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DIFFERENTIABLE RIGIDITY AND SMOOTH CONJUGACY

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Abstract. We started a study of regularity of the conjugacy between two dynamical systems with singularities. We complete this study for quasi-hyperbolic maps. We prove that if the conjugacy between two quasi-hyperbolic maps is differentiable at one point with uniform bound, then it is piecewise smooth. Furthermore, if the exponents of these two maps at all power law critical points are also the same, then the conjugacy is piecewise diffeomorphic. The degree of the smoothness of the conjugacy can be also calculated.

1. Introduction

A remarkable result in geometry is the so-called Mostow's rigidity theorem. This result assures that two closed hyperbolic 3-manifolds are isometrically equivalent if they are homeomorphically equivalent [Mo]. A closed hyperbolic 3-manifold can be viewed as the quotient space of a Kleinian group acting on the upper-half 3-space. So a homeomorphic equivalence between two closed hyperbolic 3-manifolds can be lifted to a homeomorphism of the upper-half 3-space preserving group actions. The homeomorphism can be extended to the boundary of the upper-half 3-space as a boundary map. The boundary is the complex plane and the boundary map is a quasi-conformal homeomorphism. A quasi-conformal homeomorphism of the complex plane is absolutely continuous. Following this property plus the group action is that the boundary map has no invariant line field. Thus it is a Möbius transformation.

For closed hyperbolic Riemann surfaces, the situation is quite complicated. A closed hyperbolic Riemann surface can be viewed as a Fuchsian group acting on the upper-half plane too. A homeomorphic equivalence between two closed hyperbolic Riemann surfaces can be also lifted to a homeomorphism of the upper-half plane preserving group actions. This homeomorphism can be extended to the

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boundary of the upper-half plane as a boundary map. In this case, the boundary is the real line and boundary map is a quasisymmetric homeomorphism. The complication comes from the fact that a quasisymmetric homeomorphism may not be absolutely continuous. However, this complication is a rich supply for Teichmüller theory. Actually if the boundary map is absolutely continuous, then following the idea in the proof of Mostow's rigidity theorem, it is a Möbius transformation. Furthermore, Tukia [T] proved that the boundary map is, in general, uniformly complicated at every point. Actually if the boundary map is smooth at one point, then it is absolutely continuous, therefore, it is a Möbius transformation.

Shub and Sullivan [SS] introduced this kind of study for the conjugacy between two smooth expanding circle maps. The conjugacy in this case is always quasisymmetric (refer to [J4, Chapter 3]). They proved that if the conjugacy is absolutely continuous then it is smooth. The degree of the smoothness is the same as the degree of the smoothness for both maps.

Following Shub and Sullivan, we started a similar study for dynamical systems with singularies. In the paper [J3], we investigated the conjugacy between two generalized Ulam–von Neumann transformations. A generalized Ulam–von Neumann transformation is a certain interval map with one power law type singularity (see Section 2 for more details). We proved that the conjugacy between two generalized Ulam–von Neumann transformations is smooth if they have the same type power law singularity and their asymmetries are the same and their eigenvalues at all corresponding periodic points are the same. Later, the asymmetry is removed from the result in [J5], [J6]. Moreover, in [J5], [J6], we studied the smoothness of the conjugacy between two geometrically finite one-dimensional maps. A geometrically finite one-dimensional map is a certain one-dimensional map with finitely many power law singularities and with certain Markovian property (see Section 2 for more details). In our previous paper [J1], we have proved that the conjugacy between two geometrically finite one-dimensional maps is quasisymmetric always. In [J5], [J6], we defined a smooth invariant called the scaling function for a geometrically finite one-dimensional map. We further showed that the scaling function and the exponents of power law singularities are complete smooth invariants. That is, the conjugacy between two geometrically finite one-dimensional maps are smooth if and only if they have the same scaling function and the exponents of the corresponding singularities are the same.

In this paper, we continue our study and make a more complete theory for quasi-hyperbolic maps which is a certain one-dimensional map with finitely many non-recurrent critical points (see Definition 2). We first study one point differentiable rigidity problem in dynamical systems. The problem concerns how the differentiability of the conjugacy between two dynamical systems at one point affects the global differentiability of the conjugacy. Our first theorem is a result similar to Tukia's results [T] for Fuchsian groups. This result (Theorem 1) says that if the conjugacy between two quasi-hyperbolic maps is differentiable at one point with uniform bound then it is piecewise C^1. The exceptional points are those in the closure of post singular orbits. The asymptotic behavior of the derivative of the conjugacy near those points is taken care of in the second result. Our second theorem (Theorem 2) says that, in addition to the assumption in our first theorem, if exponents at all corresponding singular points are the same, then the conjugacy is a piecewise $C^{1+\beta}$ diffeomorphism. Actually, from the proof of our second theorem, we know exactly how exponents at singular points affect the asymptotic behavior of the derivative of the conjugacy at those points. In addition, if both quasi-hyperbolic maps satisfy a certain α-Hölder condition, $0 < \alpha \leq 1$, (see Definition 1 for more precise meaning about this) and if γ is the maximum number among all the exponents at all singularities for both maps, then $\beta = \alpha/\gamma$ in our second theorem.

Following our first and second theorems, we could get more results in the study of the rigidity problem in dynamical systems. For example, we show two stronger conditions than that in Theorem 1. The first condition is given in our third result (Lemma 6). We prove that if the conjugacy is differentiable at one point with non-zero derivative and if there is a neighborhood about this point such that the absolute value of the eigenvalue at every periodic point in this neighborhood is the same as that at the corresponding periodic point, then the conjugacy is differentiable at this point with uniform bound. This result combined with our first and second theorems gives us that if the conjugacy is differentiable at one point with non-zero derivative and if the absolute value of the eigenvalue at every periodic point in a small neighborhood of this point is the same as that at the corresponding periodic point, then the conjugacy is piecewise C^1. Furthermore, if all the exponents at the corresponding singular points are also the same, then the conjugacy is a piecewise $C^{1+\beta}$ diffeomorphism for some fixed $0 < \beta \leq 1$ (Corollary 1).

The other condition is absolute continuity. We show that if the conjugacy is absolutely continuous on any small interval, then it is piecewise C^1. Furthermore, if all the exponents at the corresponding singular points are also the same, then the conjugacy is a piecewise $C^{1+\beta}$ diffeomorphism for some fixed $0 < \beta \leq 1$ (Corollary 2).

The paper is organized as follows. In Section 2, we define a quasi-hyperbolic map (Definition 2). We also provide several examples of such a map. In Section 3, we study the nonlinearity of a quasi-hyperbolic map. In this section, we prove one of our key technical results, the $C^{1+\text{Hölder}}$–Koebe–Denjoy type distortion lemma (Lemma 2) for a quasi-hyperbolic map. By using this technical result, we prove in Section 4 our main results (Theorems 1 and 2). In the same section, we show two stronger conditions than those in Theorem 1 and prove two more rigidity results (Lemma 6, Corollary 1, and Corollary 2).

2. Quasi-hyperbolic one-dimensional maps

Let M be the interval $[0,1]$ or the unit circle $\mathbf{R}/\mathbf{Z}$. Let $f\colon M \to M$ be a piecewise C^1 map. A point $c \in M$ is said to be *singular* if either $f'(c)$ does not exist or $f'(c)$ exists but $f'(c) = 0$. A singular point c is said to be *power law* if there is an interval $(c-\tau_c, c+\tau_c)$, $\tau_c > 0$, such that the restrictions of f to $(c-\tau_c, c)$ and to $(c, c+\tau_c)$ are C^1 and such that there is a real number $\gamma \geq 1$ such that the limits

$$\lim_{x\to c-} \frac{f'(x)}{|x-c|^{\gamma-1}} = B^- \quad \text{and} \quad \lim_{x\to c+} \frac{f'(x)}{|x-c|^{\gamma-1}} = B^+$$

exist and are non-zero. The number γ is called the exponent at c.

