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AN EARTHQUAKE VERSION OF THE JACKSON–ZYGMUND THEOREM

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Abstract. In this paper we present an analogy to the Jackson–Zygmund theorem on uniform approximability of Zygmund bounded continuous periodic functions by trigonometric polynomials. In our analogy the role of trigonometric polynomials is played by finite infinitesimal earthquakes.

1. Introduction

In his 1945 paper [10] Zygmund gave three proofs that the class Λ^* of real-valued continuous periodic functions f with the property

$$\left| f(x) - \frac{f(x+t)+f(x-t)}{2} \right| = O(t) \tag{1}$$

is invariant under the Hilbert transform. One of the proofs employed the Jackson–Zygmund theorem which gives a criterion for a continuous periodic function to belong to the class Λ^* in terms of the rate of its uniform approximability by trigonometric polynomials of a given degree. More precisely, assume f is continuous on $\mathbf{R}$ and $f(x+2\pi) = f(x)$. Let

$$\varepsilon_f(n) = \inf_{T_n} \|f - T_n\|_\infty, \tag{2}$$

where T_n is any trigonometric polynomial of degree less than or equal to n. That is, the infimum of the maximum difference between $f(x)$ and $T_n(x)$ is taken over all possible linear combinations T_n of the functions

$$1,\ \cos x,\ \sin x,\ \cos 2x,\ \sin 2x, \ldots, \cos nx,\ \sin nx.$$

Jackson–Zygmund theorem [10]. *A continuous periodic function f belongs to the class Λ^* if and only if there is a constant C such that for all $n \in \mathbf{N}$,*

$$\varepsilon_f(n) \le C/n. \tag{3}$$

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The necessity of this condition was shown by Jackson in [7] in 1912; he showed that the convolution of f with the Jackson kernel

$$K_n(x) = \left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^4$$

is a trigonometric polynomial T_n of degree $2n+1$ with the property that

$$\|f - T_n\|_\infty \le C/n.$$

Zygmund showed in [10] that condition (3) was also sufficient by using a scaling argument and the inequality $\|P_n''(x)\|_\infty \le n^2 \|P_n(x)\|_\infty$ for any trigonometric polynomial $P_n(x)$ of degree n. One can also find expository proofs of the Jackson–Zygmund theorem in [11] and [8, pp. 55–61].

In this paper we present an analogy to the Jackson–Zygmund theorem on uniform approximations of continuous periodic functions $f \in \Lambda^*$ by trigonometric polynomials. In our analogy the role of trigonometric polynomials is played by finite infinitesimal earthquakes, which are finite linear combinations of parabolic bump functions.

In order to state briefly our main result, we consider continuous periodic functions as continuous vector fields defined on the unit circle $\mathbf{S}^1 = \{z : |z| = 1\}$. Corresponding to such a real-valued, periodic function $\widetilde{V}$ with period 2π, the associated vector field V on $\mathbf{S}^1$ is given by the formula

$$V(e^{i\theta}) = \widetilde{V}(\theta) i e^{i\theta}. \tag{4}$$

V is said to be *Zygmund bounded* if

$$\left| V(e^{i\theta}) - \frac{V\big(e^{i(\theta+t)}\big) + V\big(e^{i(\theta-t)}\big)}{2} \right| \le Mt \tag{5}$$

for a constant $M > 0$ and for all $0 \le \theta < 2\pi$ and $0 < t < \pi$. Note that inequality (5) is equivalent to the existence of a constant $\widetilde{M}$ such that

$$\left| \widetilde{V}(\theta) - \frac{\widetilde{V}(\theta+t) + \widetilde{V}(\theta-t)}{2} \right| \le \widetilde{M} t, \tag{6}$$

and this condition coincides with condition (1) which defines the Zygmund class Λ^*.

