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BOUNDARY CORRESPONDENCE UNDER QUASICONFORMAL HARMONIC DIFFEOMORPHISMS OF A HALF-PLANE

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Abstract. It is proved that an orientation-preserving homeomorphism ψ of the real axis can be extended to a quasiconformal harmonic homeomorphism of the upper half-plane if and only if ψ is bi-Lipschitz and the Hilbert transformation of the derivative ψ' is bounded.

1. Introduction

A homeomorphism $f\colon D \mapsto G$, where D and G are subdomains of the complex plane $\mathbf{C}$, is said to be quasiconformal if f is absolutely continuous on a.e. horizontal and a.e. vertical line, and there exists a constant $K < \infty$ such that

$$\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2 \le K J_f \quad \text{a.e. on } D,$$

where J_f is the Jacobian of f (cf. [1, pp. 23–24]). If $D = G = \mathbf{U}$, where $\mathbf{U}$ denotes the upper half-plane,

$$\mathbf{U} = \{x + yi \in \mathbf{C} : y > 0\},$$

then f extends to a homeomorphisms of $\overline{\mathbf{U}}$ onto onself, where $\overline{\mathbf{U}}$ is the closure of $\mathbf{U}$ in $\mathbf{C} \cup \{\infty\}$.

We denote by $\mathrm{QC}(\mathbf{U})$ the group of all quasiconformal homeomorphisms of $\overline{\mathbf{U}}$ onto itself fixing the point ∞. By the famous theorem of Beurling and Ahlfors [1], the 'restriction' of $\mathrm{QC}(\mathbf{U})$ to the real axis $\mathbf{R}$ coincides with the class of all quasisymmetric functions, i.e., of those strictly increasing homeomorphisms ψ of $\mathbf{R}$ such that

$$\frac{1}{M} \le \frac{\psi(x+t) - \psi(x)}{\psi(x) - \psi(x-t)} \le M$$

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for some constant $M \geq 1$ and for all $x \in \mathbf{R}$ and $t > 0$.

In this paper we consider the classes

$$\mathrm{HQC}(\mathbf{U}) = \{f \in \mathrm{QC}(\mathbf{U}), f \text{ harmonic in } \mathbf{U}\}$$

and

$$\mathrm{HQS}(\mathbf{R}) = \{f|_{\mathbf{R}} : f \in \mathrm{HQC}(\mathbf{U})\},$$

where $f|_{\mathbf{R}}$ denotes the restriction of f to the real axis. The study of the analogous classes of functions on the unit disc $\mathbf{D}$ was begun by Martio [6]. Various interesting results and examples concerning that case can be found in Partyka and Sakan [7], [8]. Information on univalent (not necessarily quasiconformal) harmonic mappings can be read in [3].

The classes $\mathrm{HQC}(\mathbf{D})$ and $\mathrm{HQS}(\partial\mathbf{D})$ were characterized in [9]; in particular, f is in $\mathrm{HQC}(\mathbf{D})$ if and only if f is bi-Lipschitz. The same holds for the half-plane. In fact we can say somewhat more:

Theorem 1.1 ([5]). *Let f be a quasiconformal harmonic mapping of $\mathbf{U}$ into $\mathbf{U}$. Then the following assertions are equivalent.*

(a) *f is in $\mathrm{QC}(\mathbf{U})$.*
(b) *There are positive constants c and M such that $v(z) = cy$, and $1/M \leq u_x \leq M$ and $|u_y| \leq M$ for all $z \in \mathbf{U}$.*
(c) *f is a bi-Lipschitz mapping of $\mathbf{U}$ onto $\mathbf{U}$.*

It follows that if $\psi \in \mathrm{HQS}(\mathbf{R})$, then ψ is bi-Lipschitz, i.e., ψ is absolutely continuous and

$$1/C \leq \psi'(x) \leq C \quad (x \in \mathbf{R},\ \text{a.e.}), \tag{1.1}$$

for some constant C. However this condition is far from being sufficient for ψ to be in $\mathrm{HQS}(\mathbf{R})$, as the following theorem shows.

Theorem 1.2. *Let ψ be an increasing homeomorphism of $\mathbf{R}$. Then ψ belongs to $\mathrm{HQS}(\mathbf{R})$ if and only if it is bi-Lipschitz and the Hilbert transformation of ψ' belongs to $L^\infty(\mathbf{R})$.*

An analogous result holds in the case of the unit disc (see [9]).

The Hilbert transformation of $\phi \in L^\infty(\mathbf{R})$ is defined by

$$H\phi(x) = \lim_{\varepsilon \to 0} H_\varepsilon \phi(x),$$

where

$$H_\varepsilon \phi(x) = \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) \phi(t)\, dt.$$

It is known that the limit exists almost everywhere, but the function $H\phi$ need not be in L^∞ (cf. [4]).