For a power law singular point c, let

$$r_{c,-}(x) = \frac{f'(x)}{|x-c|^{\gamma-1}}, \quad x \in (c-\tau_c, c)$$

and

$$r_{c,+}(x) = \frac{f'(x)}{|x-c|^{\gamma-1}}, \quad x \in (c, c+\tau_c).$$

A singular point c is called critical if $\gamma > 1$.

Let SP denote the set of all singular points and let CP denote the set of all critical points. Let $\mathrm{PSO} = \bigcup_{i=1}^{\infty} f^i(\mathrm{SP})$ be the set of post-singular orbits.

Remark 1. The exponent is C^1-invariant meaning that if f and g are C^1 conjugated maps (i.e., there is a C^1 diffeomorphism h such that $h\circ f = g\circ h$), then the exponents of f and g are the same at corresponding power law critical points.

Remark 2. In general, we can define the left and right exponents at a singular point as follows. A singular point c is said to be *power law* if there is an interval $(c-\tau_c, c+\tau_c)$, $\tau_c > 0$, such that $f|(c-\tau_c, c)$ and $f|(c, c+\tau_c)$ are C^1 and there are two real numbers $\gamma^- \geq 1$ and $\gamma^+ \geq 1$ such that the limits

$$\lim_{x\to c-} \frac{f'(x)}{|x-c|^{\gamma^- -1}} = B^- \quad \text{and} \quad \lim_{x\to c+} \frac{f'(x)}{|x-c|^{\gamma^+ -1}} = B^+$$

exist and are non-zero. The numbers γ^- and γ^+ are called the left and right exponents of f at c. For a power law critical point c, let

$$r_{c,-}(x) = \frac{f'(x)}{|x-c|^{\gamma^- -1}}, \quad x \in (c-\tau_c, c)$$

and

$$r_{c,+}(x) = \frac{f'(x)}{|x-c|^{\gamma^+ -1}}, \quad x \in (c, c+\tau_c).$$

The left and right exponents are orientation-preserving C^1-invariant, meaning that if h is an orientation-preserving C^1 diffeomorphism such that

$$h \circ f = g \circ h,$$

then the left and right exponents of f and g are the same at corresponding power law critical points. Under this definition of a power law singular point, our main results (Theorems 1 and 2) will be true too if we replace the conjugacy by the orientation-preserving conjugacy.

Our f in this paper always satisfies that

(1) SP is finite (could be empty) and
(2) every singular point in SP is of power law type.

Let $\tau > 0$ be a real number. For every critical point $c \in \mathrm{CP}$, let $U_c = [c-\tau, c+\tau]$. Define

$$U(\tau) = \bigcup_{c \in CP} U_c \quad \text{and} \quad V(\tau) = \overline{M \setminus U\left(\tfrac{1}{2}\tau\right)}.$$

Denote

$$U_{c,-} = U_c \cap (c-\tau, c) \quad \text{and} \quad U_{c,+} = U_c \cap (c, c+\tau).$$

Definition 1. We call f in this paper $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if there is $\tau > 0$ such that

(i) f on every component of $M \setminus SP$ is C^1 and the derivative is α-Hölder continuous, and
(ii) every $r_{c,\pm} \mid U_{c,\pm}$ is α-Hölder continuous.

A sequence of intervals $\{I_i\}_{i=0}^n$ is called a chain (with respect to f) if

(a) $I_i \subset M \setminus SP$ for all $0 \leq i \leq n$;
(b) $f\colon I_i \to I_{i+1}$ is a C^1-diffeomorphism for every $0 \leq i \leq n-1$; and
(c) either $I_i \subseteq V(\tau)$ for all $0 \leq i \leq n-1$ or the last interval $I_n \subseteq U(\tau)$ (but in the latter case, some I_i, $0 < i < n-1$, may not be contained in $U(\tau)$ or $V(\tau)$).

Definition 2. We call f quasi-hyperbolic if

(1) f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$;
(2) $\overline{\mathrm{PSO}} \cap U(\tau) = \emptyset$ for some number $\tau > 0$; and
(3) there are two constants $C > 0$ and $0 < \mu < 1$ such that for any chain $\{I_i\}_{i=0}^n$, $|I_0| \leq C\mu^n |I_n|$.

Let $\tau > 0$ be a fixed small number in the rest of this paper.

We first show several examples of f which are quasi-hyperbolic. A point $p \in M$ is called *periodic* of period k if $f^i(p) \neq p$ for all $0 < i < k$ but $f^k(p) = p$. When $k = 1$, we also call it fixed. For a periodic point p of period k, $e_p = (f^k)'(p)$

is called the *eigenvalue* of f at p. Then p is called attractive if $|e_p| < 1$; parabolic if $|e_p| = 1$; expanding if $|e_p| > 1$. The first example comes from a theorem (see [MS, Theorem 6.3, pp. 261–262]). A critical point c of a C^2 map is called non-degenerate if $f'(c) = 0$ and $f''(c) \neq 0$.

Example 1. A C^2 map f with only non-degenerate critical points such that PSO and SP are both finite and $\mathrm{PSO} \cap \mathrm{SP} = \emptyset$ and all periodic points are expanding.

The Schwarzian derivative of a C^3 map h is defined as

$$S(h) = \frac{h'''}{h'} - \frac{3}{2}\left(\frac{h''}{h'}\right)^2.$$

We say that h has negative Schwarzian derivative if $S(h)(x) < 0$ for all x. Singer (see [Si]) proved that if f is C^3 and has negative Schwarzian derivative, then the immediate basin of every attractive or parabolic periodic orbit contains at least one critical orbit. Therefore, if f has negative Schwarzian derivative and if $\mathrm{PSO} \cap \mathrm{SP} = \emptyset$ and if $\overline{\mathrm{PSO}}$ contains neither attractive nor parabolic periodic points, then all periodic points of f are expanding. The map f is said to be preperiodic if for every singular point c, $f^m(c)$ is an expanding periodic point for some integer $m \geq 1$. Then $\overline{\mathrm{PSO}} = \mathrm{PSO}$ contains neither attractive nor parabolic periodic points. A special case of Example 1 is that

Example 2. A preperiodic C^3 map f having negative Schwarzian derivative.

Let $\mathrm{SO} = \bigcup_{n=0}^{\infty} f^n(\mathrm{SP})$ be the union of singular orbits of f. If SO is non-empty and finite, let $\eta_1 = \{I_0, \ldots, I_{k-1}\}$ be the set of the closures of intervals in $M \setminus \mathrm{SO}$, then (f, η_1) has Markovian property. That means that

(i) $I_0, \ldots, I_{k-1}$ have pairwise disjoint interiors,
(ii) the union $\bigcup_{i=0}^{k-1} I_i$ of all intervals in η_1 is M,
(iii) the restriction $f\colon I \to f(I)$ for every interval I in η_1 is homeomorphic, and
(iv) the image $f(I)$ of every interval I in η_1 is the union of some intervals in η_1.

We call η_1 a *Markov partition*. Let $g_i = (f \mid I_i)^{-1}$ be the inverse of $f\colon I_i \to f(I_i)$ for each $I_i \in \eta_1$. A sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\cdots$, $(k-1)$'s is called *admissible* if the domain $f(I_{i_l})$ of g_{i_l} contains $I_{i_{l+1}}$ for all $0 \leq l < n-1$. For an admissible sequence $w_n = i_0 \cdots i_{n-1}$ of 0's, $\cdots$, $(k-1)$'s, we can define $g_{w_n} = g_{i_0} \circ \cdots g_{i_{n-1}}$ and $I_{w_n} = g_{w_n}\big(f(I_{i_{n-1}})\big)$. Let η_n be the set of the intervals I_{w_n} for all admissible sequences of length n. It is also a Markov partition of M with respect to f. We call it the n^{th}-partition of M induced from (f, η_1). Let κ_n be the maximum of the lengths of intervals in η_n.

Definition 3. We call f *geometrically finite* if

(i) the set of singular orbits SO is non-empty and finite,
(ii) no critical point is periodic, and
(iii) there are constants $C > 0$ and $0 < \mu < 1$ such that $\kappa_n \leq C\mu^n$ for all $n > 0$.

