than $1+\tau\alpha$. Explicitly, this root is

$$\begin{aligned}
d &= (1-\beta^2)^{-1}\left[\beta^2 + \tau\alpha + \big(\beta^2-(1-\tau^2)\alpha^2\big)^{1/2}\left(1 + \frac{\beta^2\alpha^2+2\tau\alpha\beta^2}{\beta^2-(1-\tau^2)\alpha^2}\right)^{1/2}\right]\\
&\le (1-\beta^2)^{-1}\left[\beta^2 + \tau\alpha + \big(\beta^2-(1-\tau^2)\alpha^2\big)^{1/2} + \frac{\beta^2\alpha^2+2\tau\alpha\beta^2}{2\big(\beta^2-(1-\tau^2)\alpha^2\big)^{1/2}}\right]\\
&\le (1-\beta^2)^{-1}\left[\beta^2 + \tau\alpha + \big(\beta^2-(1-\tau^2)\alpha^2\big)^{1/2} + \frac{\beta^2(\beta+2\tau)}{2\tau}\right] \qquad (64)\\
&= (1-\beta^2)^{-1}\left[\left(\tau\alpha + \big(\beta^2-(1-\tau^2)\alpha^2\big)^{1/2}\right) + 2\beta^2 + \beta^3/(2\tau)\right]\\
&\le (1-\beta^2)^{-1}\left[\frac{\beta}{\sqrt{1-\tau^2}} + 2\beta^2 + \beta^3/(2\tau)\right],
\end{aligned}$$

the last inequality following from elementary calculus. The lemma follows. □

Lemma 8.5. *Let $A(r)$ be the lesser of*

$$\tanh^2 r\left[1 + \left(\tfrac{4}{3}\right)^{1/2}\operatorname{sech} r + 2\operatorname{sech}^2 r\right]^{-1} \qquad (65)$$

and

$$\begin{aligned}
&\Big[(1-2e^{-2r})\big[1-2e^{-2r} - \tfrac{16}{3}(1+e^{-2r})^2\coth^4 r\\
&\big(1+(\tfrac{4}{3})^{1/2}\operatorname{sech} r + 2\operatorname{sech}^2 r\big)^2 e^{-2r}\big]\Big]^{1/2} \qquad (66)\\
&\quad - \frac{2}{\sqrt{3}}(1+e^{-2r})\coth^2 r\left[1 + \left(\tfrac{4}{3}\right)^{1/2}\operatorname{sech} r + 2\operatorname{sech}^2 r\right]e^{-r} - e^{-r}
\end{aligned}$$

and suppose that $r > 0$ is large enough to make $A(r) > 0$.

Let H_1 and H_2 be as in Lemma 8.2, centred at w_1 and w_2 respectively. Suppose that the cylinders of radius r corresponding to H_1 and H_2 are disjoint but that H_1 and H_2 themselves intersect. Suppose also that $e^{r-s} \geq \frac{1}{2}\sqrt{3}$. If d denotes the distance from the centre of H_1 to $\nu(H_1 \cap H_2)$, then

$$d \geq C_1(r)|w_1| 2e^{-s}\left(1 - \frac{1}{2e^{2(r-s)}}\right) \tag{67}$$

where

$$C_1(r) = (1+e^{-2r})^{-1}\big(3 - 2(1+e^{-2r})^2 A(r)^{-2}\big). \tag{68}$$

Proof. By scaling we may assume that $w_1 = 1$ By Lemma 5.3, H_1 and H_2 have radii $r_1 = \operatorname{sech} s$ and $r_2 = |w_2| \operatorname{sech} t$, respectively.

