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# LINEAR DILATATION AND ABSOLUTE CONTINUITY

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**Abstract.** We show that already the local integrability of the linear dilatation of a homeomorphism guarantees that the homeomorphism is absolutely continuous on almost all lines parallel to the coordinate axes, under the assumption that the linear dilatation be finite outside a set of $\sigma$-finite $(n-1)$-measure.

## 1. Background and statement of results

Let $\Omega$ be a domain in $\mathbf{R}^n$, $n \geq 2$, and $f\colon \Omega \to \Omega' \subset \mathbf{R}^n$ a homeomorphism. For $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$ we set

$$L_f(x,r) = \max\{|f(x) - f(y)| : |x-y| \leq r\},$$
$$l_f(x,r) = \min\{|f(x) - f(y)| : |x-y| \geq r\}.$$

The linear lim sup-dilatation of $f$ at $x$ is defined as

$$H_f(x) = \limsup_{r \to 0} H_f(x,r)$$

where $H_f(x,r) = L_f(x,r)/l_f(x,r)$. Similarly we can define the linear lim inf-dilatation $h_f(x)$ by replacing lim sup with lim inf.

A well-known result of Gehring [2], [3] says that if a homeomorphism $f$ has its linear lim sup-dilatation $H_f(x)$ uniformly bounded a.e. in $\Omega \setminus E$, where $E$ has $\sigma$-finite $(n-1)$-measure, then $f$ is a quasiconformal mapping. Similarly it has been showed in [6] that if the linear lim inf-dilatation $h_f(x)$ is uniformly bounded a.e. in $\Omega \setminus E$, then $f$ is still quasiconformal. For earlier results, see [4], [5]. In particular, in both cases $f$ is ACL in $\Omega$, which means that $f$ is absolutely continuous on almost every line segment parallel to the coordinate axes in $\Omega$. In [9], Tukia conjectured that the condition $m_2\big(\{H_f(x) > K\}\big) < CK^{-\alpha}$, for some $\alpha > 3$

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and $K \geq K_0$, is sufficient for the ACL-property of a plane homeomorphism $f$. This was proven in [1] together with a space analogue. Furthermore, it was later showed in [7] that $H_f \in L^{\alpha}_{\mathrm{loc}}(\Omega)$, $\alpha > n/(n-1)$, guarantees the ACL-property. In the above results, it was also assumed that $H_f$ be finite outside a set of $\sigma$-finite $(n-1)$-measure, which seems crucial, see Remark 1.2(b).

In this paper we will show that already $H_f \in L^1_{\mathrm{loc}}(\Omega)$ is sufficient for the ACL-property. Before stating our results, let us introduce the following dilatations:

$$K_f(x) = \limsup_{r \to 0} \left( \frac{\operatorname{diam}\big(f\big(B(x,r)\big)\big)^n}{|fB(x,r)|} \right)^{1/(n-1)},$$

and

$$k_f(x) = \liminf_{r \to 0} \left( \frac{\operatorname{diam}\big(f\big(B(x,r)\big)\big)^n}{|fB(x,r)|} \right)^{1/(n-1)},$$

where $|A|$ denotes the Lebesgue measure of a set $A$. We noticed during our studies that these are more natural (and so more useful) for proving absolute continuity. At the points of differentiability with $l\big(Df(x)\big) = \min\{|Df(x)e| : |e| = 1\} > 0$ these dilatations are comparable to $H_f$ and $h_f$, respectively. It is also easy to see that $K_f$ and $k_f$ are Borel functions.

Our main theorem is the following result.

**Theorem 1.1.** *Let $f\colon \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbf{R}^n$ are domains, be a homeomorphism for which $k_f(x) < \infty$ outside a set $S$ of $\sigma$-finite $(n-1)$-measure, and suppose that $k_f \in L^1_{\mathrm{loc}}(\Omega)$. Then $f$ belongs to $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$.*

The above theorem gives us the ACL-property, since a continuous $W^{1,1}_{\mathrm{loc}}$-mapping is ACL (see Proposition I.1.2 in [8]).

**Remarks 1.2.** (a) To see that Theorem 1.1 is sharp, consider the mapping $f\colon ]0,(1/e)[\times\mathbf{R}^{n-1} \to f(]0,(1/e)[\times\mathbf{R}^{n-1})$,

$$f(x) = \big(1/\log(1/x_1), x_1 \sin(1/x_1) + x_2, x_3, \ldots, x_n\big).$$

This mapping is a non-ACL homeomorphism of $\mathbf{R}^n$ which satisfies $k_f(x) \in L^s_{\mathrm{loc}}$ for any $s < 1$.