Our next example of a quasi-hyperbolic map is a geometrically finite map (refer to [J4, Chapter 3] for more details about this). A special class of geometrically finite maps is generalized Ulam-von Neumann transformation, which is the first class we studied in this direction.

Definition 4. Suppose $M = [-1, 1]$. We call f a *generalised Ulam–von Neumann transformation* if

(I) f is geometrically finite with only one singular point 0,
(II) $f(-1) = f(1) = -1$ and $f(0) = 1$,
(III) $f \mid [-1, 0]$ is increasing and $f \mid [0, 1]$ is decreasing.

One example of a generalized Ulam–von Neumann transformation is $f(x) = 1 - 2|x|^\gamma$ for $\gamma \geq 1$. Another one is $f(x) = -1 + 2\cos\left(\frac{1}{2}\pi x\right)$. If f is a generalized Ulam–von Neumann transformation, let $I_0 = [-1, 0]$ and $I_1 = [0, 1]$. We then have that $f(I_0) = f(I_1) = M$. Thus $\eta_0 = \{I_0, I_1\}$ is a Markov partition. The post-singular orbit $\mathrm{PSO} = \bigcup_{i=1}^{\infty} f^i(0)$ is $\{-1, 1\}$. Any two generalized Ulam–von Neumann transformations f and g are topologically conjugate by an orientation-preserving homeomorphism. The conjugacy is quasisymmetric (refer to [J1] or [J4, Chapter 3]). And according to our results (Theorems 1 and 2) in this paper, it is either totally singular or smooth on $(-1, 1)$. If the conjugacy is smooth on $(-1, 1)$ and if the exponents of f and g at 0 are the same, then the conjugacy is a diffeomorphism of $[-1, 1]$.

Our last example is a circle expanding map which was first studied by Shub–Sullivan [SS]. Let $M = \mathbf{R}/\mathbf{Z}$ be the unit circle. Then f is called a circle expanding map if there are constants $C > 0$ and $\mu > 1$ such that

$$(f^n)'(x) \geq C\mu^n, \quad x \in M,\ n \geq 1.$$

An example of an expanding circle endomorphism is $x \mapsto dx \pmod 1$, where $d > 1$ is an integer. If the topological degree of f is d, then f is topologically conjugate to $x \mapsto dx \pmod 1$. A circle expanding map is quasi-hyperbolic without any singular point. The conjugacy is quasisymmetric (refer to [J4]) and, according to results in this paper, it is either a totally singular homeomorphism or a diffeomorphism of the circle.

3. Nonlinearity of a quasi-hyperbolic map

The study of nonlinearity is extremely important in dynamical systems. In one complex variable, Koebe's distortion theorem (see [Bi]) represents a beautiful result showing how nonlinearity can be controlled for all schlicht functions (or called conformal maps) defined on the unit disk. In one-dimensional dynamics, the Denjoy distortion technique becomes an important tool to estimate the nonlinearity of a C^2 diffeomorphism of the circle (see [D]). Koebe's distortion theorem represents a kind of magic in mathematics because it shows that the nonlinearity can be controlled universally for all schlicht functions on the unit disk. By a detailed analysis of the reason behind this magic, we prove a Koebe–Denjoy type distortion lemma by combining ideas of Koebe and Denjoy. This result gives us a wonderful technique to control the nonlinearity of a quasi-hyperbolic map.

Take f a quasi-hyperbolic map. Let α, $U = U(\tau)$, and $V = V(\tau)$ be as in Definition 2. This notation is fixed in the rest of the paper. Dividing M into U and V is one of the key points in this section; the set V is away from all critical points CP and the set U is away from post-singular orbit $\mathrm{PSO} = \bigcup_{n=1}^{\infty} f^n(\mathrm{SP})$. In the set V we can use the Denjoy distortion technique to control the distortion of the iterates of f (see Lemma 1); in the set U we can prove a Koebe type distortion property (see Lemma 3) to control the distortion of the iterates of f.

A chain $\mathscr{I} = \{I_i\}_{i=0}^n$ is said to be *regulated* if either $I_i \subseteq V$ or $I_i \subseteq U$ for all $0 \le i \le n$. We use $d(\,\cdot\,,\cdot\,)$ to mean the distance between two points or two sets.

Lemma 1 (The first distortion lemma). *There is a constant $C > 0$ such that for any regulated chain $\mathscr{I} = \{I_i\}_{i=0}^n$ satisfying $I_i \subseteq V$ for all $1 \le i \le n-1$ and for all x and y in I_0,*

$$\left|\log\left(\frac{|(f^n)'(x)|}{|(f^n)'(y)|}\right)\right| \le C|x_n - y_n|^{\alpha},$$

where $x_n = f^n(x)$ and $y_n = f^n(y)$.

Proof. Let ξ be the set of intervals of $V \setminus \mathrm{SP}$. Let

$$a = \inf_{x \in V\setminus \mathrm{SP}} |f'(x)| > 0$$

and let

$$b = \sup_{x \ne y \in I,\, I \in \xi} \frac{|f'(x) - f'(y)|}{|x-y|^{\alpha}} < \infty.$$

For any x and y in I_0, let $x_i = f^i(x)$ and $y_i = f^i(y)$ for $0 \le i \le n$. Then

$$\mathscr{A} = \frac{|(f^n)'(x)|}{|(f^n)'(y)|} = \prod_{i=0}^{n-1} \frac{|f'(x_i)|}{|f'(y_i)|}.$$

This implies that

$$|\log \mathscr{A}| \le \frac{b}{a} \sum_{i=0}^{n-1} |x_i - y_i|^{\alpha}.$$

From Definition 2, there are constants $C > 0$ and $0 < \mu < 1$ such that

$$|x_i - y_i| \le C\mu^{n-i}|x_n - y_n|, \quad 0 \le i \le n-1.$$

Therefore,

$$|\log \mathscr{A}| \leq \frac{bC}{a(1-\mu^{\alpha})}|x_n - y_n|^{\alpha} = B|x_n - y_n|^{\alpha}, \quad B = (bC)/\big(a(1-\mu^{\alpha})\big). \ \square$$

Lemma 2 ($C^{1+\text{Hölder}}$–Denjoy–Koebe type distortion lemma). *There are constants $C, D > 0$ such that for any regulated chain $\mathscr{I} = \{I_i\}_{i=0}^{n}$ and for all x and y in I_0,*

$$\left|\log\left(\frac{|(f^n)'(x)|}{|(f^n)'(y)|}\right)\right| \leq C|x_n - y_n|^{\alpha/\gamma} + D\frac{|x_n - y_n|}{d(\{x_n, y_n\}, \overline{\mathrm{PSO}})},$$

where $x_n = f^n(x)$ and $y_n = f^n(y)$.

Proof. For any x and y in I_0, let $x_i = f^i(x)$ and $y_i = f^i(y)$ for $0 \leq i \leq n$. The ratio $|(f^n)'(x)|/|(f^n)'(y)|$ equals the product

$$\mathscr{A} = \prod_{i=0}^{n-1} \frac{|f'(x_i)|}{|f'(y_i)|}.$$

Let (x_i, y_i) mean the open interval in I_i bounded by x_i and y_i. We divide the set of intervals $\mathscr{J} = \{(x_i, y_i)\}_{i=0}^{n-1}$ into two subsets

$$\mathscr{J}_1 = \{(x_i, y_i) \mid (x_i, y_i) \subseteq V\} \text{ and } \mathscr{J}_2 = \{(x_i, y_i) \mid (x_i, y_i) \subseteq U \text{ and } (x_i, y_i) \not\subseteq V\}.$$

Consider

$$\prod_{x_i, y_i \in \mathscr{J}_1} \frac{|f'(x_i)|}{|f'(y_i)|} \quad \text{and} \quad \prod_{x_i, y_i \in \mathscr{J}_2} \frac{|f'(x_i)|}{|f'(y_i)|}.$$

As in the proof of Lemma 1, there are constants $C_1, C_2 > 0$ such that

$$\left|\log\left(\prod_{x_i, y_i \in \mathscr{J}_1} \frac{|f'(x_i)|}{|f'(y_i)|}\right)\right| \leq C_1 \sum_{x_i, y_i \in \mathscr{J}_1} |x_i - y_i|^{\alpha} \leq C_2 \sum_{x_i, y_i \in \mathscr{J}_2} |x_i - y_i|^{\alpha}.$$

The rightmost expression here is obtained from (3) of Definition 2.