In view of Lemma 8.4 (with $\alpha = \operatorname{sech} s$, $\beta = \operatorname{sech} r$ and $\tau = \frac{1}{2}$) we may assume that

$$|w_2| \leq \coth^2 r\big[1 + \left(\tfrac{4}{3}\right)^{1/2} \operatorname{sech} r + 2\operatorname{sech}^2 r\big]. \tag{69}$$

If $r_2/r_1 = |w_2|(\operatorname{sech} t)/(\operatorname{sech} s) \leq \frac{1}{2}\sqrt{3}$, then an application of Pythagoras gives $d \geq \frac{1}{2}r_1 = \frac{1}{2}(\operatorname{sech} s)$, and again (67) holds. We therefore assume that $|w_2|(\operatorname{sech} t)/(\operatorname{sech} s) \geq \frac{1}{2}\sqrt{3}$. Using (69)

$$\begin{aligned} e^{t-s} &= \frac{(1+e^{-2s})\cosh t}{(1+e^{-2t})\cosh s} \leq (1+e^{-2r})\big(2/\sqrt{3}\big)|w_2| \\ &\leq \frac{2}{\sqrt{3}}(1+e^{-2r})\coth^2 r\left[1 + \left(\frac{4}{3}\right)^{1/2}\operatorname{sech} r + 2\operatorname{sech}^2 r\right]. \end{aligned} \tag{70}$$

Since also $e^{s-t} \leq e^{s-r} \leq \big(2/\sqrt{3}\big)$, by hypothesis, Lemma 8.2 gives

$$|w_1 - w_2| \geq 2|w_2|e^{r-s-t}A(r). \tag{71}$$

Define

$$\eta = \eta(r) = \tfrac{1}{2}e^r A(r).$$

After multiplying by a scale factor of $\eta(r)$, the hemisphere H_i has centre $w_i' = \eta(r)w_i$ and radius $r_i' = \eta(r)r_i$. By (69), and the definition of $A(r)$, we have $r_i' \leq 1$ and

$$\begin{aligned} r_1'r_2' &= \eta^2|w_1|\,|w_2|\operatorname{sech} s\,\operatorname{sech} t \\ &\leq 2\eta e^{r-s-t}|w_2|A(r) \leq \eta|w_1 - w_2|. \end{aligned} \tag{72}$$

Hence Corollary 7.1 (with $n = 3$, $k = 2$) applies to the two hemispheres scaled by η, to give, using the fact that $s \le s_0 = r + \log\bigl(2/\sqrt{3}\bigr)$,

$$\begin{aligned} d &\ge \frac{(2\eta^2 r_1^2 - 1)\sqrt{2}}{2\eta\sqrt{2\eta^2 r_1^2}} = r_1\left(1 - \frac{1}{2(\eta r_1)^2}\right) \\ &\ge (1 + e^{-2s})^{-1} 2e^{-s}\left(1 - \frac{e^{2r}(1+e^{-2r})^2}{4\eta^2}\frac{1}{2e^{2(r-s)}}\right) \\ &\ge (1 + e^{-2r})^{-1} 2e^{-s}\left(1 - \frac{1}{2e^{2(r-s)}}\right)\left(1 - \frac{e^{2r}(1+e^{-2r})^2}{4\eta^2}\frac{1}{2e^{2(r-s_0)}}\right) \\ &\quad\times\left(1 - \frac{1}{2e^{2(r-s_0)}}\right)^{-1} \\ &= 2e^{-s}\left(1 - \frac{1}{2e^{2(r-s)}}\right)\left[(1+e^{-2r})^{-1}\left(3 - \frac{e^{2r}(1+\mathrm{e}^{-2r})^2}{2\eta^2}\right)\right], \end{aligned} \tag{73}$$

which is (67). ◻

Lemma 8.6. *Let $c \ge \frac{1}{2}\sqrt{3}$. Let $A_1(r)$ be the product of*

$$\tanh^2 r\left[1 + \left(\tfrac{3}{2}\right)^{1/2}\operatorname{sech} r + \left(1 + \tfrac{1}{2}\sqrt{3}\right)\operatorname{sech}^2 r\right]^{-1} \tag{74}$$

and

$$\left[\left[(1 - 2e^{-2r})(1 - 2e^{-2r} - 6c^{-2}e^{-2r})\right]^{1/2} - \sqrt{1.5}\,c^{-1}e^{-r} - e^{-r}\right]. \tag{75}$$

Let

$$C_3(r) = (1 + e^{-2r})^{-1} A_1(r)$$

and let $r > 0$ be large enough to make $C_3(r) > 0$.