(b) The condition of $\sigma$-finiteness of $S$ is crucial. For example, if $g\colon [0,1] \to [0,1]$ is the Cantor staircase function, then $f\colon ]0,1[\times]0,1[ \to ]0,2[\times]0,1[$ defined by $f(x,y) = \big(g(x)+x, y\big)$ is a homeomorphism with $k_f(z) = 1$ almost everywhere, but $f$ is not ACL.

The following corollary summarizes the conclusions obtained from Theorem 1.1 for various distortion functions.

**Corollary 1.3.** *Let $f\colon \Omega \to \Omega'$, where $\Omega, \Omega' \subset \mathbf{R}^n$ are domains, be a homeomorphism and suppose that $S$ has $\sigma$-finite $(n-1)$-measure. Now each of the conditions below guarantees that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$.*

(1) *$K_f(x) < \infty$ outside $S$ and $K_f \in L^1_{\mathrm{loc}}(\Omega)$,*

(2) *$k_f(x) < \infty$ outside $S$ and $k_f \in L^1_{\mathrm{loc}}(\Omega)$,*

(3) *$H_f(x) < \infty$ outside $S$ and $H_f \in L^1_{\mathrm{loc}}(\Omega)$,*

(4) *$h_f(x) < \infty$ outside $S$ and $h_f \in L^{n/(n-1)}_{\mathrm{loc}}(\Omega)$.*

**Remarks 1.4.** The mapping discussed in the first part of Remarks 1.2 shows the sharpness of the integrability assumptions in (1), (2), (3). Regarding (4), we do not know if already $h_f \in L^1_{\mathrm{loc}}$ could be sufficient. This would follow if it were true that the requirement $h_f(x) < \infty$ outside $S$ and $h_f \in L^1_{\mathrm{loc}}(\Omega)$ would guarantee a.e. differentiability. We do not know if this could be the case. In any case, (4) already substantially improves on the known results from [6].

## 2. Proofs

*Proof of Theorem* 1.1. To prove that $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ we first show that $f$ is ACL. After that we show the local integrability of the partial derivatives, whose existence is guaranteed by the ACL-property.

Pick a closed cube $Q \Subset \Omega$ whose sides are parallel to the coordinate axes. Assume that $Q = Q_0 \times J_0$, where $Q_0$ is $(n-1)$-interval in $\mathbf{R}^{n-1}$, and $J_0 = [a, b] \subset \mathbf{R}$. In order to show that $f$ is ACL it suffices to show that $f$ is absolutely continuous on almost every line segment in $Q$, parallel to the coordinate axes, and by symmetry it is sufficient to consider segments parallel to the $x_n$-axis.

Next, for a Borel set $A \subset Q_0$, write

$$\Phi(A) := |f(A \times [a-d, b+d])| \le |f(Q+d)| < \infty,$$

where $d = \frac{1}{10}\operatorname{dist}(Q, \partial\Omega)$ and $Q + d = \{x \in \Omega : \operatorname{dist}(x, Q) \le d\}$. Then $\Phi$ is a finite Borel measure on $Q_0$, and hence it has a finite derivative $\Phi'(y)$ for almost all $y \in Q_0$. Denote by $E_0$ the set where $\Phi'$ does not exists or is not finite.

Next we consider the set $\mathscr{A} = \{I \subset J_0 : I$ is a finite union of closed intervals, whose interiors are mutually disjoint and whose end points are rational$\}$. This set is countable: (i) If we take all the intervals whose end points are rational, there is just a countable number of intervals. (ii) If we take all the pairs of intervals, whose endpoints are rational, there is just a countable number of pairs. (iii) If we take all triples of the same type we again have a countable numbers of triples, etc. Thus $\mathscr{A}$ is a countable union of countable sets and so countable.

Now, for almost every $y \in Q_0$, we know by the Fubini theorem that

$$\int_{\{y\}\times[a-d,b+d]} k_f(z)\, dz_n < \infty.$$

Denote the set where the above fails by $E_1$. Let us define for every $I \subset \mathscr{A}$ a function $g_I\colon Q_0 \to \mathbf{R}$,

$$g_I(y) = \int_{\{y\}\times I} k_f(z)\, dz_n.$$

By the Fubini theorem, $g_I \in L^1(Q_0)$, and thus for almost every $y \in Q_0$

$$\lim_{r\to 0} \fint_{B^{n-1}(y,r)} g_I(z)\, dz = g_I(y).$$

Denote by $E_I$ the set where this is not true. Now $E = E_0 \cup E_1 \cup \left(\bigcup_{I\in\mathscr{A}} E_I\right)$ has measure zero, because it is a countable union of sets of measure zero.

Fix $y \in Q_0 \setminus E$. We will prove that $f$ is absolutely continuous on the segment $\{y\} \times J_0$ which will prove the claim.