Suppose x_i and y_i are in $U_{j_i}(\tau)$. Denote $e_i = c_{j_i}$ as the critical point in $U_{j_i}(\tau)$ and $\kappa_i = \gamma_{j_i}$ as the exponent at e_i. Let $t_i = (\kappa_i - 1)/\kappa_i$. To estimate the product $\prod_{x_i, y_i \in \mathscr{J}_2} |f'(x_i)|/|f'(y_i)|$, we write it as the product of three factors:

$$\mathscr{B} = \prod_{x_i, y_i \in \mathscr{J}_2} \left(\frac{|x_i - e_i|^{\kappa_i}}{|f(x_i) - f(e_i)|} \frac{|f(y_i) - f(e_i)|}{|y_i - e_i|^{\kappa_i}}\right)^{t_i},$$

$$\mathscr{C} = \prod_{x_i, y_i \in \mathscr{J}_2} \frac{|y_i - e_i|^{\kappa_i - 1}}{|f'(y_i)|} \frac{|f'(x_i)|}{|x_i - e_i|^{\kappa_i - 1}},$$

and

$$\mathscr{D} = \prod_{x_i, y_i \in \mathscr{J}_2} \left(\frac{|f(x_i) - f(e_i)|}{|f(y_i) - f(e_i)|} \right)^{t_i}.$$

From Definition 1 and as in the proof of Lemma 1, there is a constant $C_3 > 0$ such that

$$|\log \mathscr{B}|, |\log \mathscr{C}| \le C_3 \sum_{x_i, y_i \in \mathscr{J}_2} |x_i - y_i|^{\alpha}.$$

The estimation of $\mathscr{D}$ is a key part in the proof. Let

$$\frac{f(x_i) - f(e_i)}{f(y_i) - f(e_i)} = 1 + \frac{f(x_i) - f(y_i)}{f(y_i) - f(e_i)}.$$

Then

$$\mathscr{D} = \exp\left(\sum_{s=1}^{r-1} \frac{1}{t_{i_s}} \log\left| 1 + \frac{f(x_{i_s}) - f(y_{i_s})}{f(y_{i_s}) - f(e_{i_s})} \right| \right)$$

where $0 \le i_1 < i_2 < \cdots < i_{r-1} < n$. Let $i_r = n$. For each i_s, $1 \le s < r$, consider the interval L_s bounded by y_{i_s} and e_{i_s} and the map $h_s = f^{i_{s+1} - i_s}$. Let $R_s \subseteq L_s$ be the maximal interval containing y_{i_s} such that h_s on R_s is $C^{1+\alpha}$ and injective. One of the endpoints of R_s is y_{i_s} and the other is a preimage, denoted as e, of a singular point $q \in \mathrm{SP}$ under f^{r_s} for some $0 \le r_s < i_{s+1} - i_s$. Let $l_s = i_{s+1} - i_s - r_s$. Then h_s on the minimal interval J_s containing x_{i_s} and R_s is $C^{1+\alpha}$ and injective and maps J_s onto an interval containing the points $y_{i_{s+1}}$, $x_{i_{s+1}}$ and $f^{l_s}(q)$. We enlarge every interval J of V into a closed interval $J' \supset J$ such that $J' \cap \mathrm{CP} = \emptyset$ and such that the length of $J' \cap U$ is greater than a constant $a > 0$. Let $V' = \bigcup_{J \in V} J'$ be the union of all these enlarged intervals and let $U' = M \setminus V'$.

We consider each $1 \le s < r - 1$ in two cases. The first is that $f^i(J_s) \subseteq V'$ for all $1 \le i < i_{s+1} - i_s$. Then as in the proof of Lemma 1, there is a constant $C_4 > 0$ such that

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{i_s})|} \le C_4 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{|y_{i_{s+1}} - f^{l_s}(q)|} \le C_4 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{A}$$

where $A > 0$ is the distance between U' and $\overline{\mathrm{PSO}}$. The other is opposite to the first one. Let $1 \le k < i_{s+1} - i_s$ be the smallest integer such that $f^k(J_s) \cap U' \ne \emptyset$. Since $f^i(J_s) \subseteq V'$ for all $1 \le i < k$, we have

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{i_s})|} \le C_4 \frac{|x_{i_s+k} - y_{i_s+k}|}{|y_{i_s+k} - f^k(e)|}.$$

Since y_{i_s+k} is in V and $f^k(e)$ is in U'

$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{k_{i_s}})|} \le C_4 \frac{|x_{i_s+k} - y_{i_s+k}|}{B}$$

where $B > 0$ is the distance between V and U'. From Definition 2, there is a constant $C_5 > 0$ such that
$$|x_{i_s+k} - y_{i_s+k}| \le C_5 |x_{i_{s+1}} - y_{i_{s+1}}|.$$
We get
$$\frac{|f(x_{i_s}) - f(y_{i_s})|}{|f(y_{i_s}) - f(c_{i_s})|} \le C_4 C_5 \frac{|x_{i_{s+1}} - y_{i_{s+1}}|}{B}.$$

For $s = r-1$, as in the proof of Lemma 1 and in the above argument, there is a constant $C_6 > 0$ such that
$$\frac{|f(x_{i_{r-1}}) - f(y_{i_{r-1}})|}{|f(y_{i_{r-1}}) - f(c_{i_{r-1}})|} \le C_6 \frac{|x_n - y_n|}{d(\{x_n, y_n\}, \overline{\mathrm{PSO}})}.$$
Therefore, there is a constant $C_7 > 0$ such that
$$|\log \mathscr{D}| \le C_7 \sum_{x_i, y_i \in \mathscr{J}_2} |x_i - y_i| + C_6 \frac{|x_n - y_n|}{d(\{x_n, y_n\}, \overline{\mathrm{PSO}})}.$$

Combining all the estimates, we have constants C_8 such that
$$\left|\log\left(\frac{|(f^n)'(x)|}{|(f^n)'(y)|}\right)\right| \le C_8 \sum_{x_i, y_i \in \mathscr{J}_2} |x_i - y_i|^{\alpha} + C_6 \frac{|x_n - y_n|}{d(\{x_n, y_n\}, \overline{\mathrm{PSO}})}.$$

From Definition 2, there are constants $C_{10} \ge C_9 > 0$ and $0 < \nu_1 \le \nu_2 < 1$ such that
$$|x_i - y_i| \le C_9 \nu_1^{i_{r-1} - i} |x_{i_{r-1}} - y_{i_{r-1}}| \le C_{10} \nu_2^{n-1} |x_n - y_n|^{1/\kappa_{r-1}}.$$
Therefore, we have constants $C, D > 0$ such that
$$\left|\log\left(\frac{|(f^n)'(x)|}{|(f^n)'(y)|}\right)\right| \le C |x_n - y_n|^{\alpha/\gamma} + D \frac{|x_n - y_n|}{d(\{x_n, y_n\}, \overline{\mathrm{PSO}})}. \ \square$$

A special case of Lemma 2 which is often used in this paper is

Lemma 3 (The second distortion lemma). *Suppose U_0 is an open set such that $d(\overline{U_0}, \overline{\mathrm{PSO}}) > 0$. Then there is a constant $C > 0$ such that for any regulated chain $\{I_i\}_{i=0}^n$ with $I_n \subseteq U_0$ and for all $x, y \in I_0$,*
$$\left|\log\left(\frac{|(f^n)'(x)|}{|(f^n)'(y)|}\right)\right| \le C |x_n - y_n|^{\alpha/\gamma},$$
where $x_n = f^n(x)$ and $y_n = f^n(y)$.

Remark 3. We also proved a similar result called *the geometric distortion theorem* for higher dimensional dynamical systems in [J2] (see also [J4, Chapter 2]). Its proof is also done by a detailed analysis of the reason behind Koebe's distortion theorem in one complex variable and by combining ideas of Koebe and Denjoy. This result is a kind of a generalization of Koebe's distortion theorem for a larger class of maps including many non-conformal maps. The reader who is interested in this result and its application to some higher dimensional dynamical systems may refer to [J4, Chapter 2].