For $1 \le i \le 3$, let A_i be geodesics at distance $2s_i \ge 2r$ from I, and distance at least $2r$ from each other. Let $H_i = \mathrm{bp}(A_i)$, and let w_i be the centre of H_i. Let d denote the distance between w_1 and $\nu(H_1 \cap H_2 \cap H_3)$, when this intersection is nonempty. If $e^{r-s_1} \ge c$, then

$$d \ge C_2(r)|w_1|(2e^{-s_1})\frac{2e^{2r} - e^{2s_1}}{e^r\sqrt{4e^{2r} - e^{2s_1}}}, \tag{76}$$

where

$$C_2(r) = \bigl(C_3(r)\bigr)^{-1}(1 + e^{-2r})^{-1}\frac{3\bigl(C_3(r)\bigr)^2 - 2}{\sqrt{\left|\left(\frac{3}{2}\right)\bigl(C_3(r)\bigr)^2 - \left(\frac{1}{2}\right)\right|}}. \tag{77}$$

Proof. The result is trivial unless $C_3(r) > \sqrt{2/3}$, so we assume this inequality does hold. By scaling we may also assume $w_1 = 1$. Let

$$\eta_1(r) = \tfrac{1}{2} e^r A_1(r).$$

In view of Lemma 8.4 (with $\alpha = \operatorname{sech} s$, $\beta = \operatorname{sech} r$ and $\tau = 1/\sqrt{3}$) we may assume for $i = 2, 3$, that

$$|w_i| \le \coth^2 r \big[1 + \big(\tfrac{3}{2}\big)^{1/2} \operatorname{sech} r + \big(1 + \tfrac{1}{2}\sqrt{3}\big) \operatorname{sech}^2 r\big]. \tag{78}$$

If, for some $i \ge 2$,

$$e^{s_i} \ge e^{s_1} \sqrt{(3/2)} \tag{79}$$

then, by (78), the radius of H_i

$$\begin{aligned} &\le \coth^2 r \big[1 + \big(\tfrac{3}{2}\big)^{1/2} \operatorname{sech} r + \big(1 + \tfrac{1}{2}\sqrt{3}\big) \operatorname{sech}^2 r\big] \operatorname{sech} s_i \\ &\le (1 + e^{-2r}) \big(\tfrac{2}{3}\big)^{1/2} \coth^2 r \\ &\quad \times \big[1 + \big(\tfrac{3}{2}\big)^{1/2} \operatorname{sech} r + \big(1 + \tfrac{1}{2}\sqrt{3}\big) \operatorname{sech}^2 r\big] \operatorname{sech} s_1, \end{aligned} \tag{80}$$

in which case, by Pythagoras,

(81)

$$\begin{aligned} d &\ge \operatorname{sech} s_1 \sqrt{1 - \tfrac{2}{3}(1 + e^{-2r})^2 \coth^4 r \big[1 + \big(\tfrac{3}{2}\big)^{1/2} \operatorname{sech} r + \big(1 + \tfrac{1}{2}\sqrt{3}\big) \operatorname{sech}^2 r\big]^2} \\ &= \frac{\operatorname{sech} s_1}{\sqrt{3}} \sqrt{3 - 2(1 + e^{-2r})^2 \coth^4 r \big[1 + \big(\tfrac{3}{2}\big)^{1/2} \operatorname{sech} r + \big(1 + \tfrac{1}{2}\sqrt{3}\big) \operatorname{sech}^2 r\big]^2} \\ &\ge \big(2/\sqrt{3}\big) e^{-s_1} (1 + e^{-2r})^{-1} \sqrt{3 - 2C_3(r)^{-2}}, \end{aligned}$$

so that (76) follows from the fact that $C_3(r) \le 1$ and the elementary inequality

$$\sqrt{3 - 2x^{-2}} \ge x^{-1} \frac{3x^2 - 2}{\sqrt{(3/2)x^2 - (1/2)}},$$

which is true for $\sqrt{2/3} < x \le 1$.