A vector field $L(z)$ on $\mathbf{S}^1$ is said to be *trivial* if it is the initial tangent vector of a curve h_ε of Möbius transformations that preserves $\mathbf{S}^1$ with h_0 equal to the identity map. Such a curve shares the same initial vector of a curve given by the formula

$$\varepsilon \mapsto \left(z \mapsto \frac{(1+\varepsilon A) z + \varepsilon B}{\varepsilon \overline{B}\, z + (1 + \varepsilon \bar{A})} \right), \tag{7}$$

where A and B are arbitrary complex constants and $z \in \mathbf{S}^1$. The tangent vector at $\varepsilon = 0$ to the curve (7) is

$$L(z) = \big(-\overline{B}\, z^2 + (A - \bar{A})z + B\big).$$

If we transform L to $\tilde{L}$ by formula (4), that is, by the change of coordinate $z \mapsto \theta$ where $z = e^{i\theta}$, after a simplification we obtain

$$\tilde{L}(\theta) = \frac{L(e^{i\theta})}{ie^{i\theta}} = a_0 + a_1 \cos\theta + b_1 \sin\theta$$

for some real constants a_0, a_1 and b_1.

Let $S_n = \{e^{2\pi i k/2^n} : k = 0, 1, 2, \ldots, 2^n - 1\}$, which is a set of 2^n equally spaced points on $\mathbf{S}^1$. Consider the open unit disk $\mathbf{D}$ bounded by $\mathbf{S}^1$ as the hyperbolic plane. Let $\mathscr{L}_n$ denote a collection of finitely many non-intersecting geodesics in $\mathbf{D}$ connecting points in S_n to points in S_n and σ_n be a measure supported on $\mathscr{L}_n$ and assigning nonnegative weights to the geodesics in $\mathscr{L}_n$. For such a measure σ_n and a point z in $\mathbf{S}^1 = \partial\mathbf{D}$, we define

$$V^+_{\sigma_n}(z) = \iint E_{ab}(z)\, d\sigma_n(a,b) \quad \text{and} \quad V^-_{\sigma_n}(z) = -\iint E_{ab}(z)\, d\sigma_n(a,b), \tag{8}$$

where

$$E_{ab}(z) = \begin{cases} \dfrac{(z-a)(z-b)}{a-b} & \text{for } z \in [a,b], \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Here, for each geodesic line $\overline{ab}$ in $\mathbf{D}$ with endpoints a and b on $\mathbf{S}^1$, we denote by $[a,b]$ the shorter arc on $\mathbf{S}^1$ with endpoints a and b. We label the endpoints so that going from a to points in the interval $[a,b]$ to b is the counterclockwise direction. In the case that a and b are antipodal, we choose either of the two possible ways to label its endpoints by a and b and again we make going from a to points of $[a,b]$ to b be the counterclockwise direction. Then $V^+_{\sigma_n}$ and $V^-_{\sigma_n}$ define Zygmund bounded vector fields on $\mathbf{S}^1$. In fact, they are the initial tangent vectors to right and left earthquake curves (see [2]).

Let V be a continuous vector field on $\mathbf{S}^1$ and $V_n = V|_{S_n}$ be the restriction of V on S_n. By the so-called finite infinitesimal earthquake theorem in [1], for each $n \in \mathbf{N}$, there exists a measure σ_n^+ supported on a finite lamination $\mathscr{L}_n^+$ and a trivial vector field L_n^+ such that

$$V_n = (V^+_{\sigma_n^+} + L_n^+)|_{S_n}$$

and another measure σ_n^- supported on another finite lamination $\mathscr{L}_n^-$ and a trivial vector field L_n^- such that

$$V_n = (V^-_{\sigma_n^-} + L_n^-)|_{S_n}.$$

Define $\delta_V(n)$ by

$$\delta_V(n) = \|(V^+_{\sigma_n^+} + L_n^+) - (V^-_{\sigma_n^-} + L_n^-)\|_\infty. \tag{10}$$

In Sections 2 and 3, we prove the following main theorem.

Theorem 1. *A continuous vector field V on $\mathbf{S}^1$ is Zygmund bounded if and only if there exists a constant $C>0$ such that for all $n\in\mathbf{N}$,*

$$\delta_V(n)\le C/2^n, \tag{11}$$

where the constant C depends on the constant M which characterizes the Zygmund boundedness of V.

