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A SIMPLY CONNECTED, HOMOGENEOUS DOMAIN THAT IS NOT A QUASIDISK

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Abstract. We construct a simply connected, homogeneous domain that is not a quasidisk. This shows that a theorem by Sarvas (1985) can not be generalized to simply connected domains instead of Jordan domains.

1. Introduction

We let $\mathscr{Q}(K)$ denote the family of K-quasiconformal self-mappings of the Riemann sphere $\overline{\mathbf{C}}$. In particular $\mathscr{Q}(1)$ denotes the family generated by the Möbius group

$$\left\{\varphi(z) = \frac{az+b}{cz+d} : ad - bc \neq 0\right\}$$

and the reflection $f(z) = \bar{z}$. We refer to [5] for further information on quasiconformal mappings. Contrary to usual convention we will allow quasiconformal mappings to be orientation-reversing as well as orientation-preserving. This peculiarity will be discussed at the end of the article.

A set $S \subset \overline{\mathbf{C}}$ is said to be homogeneous with respect to a family $\mathscr{F}$ of mappings, if for each pair of points $z_1, z_2 \in S$ there exists a mapping $f \in \mathscr{F}$ such that

$$f(S) = S \quad \text{and} \quad f(z_1) = z_2.$$

It is easy to show that there is a family $\mathscr{B}$ of conformal mappings such that the open unit disk, $\mathbf{D}$, is homogeneous with respect to it. For example take

$$\mathscr{B} = \left\{\varphi(z) = \lambda\frac{z-a}{1-\bar{a}z} : a \in \mathbf{D},\ |\lambda| = 1\right\}. \tag{1}$$

By using translations, dilations and reflections, it can be shown that any disk is homogeneous with respect to $\mathscr{Q}(1)$. The converse can also be shown to be true [2].

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Theorem 1. *A domain D is a disk if and only if it is homogeneous with respect to the family $\mathcal{Q}(1)$.*

The mappings φ in (1) distorts the boundary of $\mathbf{D}$. If we require the boundary to be left untouched, a conformal map will no longer do the job. Instead we rely on a theorem of Teichmüller. He pointed out that given a compact set E in the unit disk, any point in E can be mapped to any other point in E by a K-quasiconformal mapping f. Moreover, f can be chosen to fix the boundary of $\mathbf{D}$ [8].

We will now look at the connection between homogeneity and quasidisks. In 1977 Timo Erkama proved the following result for the boundary [1].

Theorem 2. *A simply connected domain D is a quasidisk if and only if ∂D is homogeneous with respect to the family $\mathcal{Q}(K)$ for some fixed K.*

A few years later Jukka Sarvas proved a similar theorem concerning homogeneity of the domain [7].

Theorem 3. *A Jordan domain D is a quasidisk if and only if it is homogeneous with respect to the family $\mathcal{Q}(K)$ for some fixed K.*

2. A simply connected, homogeneous domain

In Theorem 2 a simply connected domain is sufficient, while a Jordan domain is crucial in Sarvas' proof of Theorem 3. We will now show that the hypothesis of the latter theorem can not be weakened to also include simply connected domains. This result has been known for some time and a preprint by Näkki and Palka on the matter exists [6]. Their construction is based on the same technique used in Example 4.6 of [3] and some results from the field of Kleinian groups. We will present an example which does not rely on this theory.

Theorem 4. *There exists a simply connected domain D, homogeneous with respect to $\mathcal{Q}(K)$ for some fixed K, that is not a quasidisk.*

We start by constructing the domain. Let r_1 and r_2 be reflections in the circles $|z+2|=1$ and $|z-2|=1$, respectively, and let t denote the translation $t(z)=z+i$. Define $G_0=\{r_1,r_2,t,t^{-1}\}$ and let G be the group generated by the elements of G_0. Finally denote by E the closed set