We may thus assume that (79) does not hold for either $i = 2, 3$. Since, by assumption, $e^{r - s_1} \ge c$, we have $e^{|s_i - s_j|} \le c^{-1} \sqrt{(3/2)}$, whence, by Lemma 8.2,

$$\begin{aligned} |w_i - w_j| \ge 2|w_i| e^{r - s_i - s_j} \Big[&\big[(1 - 2e^{-2r})(1 - 2e^{-2r} - 6c^{-2} e^{-2r})\big]^{1/2} \\ &- \sqrt{1.5}\, c^{-1} e^{-r} - e^{-r}\Big]. \end{aligned} \tag{82}$$

After multiplying by a scale factor of $\eta_1(r)$, H_i has radius $r_i' = \eta_1(r)|w_i| \operatorname{sech} r_i$, and, using (78),

$$
\begin{aligned}
r_i' r_j' = \eta_1(r)^2 |w_i|\,|w_j| \operatorname{sech} s_i \operatorname{sech} s_j &\le 2\eta_1(r) e^{r-s_i-s_j} |w_i|\,|w_j| A_1(r) \\
&\le \eta_1(r)|w_i - w_j|.
\end{aligned}
\tag{83}
$$

Since also, from the definition of $A_1(r)$, each $r_i' \le 1$, Corollary 7.1 (with $k = n = 3$) applies to the three hemispheres scaled by $\eta_1(r)$. Scaling back again gives

$$
\begin{aligned}
d &\ge \eta_1(r)^{-1} \frac{2{\eta_1}^2(r) \operatorname{sech}^2 s_1 - 1}{\sqrt{4{\eta_1}^2(r)\operatorname{sech}^2 s_1 - 1}} \\
&\ge \big(C_3(r)\big)^{-1} (1 + e^{-2r})^{-1} (2e^{-r}) \frac{2C_3^2(r) e^{2(r-s_1)} - 1}{\sqrt{4C_3^2(r) e^{2(r-s_1)} - 1}} \\
&\ge \big(C_3(r)\big)^{-1} (1 + e^{-2r})^{-1} (2e^{-r}) \\
&\quad \times \frac{2e^{2(r-s_1)} - 1}{\sqrt{4e^{2(r-s_1)} - 1}} \frac{2c^2\big(C_3(r)\big)^2 - 1}{\sqrt{4c^2\big(C_3(r)\big)^2 - 1}} \frac{\sqrt{4c^2 - 1}}{2c^2 - 1},
\end{aligned}
\tag{84}
$$

using the assumption, $e^{r-s_1} \ge c \ge \frac{1}{2}\sqrt{3}$, and the convexity of the function

$$
-\log \frac{2e^x - 1}{\sqrt{4e^x - 1}}
$$

for $x > -\log 2$. Now (76) readily follows. □

Proof of Theorem 4.1. Let $\mathscr{P}$ be packing of $\mathbf{H}^3$ by cylinders of radius r. We may assume that $r \ge 4.8$, since the bound of the theorem is greater than one for smaller values of r. Let B be a cylinder in $\mathscr{P}$. We may assume that $B = B(r)$ with axis I. Now, by applying Lemma 8.3, we may also assume that $p(B) \subseteq B(r_0)$ with r_0 defined by (54).

Let F be a face of the boundary of $p(B)$. The projection of F onto $\partial\mathbf{H}^3$ is a polygon P. We may assume that F lies on a hemisphere with centre 1 and radius $\operatorname{sech} s$ for some $s \ge r$. Let

$$
Q = \nu^{-1}(P) \cap p(B), \quad D = \nu^{-1}(P) \cap d(B).
$$

We associate with each face F a "local" density

$$
\varrho = \varrho_F(B) = \operatorname{Vol}(B \cap D)/\operatorname{Vol}(D).
$$

(Since both volumes in the quotient are finite this definition is straightforward.) The proof has two main steps. First we show that

$$\bar{\varrho}(B) \leq \sup \varrho_F(B), \tag{85}$$

where the supremum above is taken over all faces F of $\partial p(B)$. We complete the proof by showing that each $\varrho_F(B)$, satisfies the bound (7) of the theorem.