Let $\{I_j\}_{j=1}^l$, $I_j = [a_j, b_j]$, be a union of closed intervals on $J_0$, whose interiors are mutually disjoint, and whose endpoints are rational numbers. Since $f$ is continuous, for every $j = 1, \dots, l$ there is $\delta_j$ such that

$$|f(y,a_j) - f(x)| < \frac{|f(y,a_j) - f(y,b_j)|}{4} \quad \text{when } |(y,a_j) - x| < \delta_j$$

and

$$|f(y,b_j) - f(x)| < \frac{|f(y,a_j) - f(y,b_j)|}{4} \quad \text{when } |(y,b_j) - x| < \delta_j.$$

Denote $\delta = \min_j\{\delta_j\}$.

Let $0 < r < \delta$ and let $\varepsilon > 0$. For each $k = 0, 1, 2, \dots$, write

$$A_k = \left\{ x \in B^{n-1}(y,r) \times \bigcup_j I_j : 2^k \le k_f(x) < 2^{k+1} \right\}.$$

Then $A_k$ is a Borel set,

$$B^{n-1}(y,r) \times \bigcup_j I_j \setminus S = \bigcup_k A_k$$

and for every $k$ there exists open $U_k$ such that $A_k \subset U_k$ and

$$|U_k| \le |A_k| + \frac{\varepsilon}{2^{2k}}.$$

Fix $k$. Now for every $x \in A_k$ there is $r_x > 0$ such that

(i) $0 < r_x < \frac{1}{10} \min\{r, d, |a_j - b_j|\}$,

(ii) $\operatorname{diam}(fB_x)^{n/(n-1)} < 2^{k+1} |fB_x|^{1/(n-1)}$, and

(iii) $B_x \subset U_k$.

Here $B_x = B(x, r_x)$.

By the Besicovitch covering theorem we find balls $\overline{B}_1, \overline{B}_2, \ldots$ from balls $\overline{B}(x, r_x)$ so that

$$B^{n-1}(y,r) \times \bigcup_j I_j \subset \bigcup_j \overline{B}_j \subset B^{n-1}(y,2r) \times [a-d, b+d]$$

and $\sum_j \chi_{\overline{B}_j}(x) \le C(n)$ for every $x \in \mathbf{R}^n$.

Let us define

$$\varrho(x) = \left( \sum_{j=1}^{l} |f(y,a_j) - f(y,b_j)| \right)^{-1} \sum_i \frac{\operatorname{diam}(fB_i)}{\operatorname{diam}(B_i)} \chi_{2B_i}(x).$$

The function $\varrho$ is a Borel function, because it is a countable sum of (simple) Borel functions. In the following we denote $G = \sum_{j=1}^{l} |f(y,a_j) - f(y,b_j)|$.

We wish to estimate the volume integral of $\varrho$. First of all

$$\int_{B^{n-1}(y,r)\times\cup_j I_j} \varrho(x)\,dx \ge G^{-1} \int_{B^{n-1}(y,r)} \int_{\cup_j I_j} \sum_{B_i \cap (\{z\}\times\cup_j I_j) \neq \emptyset} \times \frac{\operatorname{diam}(fB_i)}{\operatorname{diam}(B_i)} \chi_{2B_i}(z, x_n)\,dx_n\,dz.$$

Notice that

$$\int_{\cup_j I_j} \chi_{2B_i}(z, x_n)\,dx_n \ge \frac{\operatorname{diam}(B_i)}{2}$$

for the balls $B_i$ for which $B_i \cap \big(\{z\} \times \bigcup_j I_j\big) \neq \emptyset$. Moreover, for almost every $z \in B^{n-1}(y,r)$, the sets $fB_i$ cover the $f\big(\{z\} \times \bigcup_j I_j\big)$ up to a countable set, because $S$ has $\sigma$-finite $(n-1)$-measure (see Theorem 30.16 in [10]). Thus, since $r < \delta$, we have that

$$\sum_{B_i \cap (\{z\}\times\cup_j I_j) \neq \emptyset} \operatorname{diam}(fB_i) \ge \frac{G}{4} \tag{1}$$

for almost every $z \in B^{n-1}(y,r)$.

So

$$\int_{B^{n-1}(y,r)\times\cup_j I_j} \varrho(x)\,dx \ge C(n) r^{n-1}.$$

Next we establish an upper bound for this integral. Using the monotone convergence theorem, we obtain the estimate

$$\begin{aligned}\int_{B^{n-1}(y,r)\times\cup_j I_j} \varrho(x)\,dx &\le C(n)G^{-1}\sum_i \operatorname{diam}(fB_i)|B_i|^{1-(1/n)}\\ &\le C(n)G^{-1}\biggl(\sum_i |fB_i|\biggr)^{1/n}\\ &\quad\times\biggl(\sum_i\biggl(\frac{\operatorname{diam}(fB_i)^n}{|fB_i|}\biggr)^{1/(n-1)}|B_i|\biggr)^{(n-1)/n}.\end{aligned}$$

The last inequality is obtained by Hölder's inequality.