4. One-point differentiable rigidity

We use Leb to mean the Lebesgue measure on M. Our map f (or g) in this section satisfies three more technical conditions:

(1) $\overline{\mathrm{PSO}}$ has measure zero, i.e., $\mathrm{Leb}(\overline{\mathrm{PSO}}) = 0$.
(2) The set $\overline{\mathrm{PSO}}$ is not an attractor. More precisely, there is an open neighborhood $\overline{\mathrm{PSO}} \subset W \neq M$ such that for any point $p \in M$ either f^n falls into $\overline{\mathrm{PSO}}$ eventually (i.e., $\{f^n(p)\}_{n=N}^{\infty} \subseteq \overline{\mathrm{PSO}}$ for some $N > 0$) or it leaves W infinitely often (i.e., there is a subsequence $\{f^{\circ n_i}(p)\}_{i=1}^{\infty} \subseteq M \setminus W$).
(3) The map f is mixing, that is, for any intervals $I, J \subset M$, there is an integer $n \geq 0$ such that $f^n(J) \supseteq I$.

The last two conditions are invariant under topological conjugacy. Condition (3) says that $\{f^n\}_{n=0}^{\infty}$ would not be decomposed into several dynamical systems.

Denote $M_0 = M \setminus \overline{\mathrm{PSO}}$. For any point $p \in M$, let

$$\mathrm{BO}(p) = \bigcup_{n=0}^{\infty} f^{-n}(p)$$

be the set of all backward images of p. It is countable.

Suppose that f and g are two conjugated quasi-hyperbolic maps and that h is the conjugacy between f and g, i.e.,

$$h \circ f = g \circ h.$$

If h is differentiable at $p \in M_0$, then, from the last equation, h is differentiable at all points in $\mathrm{BO}(p)$.

Definition 5. We call h differentiable at $p \in M_0$ with uniform bound if there are a small neighborhood Z of p and a constant $C > 0$ such that

$$C^{-1} \leq |h'(q)| \leq C, \quad q \in \mathrm{BO}(p) \cap Z.$$

For $x \in M$, let

$$\omega(x) = \{y \in M \mid \text{there is a subsequence } f^{n_i}(x) \to y \text{ as } i \to \infty\}$$

be the set of all accumulation points of the forward orbit $\{f^{\circ n}(x)\}_{n=1}^{\infty}$. Then x is called self-recurrent if $x \in \omega(x)$.

Let $\Lambda = \bigcup_{n=0}^{\infty} f^{-n}(\overline{\mathrm{PSO}})$. Then $\mathrm{Leb}(\Lambda) = 0$. Let Ω be the set of all self-recurrent points in $M \setminus \Lambda$ of f. We say a measurable set E in M has full Lebesgue measure if $\mathrm{Leb}(E) = \mathrm{Leb}(M)$. A point p in a measurable set E_0 is said to be a density point if

$$\lim_{\mathrm{Leb}(J) \to 0} \frac{\mathrm{Leb}(E_0 \cap J)}{\mathrm{Leb}(J)} = 1$$

where J runs over all intervals containing p.

Proposition 1. *The set Ω has full Lebesgue measure in M.*

Proof. Let $\Lambda_n = \bigcup_{i=0}^n f^{-i}(\overline{\mathrm{PSO}})$. Then $\mathrm{Leb}(\Lambda_n) = 0$. Let $\Gamma_n = M \setminus \Lambda_n$. For any component J of Γ_n, let $X(J) \subseteq J$ be the set of points in J such that $\{f^{\circ n}(x)\}_{n=1}^{\infty} \cap J \neq \emptyset$. We claim $\mathrm{Leb}\big(J \setminus X(J)\big) = 0$. Let us first prove this claim.

Proof of the claim. We prove it by contradiction. Denote $A = J \setminus X(J)$. Assume $\mathrm{Leb}(A) > 0$. Almost every point in $A \setminus \Lambda$ is a density point of A. Since Λ has zero measure, we can find a point u in A such that it is a density point of A and such that there is a subsequence $\{f^{\circ n_i}(u)\}_{i=1}^{\infty}$ converging to a point q in $M \setminus W$ where W is the neighborhood defined at the beginning of this section.

Let J_0 be an open interval about q such that $C_1 = d(\overline{J_0}, \overline{\mathrm{PSO}}) > 0$. Assume that $\{f^{\circ n_i}(u)\}_{i=1}^{n} \subset J_0$. Let J_i be the interval about u such that $f^{\circ n_i}\colon J_i \to J_0$ is a $C^{1+\alpha}$-diffeomorphism. Without loss of generality, we assume that $\{f^{\circ k}(J_i)\}_{k=0}^{n_i}$ is a regulated chain for every $i \geq 1$. From Definition 2, $\mathrm{Leb}(J_i) \to 0$ as $i \to \infty$. Let $A_i = f^{\circ n_i}(A \cap J_i)$ and $A_\infty = \bigcap_{m=1}^{\infty} \bigcup_{i=m}^{\infty} A_i$. Then Lemma 3 implies that there is a constant $C_2 > 0$ such that

$$C_2^{-1} \leq \frac{|(f^{\circ n_i})'(z)|}{|(f^{\circ n_i})'(w)|} \leq C_2$$

for all $z, w \in J_i$ and all $i \geq 1$. This and the fact that

$$\lim_{i \to \infty} \frac{\mathrm{Leb}(A \cap J_i)}{\mathrm{Leb}(J_i)} = 1$$

imply $\mathrm{Leb}(A_\infty) = \mathrm{Leb}(J_0)$ (refer to the proof of Theorem 1). By the construction, the forward post-orbit of every point in A_∞ under iterates of f will not enter J. But on the other hand, because f is mixing the forward post-orbit of J_0 under iterates of f must intersect with J. This is a contradiction. We proved the claim.

Note that $\Gamma_n = M \setminus \Lambda_n$ gives a measurable partition of M and the lengths of intervals in this partition tend to zero as n goes to infinity. Thus, we have $\Omega = \bigcap_{n=1}^{\infty} \bigcup_{J \in \Gamma_n} X(J)$. It is clear that

$$M \setminus \Omega = \Lambda \cup \bigcup_{n=1}^{\infty} \bigcup_{J \in \Gamma_n} \big(J \setminus X(J)\big).$$

So

$$\mathrm{Leb}(M \setminus \Omega) \leq \mathrm{Leb}(\Lambda) + \sum_{n=1}^{\infty} \sum_{J \in \Gamma_n} \mathrm{Leb}\big(J \setminus X(J)\big) = 0.$$

In other words, $\mathrm{Leb}(\Omega) = \mathrm{Leb}(M)$. ▫

We are ready to state one of our main results in this paper now.

Theorem 1. *Suppose that f and g are conjugate quasi-hyperbolic maps and h is the conjugacy between f and g. If h is differentiable at one point $p \in M_0$ with uniform bound, then $h \mid M_0$ is C^1.*

Proof. Suppose f and g are $C^{1+\alpha}$ for $0 < \alpha \leq 1$. Let Z be an open interval about p and let $C_1 > 0$ be a constant such that

$$d\big(\overline{Z}, \overline{\mathrm{PSO}}\big),\ d\big(\overline{h(Z)}, \overline{h(\mathrm{PSO})}\big) > C_1$$

and such that

$$C_1^{-1} \leq |h'(q)| \leq C_1$$

for all q in $\mathrm{BO}(p) \cap Z$.

Let Ψ_1 be the set of intervals of $f^{-1}(Z)$ contained in Z. Inductively, let Ψ_n be the set of intervals of $f^{-n}(Z)$ contained in $Z \setminus \big(\bigcup_{I \in \Psi_{n-1}} I\big)$. Because f is mixing, there are infinitely many integers n such that Ψ_n is non-empty.