For $0 < a < b$, let $A'(a,b)$ be the union of the projections of faces of $\partial p(B)$ which lie wholly in the annulus $A(a,b)$. Each such face lies on a bisecting plane and, by Theorem 5.3 a bisecting plane with centre z has radius at most $|z|\operatorname{sech} r$. Therefore, if P is a bisecting plane, $\nu(P) \subseteq A(a,b)$ if its centre is in $A\big(a/(1-\operatorname{sech} r), b/(1+\operatorname{sech} r)\big)$, and this in turn occurs if any point of $\nu(P)$ lies in $A\big((1+\operatorname{sech} r)a/(1-\operatorname{sech} r)\big), \big((1-\operatorname{sech} r)b/(1+\operatorname{sech} r)\big)$; that is we have $A'(a,b) \supseteq A\big(a(1-2\operatorname{sech} r)^{-1}, b(1-2\operatorname{sech} r)\big)$. Hence, by (5), the volume of $B \cap \nu^{-1}\big(A(a,b)\backslash A'(a,b)\big)$ is bounded by a constant which is independent of a and b.

We now have

$$\begin{aligned}
\frac{\operatorname{vol}\big(B \cap C(a,b)\big)}{\operatorname{vol}\big(d(B) \cap C(a,b)\big)} &\leq \frac{\operatorname{vol}\big(B \cap C(a,b)\big)}{\operatorname{vol}\big(d(B) \cap \nu^{-1}\big(A'(a,b)\big)\big)} \\
&\leq \frac{\operatorname{vol}\big(B \cap \nu^{-1}\big(A'(a,b)\big)\big)}{\operatorname{vol}\big(d(B) \cap \nu^{-1}\big(A'(a,b)\big)\big)} \\
&\quad + \frac{\text{Constant}}{\operatorname{vol}\big(d(B) \cap \nu^{-1}\big(A'(a,b)\big)\big)} \\
&\leq \{\sup \varrho_F(B)\} \\
&\quad + \frac{\text{Constant}}{\operatorname{vol}\big(d(B) \cap \nu^{-1}\big(A'(a,b)\big)\big)}.
\end{aligned} \tag{86}$$

Now (85) follows by taking limits, and applying Lemma 4.1.

We now prove the required bound for the $\varrho_F(B)$.

By Lemma 8.1, we have

$$\operatorname{Vol}(D) \geq \delta(r)^{-2} \operatorname{Vol}(Q),$$

where, using Lemma 8.3,

$$\delta(r) = 1 + \frac{(1+\operatorname{sech} r)\sinh^2 r_0}{2(2\sinh^2 r - 1)(\cosh r - 1)^2} \tag{87}$$

where r_0 is as defined in (54).

Also

$$\mathrm{Vol}\,(B\cap D)=\mu(P)(\sinh^2 r)/2\le \mathrm{Area}\,(P)(\sinh^2 r)(1-\operatorname{sech} r)^{-2}/2.$$

The above estimates now give (setting, for brevity, $\varrho=\varrho_F(B)$),

$$\varrho^{-1}\ge\frac{2(1-\operatorname{sech} r)^2\mathrm{Vol}\,(Q)}{\delta(r)^2\mathrm{Area}\,(P)\sinh^2 r}. \tag{88}$$

If $e^{2s}\ge\varrho_\infty^{-1}e^{2r}$, then $\operatorname{sech} s\le 2e^{-r}\sqrt{\varrho_\infty}$, in which case

$$\mathrm{Vol}\,(Q)\ge\frac{\mathrm{Area}\,(P)}{2\operatorname{sech}^2 s}\ge\frac{\mathrm{Area}\,(P)}{8\varrho_\infty e^{-2r}},$$

so that

$$\varrho(Q)\le\frac{(1-e^{-2r})^2\varrho_\infty}{(1-\operatorname{sech} r)^2}\delta(r)^2.$$

A straightforward calculation shows that Theorem 4.1 holds in this case. Therefore we assume henceforth that $e^{2s}\le\varrho_\infty^{-1}e^{2r}$.