$$E=\left\{z=x+iy : |x|\le 2-\sqrt{1-y^2},\quad |y|\le \tfrac{1}{2}\right\}.$$

See Figure 1. The domain D will then be given by the union

$$D=\bigcup_{g\in G} g(E).$$

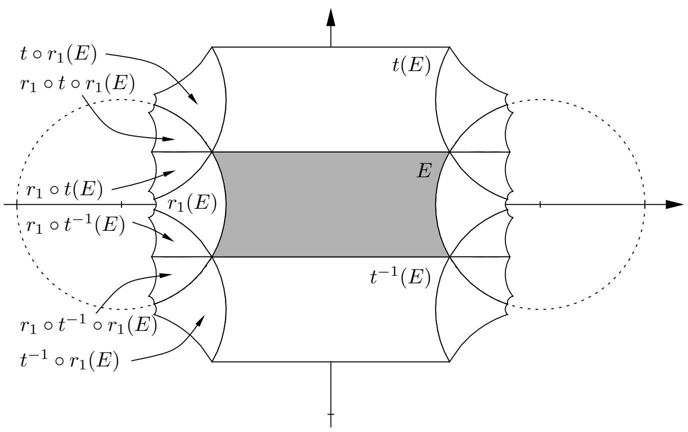

Figure 1. The set E, the two circles we reflect in and some of the other components of D.

The domain D will be an infinite "strip" and is indicated in Figure 2.

Let us introduce some more notation. A set $P = g(E)$ with $g \in G$ is called a component of D. If $g = g_n \circ \dots \circ g_1$ and n is minimal, P is called an nth generation component and is denoted $P^{(n)}$. We also define

$$D_n = \operatorname{int}\Bigl(\bigcup_{i\le n} P^{(i)}\Bigr). \tag{2}$$

Observe that we can introduce a metric d_c on the set of components in the following way: Let $P_1^{(i)} = \varphi(E)$ and $P_2^{(j)} = \psi(E)$ with $\varphi, \psi \in G$, and let n be the generation of $P = \psi\circ\varphi^{-1}(E)$. Define $d_c(P_1, P_2) = n$. Also note that for $P^{(n)} = g(E) = g_n \circ\dots\circ g_1(E)$ the union of the n components

$$\bigcup_{i=1}^{n} g_n \circ \dots \circ g_i(E)$$

is path connected. Thus, $d_c(P_1, P_2) = n$ and the minimality of n implies that the shortest path from P_1 to P_2 crosses $n-1$ other components.

Before we go on to show that D is a simply connected, homogeneous domain, we prove two key lemmas.

Lemma 1. *D_n as defined in* (2) *is a simply connected domain for all* $n = 0, 1, 2, \dots$.

Proof. We use induction. Clearly $D_0 = \operatorname{int}(E)$ is simply connected. Assume D_{n-1} is simply connected. We will show that $\overline{D}_{n-1} \cup P^{(n)}$ is simply connected for all $P^{(n)}$. Then also D_n will be simply connected as two different nth generation components do not have any point in common that is not already in $\overline{D}_{n-1}$.

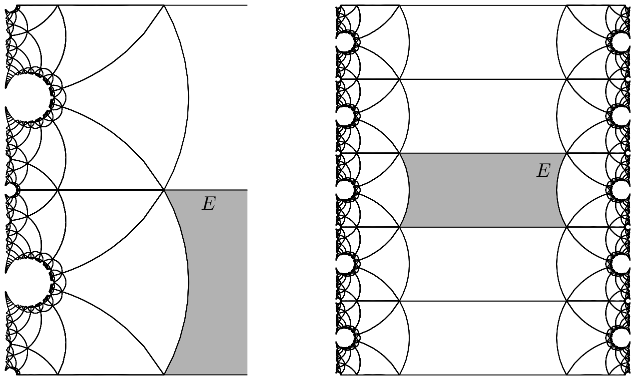

Figure 2. The domain D and its components. To the left is a blown-up image showing more detail.

Obviously $\overline{D}_{n-1} \cup P^{(n)}$ is connected. We must make sure that the addition of $P^{(n)}$ does not split the boundary into two or more parts, or equivalently we must see that the set $\overline{D}_{n-1} \cap P^{(n)}$ is connected. Seeking a contradiction we assume that $\overline{D}_{n-1} \cap P^{(n)}$ is disconnected. $P^{(n)}$ is surrounded by four components of generation $n-1$ or $n+1$. We can illustrate the situation as follows:

$$\begin{array}{ccc} & P_1^{(n-1)} & \\ P^{(n+1)} & P^{(n)} & P^{(n+1)} \\ & P_2^{(n-1)} & \end{array}$$