2. Proof of Theorem 1.1.

Lemma 2.1. *If $f = u + iv: \mathbf{U} \mapsto \mathbf{U}$ is a quasiconformal mapping of class C^1 such that $v(z) \equiv cy$ for some constant $c > 0$, then f is onto and bi-Lipschitz.*

Proof. Since f is quasiconformal the following inequality holds:

$$|\nabla u(z)|^2 + |\nabla v(z)|^2 \leq K J_f(z),$$

where J_f is the Jacobi determinant of f and K is a constant independent of $z \in \mathbf{U}$. In this case this means that

$$u_x^2 + u_y^2 + c^2 \leq K u_x,$$

where u_x and u_y are the partial derivatives of u. It follows that

$$u_x \geq c^2/K \quad \text{and} \quad u_x \leq K \tag{2.1}$$

and hence that $|u_y| \leq K$. This implies that the function f satisfies a Lipschitz condition.

On the other hand, it follows from (2.1) and the hypothesis $v(z) = cy$ that f is *onto*. Since the inverse mapping is of the same form as f the above argument shows that f^{-1} satisfies a Lipschitz condition. The result follows. □

Proof of Theorem 1.1. Assuming (a) we have that v is a positive harmonic function on $\mathbf{U}$ and therefore, by the Riesz–Herglotz theorem (see [2, Theorem 7.20]), v has the form

$$v(z) = cy + \pi^{-1} \int_{-\infty}^{+\infty} P(z,t)\, d\mu(t),$$

where c is a non-negative constant, μ is a non-decreasing function on $\mathbf{R}$, and P is the Poisson kernel,

$$P(z,t) = \frac{y}{|z-t|^2} \qquad (z = x + iy \in \mathbf{U},\ t \in \mathbf{R}).$$

Therefore

$$\begin{aligned} v(z) &\geq cy + \pi^{-1} \int_x^{x+y} P(z,t)\, d\mu(t) \\ &\geq cy + \pi^{-1} \int_x^{x+y} \frac{y}{2y^2}\, d\mu(t) \\ &= cy + \pi^{-1} \frac{\mu(x+y) - \mu(x)}{y} \geq 0. \end{aligned} \tag{2.2}$$

On the other hand, since f is quasiconformal it is continuous up to the boundary and in particular $v(x,y)\to 0$ as $y\to 0$ for any fixed $x\in\mathbf{R}$. From this and (2.2) it follows that the right derivative of μ vanishes everywhere. That the left derivative vanishes everywhere can be proved in a similar way. Hence μ is constant, and this proves that $v(z)=cy$ for some $c>0$. Now Lemma 2.1 and its proof conclude the proof that (a) implies (b).

That (a) implies (c) follows from Lemma 2.1 and the implication '(a) implies (b)'. That (b) implies (a) follows from the definition of quasiconformality. Finally, it is well known that that (c) implies (a). This completes the proof of Theorem 1.1. □

3. A representation of HQC(U)

For a harmonic mapping $f=u+iv$ defined on $\mathbf{U}$ let

$$f(i)=b+ic,\quad \varphi(z)=\partial u(z):=\frac{\partial u}{\partial z}=\frac{1}{2}(u_x-iu_y). \tag{3.1}$$

Since the function φ is holomorphic and

$$u(z)-u(i)=2\operatorname{Re}\int_i^z\varphi(\zeta)\,d\zeta,$$

we have the following reformulation of Theorem 1.1.

Theorem 3.1. *Each $f\in\mathrm{HQC}(\mathbf{U})$ has a unique representation of the form*

$$f(z)=2\operatorname{Re}\int_i^z\varphi(\zeta)\,d\zeta+b+ic\operatorname{Im}(z), \tag{3.2}$$

where

(i) *$b+ic$ is a point in $\mathbf{U}$,*

(ii) *φ is a holomorphic function on $\mathbf{U}$ such that $\varphi(\mathbf{U})$ is a relatively compact subset of the right half-plane $\mathbf{H}$.*

Conversely, if (i) and (ii) are satisfied, then the function f defined by (3.2) belongs to HCQ($\mathbf{U}$).

4. Proof of Theorem 1.2

Let U be a real-valued function harmonic in $\mathbf{U}$. Then there exists a unique harmonic function V, called the harmonic conjugate of U, such that $V(i)=0$ and that the function $U+iV$ is analytic in $\mathbf{U}$.

Let U be the Poisson integral of $\phi\in L^\infty(\mathbf{R})$, i.e., the harmonic function on $\mathbf{U}$ defined by

$$U(z)=\frac{1}{\pi}\int_{-\infty}^{+\infty}P(z,t)\phi(t)\,dt. \tag{4.1}$$

Then the harmonic conjugate of U is given by

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\Big(Q(z,t) + \frac{t}{t^2+1}\Big)\phi(t)\,dt, \tag{4.2}$$

where

$$Q(z,t) = \frac{x-t}{|z-t|^2}$$

is the conjugate Poisson kernel. We have $F = U + iV$, where F is the analytic function defined by

$$F(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty}\Big(\frac{i}{z-t} + \frac{t}{t^2+1}\Big)\phi(t)\,dt.$$

The Hilbert transformation of ϕ and the harmonic conjugate of U are connected by the formulae

$$\lim_{y\to 0}\big(H_y\phi(x) - V(x+iy)\big) = 0 \tag{4.3}$$

and

$$V(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} P(z,t)H(\phi)(t)\,dt. \tag{4.4}$$

It is a simple but important fact that if a function u is harmonic in $\mathbf{U}$, then so is $u_x := \partial u/\partial x$, and the harmonic conjugate of u_x is equal to $u_y(i) - u_y$.