Let $h(t)$ be the vertical distance from $t\in P$ to the boundary of $p(B)$. From (88), ϱ^{-1} is at least $2\delta(r)^{-2}(1-\operatorname{sech} r)^2/\sinh^2 r$ of the mean value of $1/\big(2h(t)^2\big)$ over the polygon P.

Explicitly, for $(x,y)\in P$, we have $h\big((x,y)\big)=\sqrt{\operatorname{sech}^2 s-(x-1)^2-y^2}$. If $1\notin\nu(P)$, then we clearly decrease the mean of h^{-2} by extending P to the convex hull of P and 1. We may thus assume that $1\in P$. We next subdivide P into triangles as shown in Figure 7(a). If such a triangle has an obtuse angle other than at 1, then it can be extended as shown in Figure 7(b) to a right angled triangle without increasing the mean value of $1/\big(2h(t)^2\big)$. Otherwise it can be subdivided into two right angled triangles.

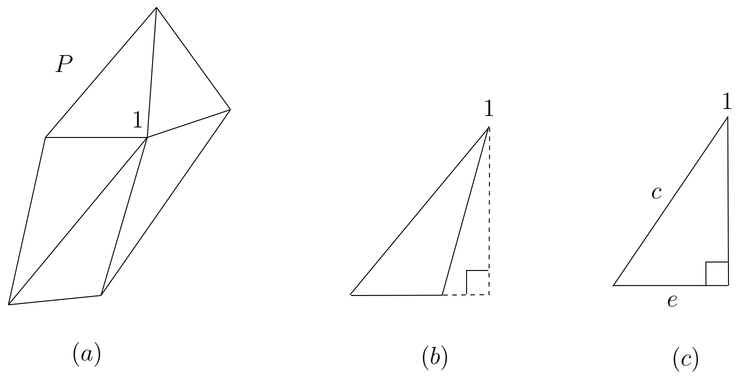

Figure 7.

Thus, in any case, a lower bound for the mean value of $1/\big(2h(t)^2\big)$ over P is obtained by minimizing the mean of the same function over T, where T is the triangle in Figure 7(c), where the lengths e and c are bounded below by Lemmas 8.5 and 8.6, respectively (setting $c = \sqrt{\varrho_\infty}$ in the latter).

Specifically, if we let $\xi = e^{2(r-s)}$, and set

$$e_0 = 2e^{-s}\left(1 - \frac{e^{2s}}{2e^{2r}}\right) = 2e^{-s}\left(1 - \frac{1}{2\xi}\right)$$

and

$$c_0 = 2e^{-r}\frac{2e^{2(r-s)} - 1}{\sqrt{4e^{2(r-s)} - 1}} = 2e^{-s}\frac{2\xi - 1}{\sqrt{\xi(4\xi - 1)}},$$

then we have

$$e \geq C(r)e_0 \tag{89}$$

and

$$c \geq C(r)c_0 \tag{90}$$

where $C(r)$ is the lesser of $C_1(r)$ and $C_2(r)$, defined in (68) and (77) respectively (in fact $C(r) = C_2(r)$ for $r \geq 4.8$). Since the mean value of $1/\big(2h(t)^2\big)$ over T is evidently an increasing function of both c and e (the latter is clear if two triangles with different values of e are drawn on the same hypotenuse), we minimize this mean by taking equality in (89) and (90).