The black lines mark $\overline{D}_{n-1} \cap P^{(n)}$. Let $g = g_n \circ \cdots \circ g_1$ be such that $P^{(n)} = g(E)$ with $g_i \in G_0$. Then

$$\begin{array}{ccc} & P_1^{(n-1)} & \\ P^{(n+1)} & P^{(n)} & P^{(n+1)} \\ & P_2^{(n-1)} & \end{array} \xrightarrow{g^{-1}} \begin{array}{ccc} & P_1^{(1)} & \\ P^{(1)} & P^{(0)} = E & P^{(1)} \\ & P_2^{(1)} & \end{array}$$

because there are no components of generation -1. Thus there are $\varphi_j \in G_0$ such that $P_j^{(1)} = \varphi_j(E)$ for $j = 1, 2$. Moreover, φ_1 and φ_2 can be chosen such that either $\varphi_1 = t$ and $\varphi_2 = t^{-1}$, or $\varphi_1 = r_1$ and $\varphi_2 = r_2$. But then

$$P_j = P_j^{(n-1)} = g\bigl(\varphi_j(E)\bigr) = g_n \circ \cdots \circ g_1 \circ \varphi_j(E). \tag{3}$$

The components P_j are of generation $n-1$, hence some of the terms in (3) must cancel for both $j = 1, 2$. To make the remaining discussion less abstract, assume $g_1 = t$ and $\varphi_1 = t$, $\varphi_2 = t^{-1}$. The other cases may be treated equally.

Clearly $P_2 = g_n \circ \cdots \circ g_2 \circ t \circ t^{-1}(E) = g_n \circ \cdots \circ g_2(E)$ is of generation $n-1$. Let $h^{-1} = g_n \circ \cdots \circ g_2$ and consider $h(P_1) = h \circ g_n \circ \cdots \circ g_2 \circ t \circ t(E) = t \circ t(E)$. We want to find the shortest path from $h(E)$ to $t \circ t(E)$ and hence calculate $d_c\big(h(E), t \circ t(E)\big)$. As $g = g_n \circ \cdots \circ g_1$ is minimal, we know that the shortest path from $h(E)$ to $t(E)$ goes through E and that $d_c\big(h(E), t(E)\big) = n$. This implies that $h(E)$ does not lie above E, or in other words that $h(E)$ lies below the line $y = \frac{1}{2}$. It is then easy to realize that $d_c\big(h(E), t \circ t(E)\big) = n+1$ by considering Figure 2. The reason is that a shortest path would use the components $t^n(E)$, $n \in \mathbf{Z}$ as much as possible.

This contradicts the assumption that P_1 is of generation $n-1$ and therefore also that $\overline{D}_{n-1} \cup P^{(n)}$ is disconnected. □

Lemma 2. *D is contained in the infinite strip $\{z : |\operatorname{Re} z| < 2\}$.*

Proof. Let L and R denote the two halfplanes $L = \{z \in \mathbf{C} : \operatorname{Re} z \le -2\}$ and $R = \{z \in \mathbf{C} : \operatorname{Re} z \ge 2\}$. We will show that $g(L \cup R)$ does not meet E for any $g \in G$. Let Λ be the set of all disks $g(R)$, $g \in G$, that touch L (excluding R itself) and define $L_0 = r_1(R) \in \Lambda$. We claim that L_0 is the biggest disk in Λ. Any disk $L_\lambda \in \Lambda$ can be written

$$L_\lambda = t^{n_{m+1}} \circ r_1 \circ t^{n_m} \circ \cdots \circ r_1 \circ t^{n_1}, \qquad n_i \in \mathbf{Z}.$$

Represent a disk L_λ by the pair (a, ϱ) where $a = \operatorname{Im} z_\lambda$ and z_λ is the center of L_λ. The value ϱ denotes the radius of L_λ. To calculate ϱ observe that the reflection r_1 takes the disk (a, ϱ) to $(1/a, \varrho/a^2)$. Define the sequence $\big((a_i, \varrho_i)\big)_{i=0}^{m}$ by

$$L_0 \sim (a_0, \varrho_0) \overset{r_1 \circ t^{n_1}}{\longrightarrow} (a_1, \varrho_1) \longrightarrow \cdots \overset{r_1 \circ t^{n_m}}{\longrightarrow} (a_m, \varrho_m).$$