Suppose Ψ_n is non-empty. Then for any interval $I \in \Psi_n$, $f^n\colon I \to Z$ and $g^n\colon h(I) \to h(Z)$ are $C^{1+\alpha}$-diffeomorphisms. Moreover we have that

$$\frac{|h(I)|}{|I|} = \frac{|(f^n)'(x)|}{\big|(g^n)'\big(h(y)\big)\big|} \frac{|h(Z)|}{|Z|}$$

for some $x, y \in I$. Without loss of generality, we assume that $\{f^k(I)\}_{k=0}^n$ and $\{g^k(h(J))\}_{k=0}^n$ are regulated chains.

Let $q \in I \subseteq Z$ be the preimage of p under $f^n\colon I \to Z$. Then

$$\frac{(f^n)'(q)}{(g^n)'\big(h(q)\big)} = \frac{h'(q)}{h'(p)}.$$

So we have that

$$C_1^{-2} \leq \frac{|(f^n)'(q)|}{\big|(g^n)'\big(h(q)\big)\big|} \leq C_1^2.$$

Therefore,

$$C_1^{-2} \frac{|(f^n)'(x)|}{|(f^n)'(q)|} \frac{\big|(g^n)'\big(h(q)\big)\big|}{\big|(g^n)'\big(h(y)\big)\big|} \frac{|h(Z)|}{|Z|} \leq \frac{|h(I)|}{|I|} \leq C_1^2 \frac{|(f^n)'(x)|}{|(f^n)'(q)|} \frac{\big|(g^n)'\big(h(q)\big)\big|}{\big|(g^n)'\big(h(y)\big)\big|} \frac{|h(Z)|}{|Z|}.$$

Now Lemma 3 implies that there is a constant $C_2 > 0$ such that

$$C_2^{-1} \leq \frac{|(f^n)'(x)|}{|(f^n)'(q)|} \leq C_2 \quad \text{and} \quad C_2^{-1} \leq \frac{\big|(g^n)'\big(h(q)\big)\big|}{\big|(g^n)'\big(h(y)\big)\big|} \leq C_2.$$

So we have a constant $C_3 > 0$ such that

$$C_3^{-1} \leq \frac{|h(I)|}{|I|} \leq C_3.$$

Now we are going to prove that $h \mid Z$ is bi-Lipschitz. Suppose $x < y$ are in Z. Let $\Psi_1(x,y)$ be the set of intervals of $f^{-1}(Z)$ contained in $[x,y]$. Inductively, let $\Psi_n(x,y)$ be the set of intervals of $f^{-n}(Z)$ contained in $[x,y] \setminus \left(\bigcup_{I\in\Psi_n} I\right)$. Then

$$\bigcup_{n=1}^{\infty} \bigcup_{I\in\Psi_n(x,y)} I$$

is the union of pairwise disjoint intervals and its closure is $[x,y]$. Let

$$A = [x,y] \setminus \left(\bigcup_{n=1}^{\infty} \bigcup_{I\in\Psi_n(x,y)} I\right).$$

Since every point $z \neq x, y$ in A is not self-recurrent, now Proposition 1 implies that $\mathrm{Leb}(A) = 0$. Hence

$$\mathrm{Leb}\left(\bigcup_{n=1}^{\infty} \bigcup_{I\in\Psi_n(x,y)} I\right) = \sum_{n=1}^{\infty} \sum_{I\in\Psi_n(x,y)} |I| = [x,y].$$

Similarly,

$$m\left(\bigcup_{n=1}^{\infty} \bigcup_{I\in\Psi_n(x,y)} h(I)\right) = \sum_{n=1}^{\infty} \sum_{I\in\Psi_n(x,y)} |h(I)| = h([x,y]).$$

The additive formula implies that

$$C_3^{-1} \leq \frac{|h(x)-h(y)|}{|x-y|} \leq C_3.$$

Therefore, $h \mid Z$ is bi-Lipschitz.

Since $h \mid Z$ is bi-Lipschitz, h' exists a.e. in Z and is integrable (refer to [Br], [E]). Since $(h \mid Z)'(x)$ is measurable, $h \mid Z$ is a homeomorphism, and $\mathrm{Leb}(\Lambda) = 0$, we can find a point p_0 in $Z \setminus \Lambda$ and a subset E_0 containing p_0 such that (refer to [Br], [E])

(1) $h \mid Z$ is differentiable at every point in E_0;
(2) p_0 is a density point of E_0;
(3) $h'(p_0) \neq 0$; and
(4) the derivative $h' \mid E_0$ is continuous at p_0.

From the beginning of this section, we know that there is a subsequence $\{f^{n_k}(p_0)\}_{k=1}^{\infty} \subseteq M \setminus W$ converging to a point q_0 in $M \setminus W$. Let $I_0 = (a,b)$ be an open interval about q_0 such that $C_4 = d(\overline{I_0}, \overline{\mathrm{PSO}}) > 0$. There is a sequence of intervals $\{I_k\}_{k=1}^{\infty}$ such that $p_0 \in I_k \subseteq Z$ and $f^{n_k}\colon I_k \to I_0$ is a $C^{1+\alpha}$ diffeomorphism. Without loss of generality, we may assume that $\{I_{l,k} = f^l(I_k)\}_{l=0}^{n_k}$ is a regulated chain for every $k \geq 1$. From Definition 2, $\mathrm{Leb}(I_k)$ goes to zero as k tends to infinity. By Lemma 3, there is a constant $C_5 > 0$, such that

$$\left|\log\left(\frac{|(f^{n_k})'(w)|}{|(f^{n_k})'(z)|}\right)\right| \leq C_5$$

for any w and z in I_k and all $k \geq 1$.

For any positive integer s, there is an integer $N_s > 0$ such that

$$\frac{\mathrm{Leb}(E_0 \cap I_k)}{\mathrm{Leb}(I_k)} \geq 1 - \frac{1}{s}$$

for all $k > N_s$. Let $E_k = f^{n_k}(E_0 \cap I_k)$. Then h is differentiable at every point in E_k and there is a constant $C_6 > 0$ such that

$$\frac{\mathrm{Leb}(E_k \cap I_0)}{\mathrm{Leb}(I_0)} \geq 1 - \frac{C_6}{s}$$

for all $k > N_s$ because $\{f^{n_k} \mid I_k\}_{k=1}^{\infty}$ have uniformly bounded distortion. Let $E = \bigcap_{s=1}^{\infty} \bigcup_{k>N_s} E_k$. Then E has full measure in I_0 and h is differentiable at every point in E with non-zero derivative.

Next, we are going to prove that $h' \mid E$ is uniformly continuous. For any x and y in E, let z_k and w_k be the preimages of x and y under the diffeomorphism $f^{n_k}\colon I_k \to I_0$. Then z_k and w_k are in E_0. From $h \circ f = g \circ h$, we have that

$$h'(x) = \frac{(g^{n_k})'\big(h(z_k)\big)}{(f^{n_k})'(z_k)} h'(z_k)$$

and

$$h'(y) = \frac{(g^{n_k})'\big(h(w_k)\big)}{(f^{n_k})'(w_k)} h'(w_k).$$

So

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \leq \left|\log\left|\frac{(g^{n_k})'\big(h(z_k)\big)}{(g^{n_k})'\big(h(w_k)\big)}\right|\right| + \left|\log\left|\frac{(f^{n_k})'(w_k)}{(f^{n_k})'(z_k)}\right|\right| + \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right|.$$

Applying Lemma 3 to both f and g, we can find a constant $C_7 > 0$ such that

$$\left|\log\left|\frac{(f^{n_k})'(w_k)}{(f^{n_k})'(z_k)}\right|\right| \leq C_7 |x - y|^{\alpha}$$

and

$$\left|\log\left|\frac{(g^{n_k})'\big(h(z_k)\big)}{(g^{n_k})'\big(h(w_k)\big)}\right|\right| \le C_7|h(x)-h(y)|^\alpha$$

for all $k \ge 1$. Therefore,

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \le C_7\big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\big) + \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right|$$

for all $k \ge 1$. Since $h' \mid E_0$ is continuous at p_0, the last term in the last inequality tends to zero as k goes to infinity. Hence

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \le C_7\Big(|x-y|^\alpha + |h(x)-h(y)|^\alpha\Big).$$

This means that $h' \mid E$ is uniformly continuous. So it can be extended to a continuous function ϕ on I_0. Because $h \mid I_0$ is absolutely continuous and E has full measure,

$$h(x) = h(a) + \int_a^x h'(x)\,dx = h(a) + \int_a^x \phi(x)\,dx$$

on I_0. This implies that $h \mid I_0$ is actually C^1.