The integral of $1/\big(2h(t)^2\big)$ over T is the volume of the tetrahedron $S_{\alpha,\beta}$, with three dihedral right angles and three other dihedral angles (Figure 8), where α and β are the acute angles given by $\cos\alpha = \big(C(r)e_0\big)/\big(C(r)c_0\big) = e_0/c_0$, $\cos\beta = C(r)e_0/\operatorname{sech} s$. This volume is given by

$$\operatorname{vol}(S_{\alpha,\beta}) = \tfrac{1}{4}\big[\Pi(\alpha+\beta) + \Pi(\alpha-\beta) + 2\Pi\big(\tfrac{1}{2}\pi - \alpha\big)\big]$$

(see e.g. [Ra, Section 10.4]) where $\Pi(\theta)$ is the Lobachevsky function defined by

$$\Pi(\theta) = -\int_0^\theta \log|2\sin(t)|\,\mathrm{d}t. \tag{91}$$

Set $\beta_0 = 2\alpha$. We have

$$\cos\alpha = \frac{\sqrt{\xi(4\xi-1)}}{2\xi}, \quad \cos\beta_0 = \frac{2\xi - 1}{2\xi}.$$

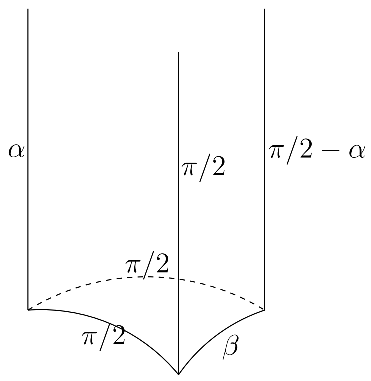

Figure 8.

Clearly $\frac{1}{6}\pi \le \alpha < \beta_0 < \beta < \frac{1}{2}\pi$, the third inequality holding because $C(r) < e^s \frac{1}{2}(\operatorname{sech} s)$. Now, since $\Pi(-\theta) = \Pi(\pi-\theta) = -\Pi(\theta)$ [Ra],

$$
\begin{aligned}
\Pi(\alpha+\beta) + \Pi(\alpha-\beta) + 2\Pi\bigl(\tfrac{1}{2}\pi-\alpha\bigr) &= \Pi\bigl(3\alpha+(\beta-\beta_0)\bigr) \\
&\quad - \Pi\bigl(\alpha+(\beta-\beta_0)\bigr) + 2\Pi\bigl(\tfrac{1}{2}\pi-\alpha\bigr) \\
&= -\Pi(\pi-3\alpha) - \Pi(\alpha) + 2\Pi\bigl(\tfrac{1}{2}\pi-\alpha\bigr) \\
&\quad - \int_{\alpha+\beta_0}^{\alpha+\beta} \log|2\sin t|\,\mathrm{d}t \\
&\quad + \int_{\alpha}^{\alpha+(\beta-\beta_0)} \log|2\sin t|\,\mathrm{d}t \qquad (92)\\
&\ge -\Pi(\pi-3\alpha) - \Pi(\alpha) \\
&\quad + 2\Pi\bigl(\tfrac{1}{2}\pi-\alpha\bigr) - (\beta-\beta_0)\log(2).
\end{aligned}
$$

We have

$$
\begin{aligned}
\beta-\beta_0 \le \frac{\cos\beta_0-\cos\beta}{\sin\beta_0} &= \frac{2\xi-1}{\sqrt{4\xi-1}}\bigl(1-(1+e^{-2s})C(r)\bigr) \\
&\le \frac{2\xi-1}{\sqrt{4\xi-1}}\bigl(1-C(r)\bigr),
\end{aligned}
\tag{93}
$$

whence

$$
\begin{aligned}
\operatorname{vol}(S_{\alpha,\beta}) \ge \frac{1}{4}&\bigl[-\Pi(\pi-3\alpha) - \Pi(\alpha) + 2\Pi\bigl(\tfrac{1}{2}\pi-\alpha\bigr)\bigr] \\
&- \frac{1}{4}\log(2)\frac{2\xi-1}{\sqrt{4\xi-1}}\bigl(1-C(r)\bigr).
\end{aligned}
\tag{94}
$$

A change of variable gives

$$\Pi(\theta) = -\int_0^{\sin\theta} \frac{\log(2t)}{\sqrt{1-t^2}}\,\mathrm{d}t$$

for $\theta \in \left[0, \frac{1}{2}\pi\right]$.