For the first term we have that

$$\sum_j |fB_j| \le C(n)|f(B^{n-1}(y,2r)\times[a-d,b+d])| = C(n)\Phi\bigl(B^{n-1}(y,2r)\bigr)$$

because the overlapping of the balls was bounded. The approximation of the second term is a little bit trickier. First notice that $|B_i| = (|B_i\cap A_k| + |B_i\setminus A_k|)$. Thus

$$\begin{aligned}\sum_i\biggl(\frac{\operatorname{diam}(fB_i)^n}{|fB_i|}\biggr)^{1/(n-1)}|B_i| &\le 2\sum_k\sum_{x_i\in A_k} 2^k|B_i\cap A_k| + 2\sum_k\sum_{x_i\in A_k}2^k|B_i\setminus A_k|\\ &\le \int_{B^{n-1}(y,r)\times\cup_j I_j} k_f(x)\,dx + \varepsilon.\end{aligned}$$

Here $x_i$ is the center of the ball $B_i$.

Combining the lower bound and the upper bound and remembering that $\varepsilon$ was arbitrary, we obtain the inequality

$$\begin{aligned}\sum_{j=1}^{l}|f(y,a_j)-f(y,b_j)| &\le C(n)\biggl(\frac{\Phi\bigl(B^{n-1}(y,2r)\bigr)}{(2r)^{n-1}}\biggr)^{1/n}\\ &\quad\times\biggl(\fint_{B^{n-1}(y,r)}\int_{\cup_j I_j} k_f(z,x_n)\,dx_n\,dz\biggr)^{(n-1)/n}.\end{aligned}$$

Letting finally $r$ tend to zero, we arrive at

$$\sum_{j=1}^{l}|f(y,a_j)-f(y,b_j)| \le C(n)\bigl(\Phi'(y)\bigr)^{1/n}\biggl(\int_{\{y\}\times\cup_j I_j} k_f(x)\,dx_n\biggr)^{(n-1)/n}.$$

This estimate holds for rational $a_i$, $b_i$. By continuity, it then holds for all $a_i, b_i \in \mathbf{R}$. Thus $f$ is absolutely continuous on $\{y\} \times J_0$.

To see that partial derivatives, which now exists a.e., are locally integrable, notice that for a.e. $x$ we have

$$|\partial_i f_j(x)| \le \liminf_{r \to 0} \frac{\operatorname{diam} fB(x,r)}{r},$$

when $i, j \in \{1, \dots, n\}$. Since

$$\frac{\operatorname{diam} fB(x,r)}{r} \le \left( \frac{|fB(x,r)|}{r^n} \right)^{1/n} \frac{\operatorname{diam} fB(x,r)}{|fB(x,r)|^{1/n}},$$

we conclude that for a.e. $x$ we have

$$|\partial_i f_j(x)| \le \mu_f(x)^{1/n} k_f(x)^{(n-1)/n}, \quad i, j \in \{1, \dots, n\},$$

where $\mu_f$ is the measure derivative of the measure $m(A) := |f(A)|$. Since the measure derivative and $k_f$ are locally integrable, it follows by Hölder's inequality that the partial derivatives are locally integrable. □

**Remark 2.1.** The assumption that $S$ have $\sigma$-finite $(n-1)$-measure was only used to guarantee that (1) holds. In fact, this estimate follows even if we only know that the one-dimensional measure of $f(L \cap S)$ is zero for (almost every) line $L$ parallel to the coordinate axes. Thus, instead of an exceptional set $S$ of $\sigma$-finite $(n-1)$-measure, we could allow for any exceptional set $S$ with this property.

*Proof of Corollary* 1.3. First, Part (2) is the previous theorem.

Next, Part (1) is a trivial corollary of (2).

For (3), we first notice that always, for every $x$, we have

$$K_f(x) \le C(n) H_f(x)^{n/(n-1)},$$

and thus $K_f$ is finite outside a set of $\sigma$-finite $(n-1)$-measure. Secondly, since $H_f$ is finite almost everywhere, our mapping is differentiable almost everywhere; for this see for example [7]. Now, in the set where $f$ is differentiable and

$$\min_{|e|=1} |Df(x)e| > 0,$$

it is easy to see that $K_f(x) \le C(n) H_f(x)$. Set

$$\widetilde{S} = S \cup \{x \in \Omega : f \text{ differentiable at } x \text{ with } |Df(x)| = 0\}.$$

Then the one-dimensional measure of $f(\widetilde{S} \cap L)$ is zero for each line $L$. The claim follows from Theorem 1.1 and Remark 2.1.

Part (4) follows from Theorem 1.1 and the fact that we always have $k_f \le C(n) h_f^{n/(n-1)}$. □

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