As the translation $t^{n_{m+1}}$ does not affect the radius of L_λ we have $\varrho = \varrho_m$. Obviously $a_i \in \mathbf{Q}$ so write $a_i = b_i / c_i$ with $b_i, c_i \in \mathbf{Z}$ and let $a_0 = b_0 = 0$ and $c_0 = 1$. Then

$$a_i = \frac{1}{a_{i-1} + n_i} = \frac{1}{\dfrac{b_{i-1}}{c_{i-1}} + n_i} = \frac{c_{i-1}}{b_{i-1} + c_{i-1} n_i}$$

or $b_i = c_{i-1}$ and $c_i = b_{i-1} + c_{i-1} n_i$. We now claim that $\varrho_i = \varrho_0 / c_i^2$ and prove this by induction. For $i = 0$ the claim is obvious. Assume the relation holds for $i = k-1$. Then for $i = k$,

$$\varrho_k = \frac{\varrho_{k-1}}{(a_{k-1} + n_k)^2} = \varrho_{k-1} \cdot a_k^2 = \frac{\varrho_0}{c_{k-1}^2} \cdot \frac{b_k^2}{c_k^2} = \frac{\varrho_0}{c_{k-1}^2} \cdot \frac{c_{k-1}^2}{c_k^2} = \frac{\varrho_0}{c_k^2}.$$

As $c_m \in \mathbf{Z}$ we get that $\varrho = \varrho_m$ is either infinite, or smaller than or equal to ϱ_0 which is the radius of L_0. The case when ϱ is infinite is when L_0 is mapped on the halfplane R. Since R is not in Λ, L_0 is the biggest disk in Λ.

Using that L_0 is both the biggest disk in Λ and the disk in Λ closest to the point $z=2$, shows that the bulb $\tilde{L}_0$ containing L_0 is the bulb in $\bigcup_{g\in G} g(L\cup R)$ coming closest to E. Now define the sequence of points (z_i) by

$$z_0=-2, \quad z_1=r_1(2), \quad z_2=r_1\circ r_2(-2), \quad z_3=r_1\circ r_2\circ r_1(2), \quad \dots .$$

Then $\lim_{i\to\infty} z_i$ will be the point in $\tilde{L}_0$ furthest away from $z=-2\in L$. Let d_i be the distance $d_i=|2-z_i|$. Then

$$d_0=4, \qquad d_i=4-\frac{1}{d_{i-1}}, \quad i\geq 1.$$

Thus $\lim_{i\to\infty} d_i$ is the attracting fixed point of $f=4-1/f$, which is $2+\sqrt{3}$. So the bulb $\tilde{L}_0$ reaches a distance of $4-\bigl(2+\sqrt{3}\,\bigr)=2-\sqrt{3}<1$ away from $z=-2$ and hence does not meet E. □

Having proved these lemmas, we are ready to prove Theorem 4. We first show that D is open. Let F be the union of images of E under the identity and the sixteen mappings

$$r_j, \quad t, \quad t^{-1}, \quad r_j\circ t, \quad r_j\circ t^{-1}, \quad t\circ r_j, \quad t^{-1}\circ r_j, \quad r_j\circ t\circ r_j, \quad r_j\circ t^{-1}\circ r_j$$

for $j=1,2$. See Figure 1. Take F_0 to be the interior of F. Then F_0 is an open set which properly contains E. Hence we can write $D=\bigcup_{g\in G} g(F_0)$ and it follows that D is open, because it is the union of open sets.

To see that D is simply connected observe that $D=\cup D_n$ and $D_n\subseteq D_{n+1}$, where D_n was defined in (2). As each D_n is a simply connected domain, the complements $D_n^*=\overline{\mathbf{C}}\setminus D_n$ will be closed, compact and connected sets. Since $(D_n^*)_n$ is a decreasing sequence, $D_n^*\subseteq D_{n-1}^*$, with $D^*=\cap D_n^*$, D^* is also closed, compact and connected. Hence D is simply connected.