Define

$$\begin{aligned} I(\xi) &= 4\mathrm{vol}\,(S_{\alpha,\beta_0}) \\ &= \int_0^{\sin 3\alpha} \frac{\log(2t)}{\sqrt{1-t^2}}\,\mathrm{d}t + \int_0^{\sin\alpha} \frac{\log(2t)}{\sqrt{1-t^2}}\,\mathrm{d}t \\ &\quad - 2\int_0^{\cos\alpha} \frac{\log(2t)}{\sqrt{1-t^2}}\,\mathrm{d}t. \end{aligned} \tag{95}$$

Recall from Section 6 that we have used V to denote the volume of the regular ideal tetrahedron, and that $\varrho_\infty = \sqrt{3}/(2V)$. Since this tetrahedron can be subdivided into 24 copies of $S_{\pi/6,\pi/3}$ [M], we have, in particular,

$$I(1) = V/6 = \sqrt{3}/(12\varrho_\infty).$$

Since we are taking equality to hold in (89) and (90), the area of the triangle in Figure 8 is

$$\frac{1}{2}e(c^2-e^2)^{1/2} = \frac{1}{2}C(r)^2 e_0(c_0^2-e_0^2)^{1/2} = \frac{e^{-2r}(2\xi-1)^3}{2\xi\sqrt{4\xi-1}}\,C(r)^2, \tag{96}$$

whence

$$\begin{aligned} \varrho^{-1} &\geq e^{2r}C(r)^{-2}\delta(r)^{-2}\mathrm{cosech}^2 r(1-\mathrm{sech}\,r)^2 \\ &\quad \times \left(I(\xi) - \log(2)\frac{2\xi-1}{\sqrt{4\xi-1}}\,(1-C(r))\right)\left(\frac{\xi\sqrt{4\xi-1}}{(2\xi-1)^3}\right). \end{aligned} \tag{97}$$

Differentiating the above function with respect to ξ, gives a positive constant (with respect to ξ) multiple of

$$\begin{aligned} &(2\xi-1)^{-4}(4\xi-1)^{-1}\Big[\big[(4\xi-1)^{3/2}I'(\xi) - 4\xi\big(1-C(r)\big)\log(2)\big](2\xi-1)\xi \\ &\qquad + \big[(4\xi-1)^{1/2}I(\xi) - (2\xi-1)\big(1-C(r)\big)\log(2)\big](-12\xi^2-2\xi+1)\Big], \end{aligned} \tag{98}$$

where in turn

$$I'(\xi) = \frac{1}{4\xi^{5/2}}\left[3(\xi-1)\frac{\log(2\sin 3\alpha)}{\cos 3\alpha} - \xi\frac{\log(2\sin\alpha)}{\cos\alpha} - \frac{2\xi}{\sqrt{4\xi-1}}\,\frac{\log(2\cos\alpha)}{\sin\alpha}\right].$$

Calculation shows that the derivative (98) is negative throughout $[\varrho_\infty, 1]$ when $C(r) = 0.7$, and hence when $C(r) \geq 0.7$. Further calculation shows that this holds when $r \geq 4.8$.

Hence an upper bound for ϱ may be obtained by setting $\xi = 1$ in (97), giving

$$\varrho \leq \frac{C(r)^2 \delta(r)^2 (1 - e^{-2r})^2 \varrho_\infty}{(1 - \operatorname{sech} r)^2 \big(1 - 4\varrho_\infty \log(2)\big(1 - C(r)\big)\big)}.$$

Finally, more calculations show that

$$\varrho \leq (1 + 23e^{-r})\varrho_\infty.$$

In view of (85), Theorem 4.1 follows. □

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Received 13 March 2002