Now for any $x \in M_0$, let $J \subset M_0$ be an open interval about x. By the mixing condition, there are an integer $n > 0$ and an open interval $J_0 \subset I_0$ such that f^n: $J_0 \to J$ is a C^1 diffeomorphism. By the equation $h \circ f = g \circ h$, we have that $h \mid J$ is C^1. Therefore, $h \mid M_0$ is C^1. □

Consider the conjugacy $h(x) = (2/\pi)\arcsin(x)$ between $f(x) = 1-2x^2$ and $g(x) = 1-2|x|$ on $[-1,1]$. The maps h and h^{-1} are both C^1 on $(-1,1)$. But h' is not uniformly continuous because the exponents of f and g at 0 are different. Note that the exponent at a singular point is invariant under C^1 conjugacy. Furthermore, we have the following result to enhance Theorem 1.

Theorem 2. *Suppose f and g and h are the same as those in Theorem 1. If h is differentiable at one point in M_0 with uniform bound and all the exponents of f and g at the corresponding singular points are the same, then h restricted to the closure of every interval of M_0 is a $C^{1+\beta}$ diffeomorphism for some $0 < \beta \le 1$.*

We prove this theorem by two lemmas. Let f and g be the maps and h be the conjugacy in Theorem 2. Suppose f and g are both $C^{1+\alpha}$. Let $U = U(\tau)$ and $V = V(\tau)$ be the sets satisfied by both f and g. Let I_0 and q_0 be the interval and the point found in the proof of Theorem 1. Denote $\mathrm{CP} = \{c_1, \dots, c_d\}$ as the set of critical points and $\Gamma = \{\gamma_1, \dots, \gamma_d\}$ as the corresponding exponents. Let $\gamma = \max\{\gamma_i \mid 1 \le i \le d\}$. Let $U_i(\tau) = [c_i - \tau, c_i + \tau]$ for $c_i \in \mathrm{CP}$.

Remark 4. Under the above assumption, $\beta = \alpha/\gamma$ in Theorem 2.

Lemma 4. *The map $h \mid U_i(\tau)$ is $C^{1+\alpha}$.*

Proof. We use the same notation as in the proof of Theorem 1. Denote $J = U_i(\tau) = [d, e]$. Since f has mixing condition and $J \cap \overline{\mathrm{PSO}} = \emptyset$, there is a preimage $J_k \subset I_0$ of J under f^{n_k} such that J_k tends to q_0 as $k \to \infty$ and such that $f^{n_k}\colon J_k \to J$ is a C^1 diffeomorphism where $\{n_k\}_{k=1}^{\infty}$ is a subset of the positive integers. Let F_k be the inverse of $f^{n_k}\colon J_k \to J$. From the equation $h \circ f = g \circ h$, we have that $h \mid J = g^{n_k} \circ h \circ F_k$. So $h \mid J$ is C^1 and

$$h'(x) = \frac{(g^{n_k})'\big(h(z_k)\big)}{(f^{n_k})'(z_k)} h'(z_k) \neq 0$$

for any x in J where $z_k = F_k(x)$.

Without loss of generality, we assume that $\{J_{i,k} = f^{n_k - i}(J_k)\}_{i=0}^{n_k}$ are all regulated chains for all $k > 0$. For any x and y in J, let z_k and w_k be the preimage of x and y under the diffeomorphism $f^{n_k}\colon J_k \to J$. Since

$$\frac{h'(x)}{h'(y)} = \frac{(g^{n_k})'\big(h(z_k)\big)}{(f^{n_k})'(z_k)} \cdot \frac{(f^{n_k})'(w_k)}{(g^{n_k})'\big(h(w_k)\big)} \cdot \frac{h'(z_k)}{h'(w_k)},$$

from Lemma 3, there is a constant $C > 0$ such that

$$\begin{aligned} \left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| &\leq \left|\log\left|\frac{(g^{n_k})'\big(h(z_k)\big)}{(g^{n_k})'\big(h(w_k)\big)}\right|\right| + \left|\log\left|\frac{(f^{n_k})'(w_k)}{(f^{n_k})'(z_k)}\right|\right| + \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right| \\ &\leq C\big(|x-y|^{\alpha} + |h(x) - h(y)|^{\alpha}\big) + \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right|. \end{aligned}$$

Because $h'(z_k), h'(w_k) \to h'(q_0)$ as $k \to \infty$, we have

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \leq C\big(|x-y|^{\alpha} + |h(x) - h(y)|^{\alpha}\big).$$

This implies that $h \mid J$ is actually $C^{1+\alpha}$. □

Lemma 5. *The restriction of h to the closure of every interval J of $V \setminus \overline{\mathrm{PSO}}$ is $C^{1+(\alpha/\gamma)}$.*

Proof. We always use C to denote a positive constant (although it may be different in different formulas). Since f has mixing condition, we can find a subsequence $\{n_k\}_{k=1}^{\infty}$ of the positive integers and intervals $J_k \subset I_0$ such that J_k tends to q_0 as $k \to \infty$ and such that $f^{n_k}\colon J_k \to J$ is a C^1 diffeomorphism. Let

F_k be the inverse of $f^{n_k}\colon J_k \to J$. From the equation $h\circ f = g\circ h$, we have that $h \mid J = g^{n_k}\circ h\circ F_k$. So $h\mid J$ is C^1 and

$$h'(x) = \frac{(g^{n_k})'\big(h(z_k)\big)}{(f^{n_k})'(z_k)} h'(z_k) \neq 0$$

for any x in J where $z_k = F_k(x)$.

Let $J_{i,k} = f^{n_k - i}(J_k)$ for $0 \leq i \leq n_k$. Without loss of generality, we assume that $\{J_{i,k}\}_{i=0}^{n_k}$ are regulated chains for all $k > 0$. For any x and y in J, let z_k and w_k be the preimage of x and y under the diffeomorphism $f^{n_k}\colon J_k \to J$. From the equation $h\circ f = g\circ h$, we have

$$\frac{h'(x)}{h'(y)} = \frac{(g^{n_k})'\big(h(z_k)\big)}{(f^{n_k})'(w_k)} \cdot \frac{(f^{n_k})'(w_k)}{(g^{n_k})'\big(h(w_k)\big)} \cdot \frac{h'(z_k)}{h'(w_k)}.$$

Thus,

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \leq \left|\log\left(\frac{\big|(g^{n_k})'\big(h(z_k)\big)\big|}{\big|(g^{n_k})'\big(h(w_k)\big)\big|}\right)\right| + \left|\log\left(\frac{|(f^{n_k})'(w_k)|}{|(f^{n_k})'(z_k)|}\right)\right| + \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right|.$$

Let $n = n_k$ and let $m = m(n_k) > 0$ be the smallest integer such that $J_{m,k} \subseteq U_j(\tau) \subseteq U$ for some $1 \leq j \leq d$. Let $x_i = f^{n_k - i}(z_k)$ and $y_i = f^{n_k-i}(w_k)$ for all $0 \leq i \leq n_k$. Then

$$\begin{aligned}
\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right| \leq{}& \left|\sum_{i=1}^{m-1}\big(\log|f'(y_i)| - \log|f'(x_i)|\big)\right| \\
&+ \left|\sum_{i=1}^{m-1}\Big(\log\big|g'\big(h(x_i)\big)\big| - \log\big|g'\big(h(y_i)\big)\big|\Big)\right| \\
&+ \left|\log\left(\frac{\big|g'\big(h(x_m)\big)\big|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{\big|g'\big(h(y_m)\big)\big|}\right)\right| \\
&+ \left|\sum_{i=m+1}^{n}\big(\log|f'(y_i)| - \log|f'(x_i)|\big)\right| \\
&+ \left|\sum_{i=m+1}^{n}\Big(\log\big|g'\big(h(x_i)\big)\big| - \log\big|g'\big(h(y_i)\big)\big|\Big)\right| \\
&+ \left|\log\left(\frac{h'(z_k)}{h'(w_k)}\right)\right|.
\end{aligned}$$