Next we prove the homogeneity of D. By Riemann's mapping theorem $F_0\subseteq D$ can be mapped onto the unit disk by a conformal mapping f. E is a compact subset of F_0, so $f(E)$ is a compact subset of $f(F_0)=\mathbf{D}$. So by Teichmüller's theorem [8], for any pair of points $z_1,z_2\in f(E)$ there exists a K-quasiconformal self-mapping h of $\overline{\mathbf{C}}$ which is the identity outside $f(F_0)$ and maps z_1 to z_2. From the construction of D, for any point $z\in D$ there exists a mapping $g\in G$ and a point $w\in E$ such that $z=g(w)$. Altogether, it follows that D is homogeneous with respect to the family

$$\mathscr{F}=G\circ f^{-1}\circ\{h\}\circ f\circ G\subseteq \mathscr{Q}(K).$$

To complete the proof, we show that D is not locally connected at $\infty\in\partial D$. It will follow that D is not a Jordan domain and hence not a quasidisk. We construct a neighborhood of ∞ in the following way: For $n\in\mathbf{N}$, let

$$C_n=\{z:|\operatorname{Re} z|\leq 2n,\ |\operatorname{Im} z|\leq n\}.$$

Then C_n^* is a neighborhood of ∞. Fix $n \geq 1$. For any neighborhood $U \subseteq C_n^*$ there is a number $m \in \mathbf{N}$ such that the points $z_1 = (n+m)i$ and $z_2 = -(n+m)i$ are in $U \cap D$. However, the two points can not be connected in $C_n^* \cap D$ because D is contained in the strip $\{z : |\operatorname{Re} z| < 2\}$. Thus D is not locally connected at ∞. □

3. Remarks

To end things off we provide a few comments. First of all, for the construction to work as it is, we must admit $\mathcal{Q}(K)$ to contain both orientation-preserving and orientation-reversing mappings. Usually a K-quasiconformal mapping is orientation-preserving by definition. In particular this is the case in the theorems of Erkama and Sarvas.

The example is however easily modified to use a family of orientation-preserving mappings only. The orientation is reversed each time we apply a reflection, r_j. So for $g \in G$ to be orientation-preserving, it must consist of an even number of reflections.

Let $\widetilde{E} = E \cup r_1(E)$. In the example we found for each $z \in D$ a mapping $g \in G$ and a point $w \in E$ with $z = g(w)$. If g consists of an even number of reflections, define $\tilde{g} = g$ and $\widetilde{w} = w$. If not, let $\tilde{g} = g \circ r_1$ and $\widetilde{w} = r_1(w)$. Then $\tilde{g} \in G$ is an orientation-preserving mapping such that $z = \tilde{g}(\widetilde{w})$ for some point $\widetilde{w} \in \widetilde{E}$. $\widetilde{E}$ will be a compact subset of some $\widetilde{F}_0 \subset D$. Thus the same construction of a homogeneous family as before can be carried out, but now with $\widetilde{E}$, $\tilde{g}$ and $\widetilde{w}$ instead of E, g and w.

As a second comment, we remark that the same construction can be done in more generality. Define r_1 and r_2 to be the reflections in the circles $|z+b| = a$ and $|z-b| = a$, and t to be the translation $t(z) = z + ia$. Then for any $0 < a < b < \infty$ and with

$$E = \left\{ z = x + iy : |x| \leq b - \sqrt{a^2 - y^2},\ |y| \leq \tfrac{1}{2}a \right\}$$

a similar example can be carried out.

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References

[1] Erkama, T.: Quasiconformally homogeneous curves. - Michigan Math. J. 24, 1977, 157–159.

[2] Erkama, T.: Möbius automorphisms of plane domains. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 155–162.

[3] Gehring, F. W., and B. P. Palka: Quasiconformally homogeneous domains. - J. Anal. Math. 30, 1976, 172–199.

[4] Hjelle, G. A.: Quasidisks – examples and counterexamples. - Diploma Thesis, Norwegian University of Science and Technology (NTNU), 2002.

[5] Lehto, O., and K. I. Virtanen: Quasiconformal Mappings in the Plane. - Springer-Verlag, Berlin, 1973.

[6] Näkki, R., and B. P. Palka: Preprint.

[7] Sarvas, J.: Boundary of a homogeneous Jordan domain. - Ann. Acad. Sci. Fenn. Ser. A I Math. 10, 1985, 511–514.

[8] Teichmüller, O.: Ein Verschiebungssatz der quasikonformen Abbildung. - Deutsche Mathematik 7, 1944, 336–343.

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