All the above facts can be found in Garnett [4]. 'Only if' part of Theorem 1.2 is a consequence of the following two lemmas.

Lemma 4.1. *If $f \in \mathrm{HQC}(\mathbf{U})$, then the restriction ψ of f to the real axis is bi-Lipschitz and we have the relations*

$$u_x(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} P(z,t)\psi'(t)\,dt \tag{4.5}$$

and

$$u_y(z) - u_y(i) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\Big(Q(z,t) + \frac{t}{t^2+1}\Big)\psi'(t)\,dt. \tag{4.6}$$

Proof. The function ψ is bi-Lipschitz on $\mathbf{R}$ because the mapping f is bi-Lipschitz on $\mathbf{U}$. The function u_x is bounded on $\mathbf{U}$ and therefore, by Fatou's theorem, there exists the limit

$$\lim_{y\to 0} u_x(x,y) = \phi(x)$$

for almost all $x \in \mathbf{R}$. Furthermore, we have

$$u_x(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} P(z,t)\phi(t)\,dt.$$

Thus in order to prove (4.5) we have to prove that $\psi'(x) = \phi(x)$ almost everywhere. We start from the relation

$$u(x,y) - u(0,y) = \int_0^x u_t(t,y)\,dt.$$

Since $u_t(t,y)$ is bounded we have

$$\lim_{y\to 0}\int_0^x u_t(t,y)\,dt = \int_0^x \phi(t)\,dt,$$

by the dominated convergence theorem. On the other hand,

$$\lim_{y\to 0}\big(u(x,y) - u(0,y)\big) = \psi(x) - \psi(0)$$

and therefore

$$\psi(x) - \psi(0) = \int_0^x \phi(t)\,dt,$$

which proves (4.5).

The validity of (4.6) now follows from (4.2) and the fact that the function $V = -\big(u_y - u_y(i)\big)$ is equal to the harmonic conjugate of $U = u_x$. ▫

Lemma 4.2. *If $f = u + iv \in \mathrm{HQC}(\mathbf{R})$, and $\psi = f|_{\mathbf{R}}$, then the function $H(\psi')$ belongs to $L^\infty(\mathbf{R})$ and the following equality holds*:

$$u_y(z) - u_y(i) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} P(z,t)H(\psi')(t)\,dt. \tag{4.7}$$

Proof. By Theorem 1.1 the function $V(z) = u_y(i) - u_y(z)$ is bounded on $\mathbf{U}$. Since V is the harmonic conjugate of $U = u_x$, we can use (4.5) and (4.3) ($\phi = \psi'$) to conclude that $H(\psi')$ is in L^∞. Then formula (4.7) follows from (4.4). ▫

To prove the 'if' part of Theorem 1.2, assume that ψ is bi-Lipschitz, $|H(\psi')| \le M$=const. a.e. on $\mathbf{R}$, and define U by (4.1), where $\phi = \psi'$. Let V be the harmonic conjugate of U, let $\varphi(z) = \frac{1}{2}\big(U(z) + iV(z)\big)$ and define f by

$$f(z) = 2\,\mathrm{Re}\int_i^z \varphi(\zeta)\,d\zeta + i\,\mathrm{Im}(z).$$

From the inequality $1/C \le \psi' \le C$ and (4.1) it follows that $1/C \le U(z) \le C$ ($z \in \mathbf{U}$). Since $|V(z)| \le M$, by (4.4), we see that the function φ maps $\mathbf{U}$ onto a relatively compact subset of $\mathbf{H}$. From Theorem 3.1 it follows that $f \in \mathrm{HQC}(\mathbf{U})$. Then by Lemma 4.1, the restriction $\tilde\psi$ of F to $\mathbf{R}$ is a bi-Lipschitz function and, since $(\mathrm{Re}\, f)_x = U$ on $\mathbf{U}$, $\tilde\psi' = \psi'$ a.e. on $\mathbf{R}$. Since ψ and $\tilde\psi$ are absolutely continuous, $\psi = \tilde\psi + a$ for some $a \in \mathbf{R}$. Thus defining $\tilde f = f + a$ we see that $\psi = \tilde f|_{\mathbf{R}} \in \mathrm{HQC}(\mathbf{R})$, which completes the proof.

5. A question

The set $\mathrm{QS}(\mathbf{R})$ is a group with respect to composition. Is this true for $\mathrm{HQS}(\mathbf{R})$?

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