The last term tends to zero as k goes to infinity. We estimate the first five terms. From Lemma 1, there is a constant $C > 0$ such that

$$\left|\sum_{i=1}^{m-1}\big(\log|f'(y_i)| - \log|f'(x_i)|\big)\right| \leq C|x-y|^{\alpha}$$

and

$$\Big|\sum_{i=1}^{m-1}\big(\log\big|g'\big(h(x_i)\big)\big|-\log\big|g'\big(h(y_i)\big)\big|\big)\Big|\le C|h(x)-h(y)|^{\alpha}.$$

From Lemma 3, there are constants $C>0$ such that

$$\begin{aligned}\Big|\sum_{i=m+1}^{n}\big(\log|f'(y_i)|-\log|f'(x_i)|\big)\Big| &\le C|x_m-y_m|^{\alpha}\\ &\le C|x_{m-1}-y_{m-1}|^{\alpha/\gamma_j}\le C|x-y|^{\alpha/\gamma_j}.\end{aligned}$$

Similarly,

$$\Big|\sum_{i=m+1}^{n}\big(\log\big|g'\big(h(x_i)\big)\big|-\log\big|g'\big(h(y_i)\big)\big|\big)\Big|\le C|h(x)-h(y)|^{\alpha/\gamma_j}.$$

Now we consider

$$\mathscr{S}=\frac{\big|g'\big(h(x_m)\big)\big|}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{\big|g'\big(h(y_m)\big)\big|}.$$

Define

$$\mathscr{S}=\mathscr{S}_1\cdot\mathscr{S}_2\cdot\mathscr{S}_3$$

where

$$\mathscr{S}_1=\frac{\big|g'\big(h(x_m)\big)\big|}{|h(x_m)-h(c_j)|^{\gamma_j-1}}\cdot\frac{|h(y_m)-h(c_j)|^{\gamma_j-1}}{\big|g'\big(h(y_m)\big)\big|},$$

$$\mathscr{S}_2=\frac{|x_m-c_j|^{\gamma_j-1}}{|f'(x_m)|}\cdot\frac{|f'(y_m)|}{|y_m-c_j|^{\gamma_j-1}},$$

and

$$\mathscr{S}_3=\left(\frac{|h(x_m)-h(c_j)|}{|x_m-c_j|}\right)^{\gamma_j-1}\cdot\left(\frac{|y_m-c_j|}{|h(y_m)-h(c_j)|}\right)^{\gamma_j-1}.$$

Now Lemma 4 implies that

$$|\log S_3|\le C|x_m-y_m|^{\alpha}\le C|x_{m-1}-y_{m-1}|^{\alpha/\gamma_j}\le C|x-y|^{\alpha/\gamma_j}.$$

From Definition 1,

$$|\log S_2|\le C|x_m-y_m|^{\alpha}\le C|x_{m-1}-y_{m-1}|^{\alpha/\gamma_j}\le C|x-y|^{\alpha/\gamma_j}$$

and

$$|\log S_1|\le C|h(x_m)-h(y_m)|^{\alpha}\le C|h(x_{m-1})-h(y_{m-1})|^{\alpha/\gamma_j}\le C|h(x)-h(y)|^{\alpha/\gamma_j}.$$

Thus, as k goes to infinity, we have that

$$\left|\log\left(\frac{h'(x)}{h'(y)}\right)\right|\le C(|x-y|^{\alpha/\gamma}+|h(x)-h(y)|^{\alpha/\gamma}).$$

This implies that $h'\mid J$ is actually $C^{\alpha/\gamma}$. So is $h'\mid \bar{J}$. We have that $h\mid \bar{J}$ is $C^{1+\alpha/\gamma}$. □

Proof of Theorem 2. Both Lemmas 4 and 5 for h and h^{-1} complete the proof. ▫

We can use eigenvalues of f and g at corresponding periodic points to verify the condition, differentiable at one point with uniform bound, in Theorems 1 and 2.

Lemma 6. *Suppose f and g and h are those in Theorem* 1. *If h is differentiable at a point p in M_0 with non-zero derivative and if there is an open interval Y about p such that the absolute values of the eigenvalues of f and g at periodic points in Y and at corresponding periodic points in $h(Y)$ are the same, then h is differentiable at p with uniform bound.*

Proof. Let $\mathrm{BO}(p)$ be the backward orbit of p. Assume that $C_1 = d(\overline{Y}, \overline{\mathrm{PSO}}) > 0$. From Definition 2, there is an integer $N \geq 0$ and an open interval Z about p such that every component of $f^{-n}(Y)$ which intersects with Z is contained in Y for $n > N$.

For any $q \in \mathrm{BO}(p) \cap Z$, we have $f^n(q) = p$ for some $n \geq 0$. Since there are only a finite number of points in $\bigcup_{i=0}^{N} f^{-i}(p)$, we need only consider $n > N$. There is an open interval $q \in J_q$ such that $f^n\colon J_q \to Y$ is a C^1-diffeomorphism where $J_q \subseteq Y$. Therefore, there is a fixed point r of f^n in J_q. By the assumption, we have that

$$|(f^n)'(r)| = \big|(g^n)'\big(h(r)\big)\big|.$$

From the equation $h \circ f = g \circ h$, we also have that

$$h'(q) = \frac{(f^n)'(q)}{(g^n)'\big(h(q)\big)} h'(p).$$

These imply that

$$|h'(q)| = \frac{|(f^n)'(q)|}{|(f^n)'(r)|} \frac{\big|(g^n)'\big(h(r)\big)\big|}{\big|(g^n)'\big(h(q)\big)\big|} |h'(p)|.$$

Without loss of generality, we assume that $\{f^k(J_q)\}_{k=0}^{n}$ is a regulated chain. Now applying Lemma 3, there is a constant $C_2 > 0$ such that

$$C_2^{-1} \leq \frac{|(f^n)'(q)|}{|(f^n)'(r)|} \leq C_2$$

and

$$C_2^{-1} \leq \frac{\big|(g^n)'\big(h(r)\big)\big|}{\big|(g^n)'\big(h(q)\big)\big|} \leq C_2.$$

But $h'(p) \neq 0$. We get a constant $C_3 > 0$ such that

$$C_3^{-1} \leq |h'(q)| \leq C_3$$

for all $q \in \mathrm{BO}(p) \cap Z$. ▫

The above lemma combined with Theorems 1 and 2 gives us the following result.

Corollary 1. *Suppose f and g and h are those in Theorem* 1*. If h is differentiable at one point p in M_0 with non-zero derivative and if the absolute values of the eigenvalues of f and g at periodic points in a small neighborhood Y of p and at corresponding periodic points in $h(Y)$ are the same, then $h \mid M_0$ is C^1. Furthermore, if all the exponents of f and g at the corresponding singular points are also the same, then h restricted on the closure of every interval of M_0 is a $C^{1+\beta}$ diffeomorphism for some fixed $0 < \beta \le 1$.*

The following rigidity result can now be obtained from Theorems 1, 2, and Corollary 1.

Corollary 2. *Suppose f and g and h are those in Theorem* 1*. If there is a small interval Y of M such that $h \mid Y$ is absolutely continuous, then $h \mid M_0$ is C^1. Furthermore, if all the exponents of f and g at the corresponding singular points are also the same, then h restricted on the closure of every interval of M_0 is $C^{1+\beta}$ for some fixed $0 < \beta \le 1$.*

Proof. Since $h \mid Y$ is absolutely continuous, it is differentiable a.e. on Y. Moreover, since h is a homeomorphism, there is a positive measure set in Z such that h is differentiable with non-zero derivatives. An absolutely continuous map preserves the absolute values of all the eigenvalues of f and g at periodic points in $Y \cap M_0$ and at corresponding periodic points in $h(Y \cap M_0)$. Therefore, the corollary follows from Corollary 1. □

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