In the course of proving the necessity, we obtain

$$\|V-(V^+_{\sigma_n^+}+L_n^+)\|_\infty\le C/2^n \tag{12}$$

and

$$\|V-(V^-_{\sigma_n^-}+L_n^-)\|_\infty\le C/2^n. \tag{13}$$

Conversely, in Section 4 we give a counterexample to show that neither (12) nor (13) is sufficient by itself to imply that V is Zygmund bounded.

The supporting geodesics of σ_m^+ (resp. σ_m^-) have no relation to the supporting geodesics of any other σ_n^+'s (resp. σ_n^-'s). However, we show in Section 5 that it is possible to approximate Zygmund bounded vector fields on $\mathbf{S}^1$ in the L_∞-norm by finite sums of parabolic bump functions in such a way that the supports of the measures σ_n^+ are increasing sequences of nested sets and such that inequality (12) holds.

2. Proof of necessity

Let $S_n=\{e^{2\pi ik/2^n}: k=0,1,2,\dots,2^n-1\}$.

Lemma 1. *If a Zygmund bounded vector field V on $\mathbf{S}^1$ vanishes on S_n for some $n\in\mathbf{N}$, then*

$$\|V\|_\infty\le C/2^n$$

for a constant $C>0$ only depending on the Zygmund constant of V.

Proof. Denote by $t_{n+k}=1/2^{n+k}$, $k\in\mathbf{N}$.

Clearly, $\|V|_{S_n}\|_\infty=0$. For any $e^{2\pi ix}\in S_{n+1}\setminus S_n$, $e^{2\pi i(x\pm t_{n+1})}\in S_n$ and then $V\big(e^{2\pi i(x\pm t_{n+1})}\big)=0$. Hence

$$|V(e^{2\pi ix})|=\left|V(e^{2\pi ix})-\frac{V(e^{2\pi i(x+t_{n+1})})+V(e^{2\pi i(x-t_{n+1})})}{2}\right|\le Ct_{n+1}.$$

Therefore $\|V|_{S_{n+1}}\|_\infty\le Ct_{n+1}$.

Now for any $e^{2\pi ix} \in S_{n+2} \setminus S_{n+1}$, $e^{2\pi i(x \pm t_{n+2})} \in S_{n+1}$. Then

$$\begin{aligned}|V(e^{2\pi ix})| &\le \left| V(e^{2\pi ix}) - \frac{V(e^{2\pi i(x+t_{n+2})}) + V(e^{2\pi i(x-t_{n+2})})}{2} \right| \\ &\quad + \left| \frac{V(e^{2\pi i(x+t_{n+2})}) + V(e^{2\pi i(x-t_{n+2})})}{2} \right| \\ &\le Ct_{n+2} + \|V|_{S_{n+1}}\|_\infty \le Ct_{n+2} + Ct_{n+1}.\end{aligned}$$

Hence

$$\|V|_{S_{n+2}}\|_\infty \le C(t_{n+1} + t_{n+2}).$$

Inductively, we obtain

$$\|V|_{S_{n+k}}\|_\infty \le C \sum_{i=1}^{k} t_{n+i}.$$

Let $S = \bigcup_{n=1}^{\infty} S_n$. Then

$$\|V|_S\|_\infty \le C \sum_{i=1}^{\infty} t_{n+i} = \frac{C}{2^n}.$$

By the continuity of V and the density of S in $\mathbf{S}^1$, we obtain

$$\|V\|_\infty \le \frac{C}{2^n}. \quad \square$$

By a *finite earthquake measure* σ we mean a collection of nonnegative weights assigned to finitely many nonintersecting geodesics in the hyperbolic plane $\mathbf{D}$. Let σ be such a measure and $\mathscr{L}$ denote the collection of the finitely many geodesics which support σ. Let β be a closed hyperbolic geodesic segment in $\mathbf{D}$ with hyperbolic length $l(\beta)$ less than or equal to 1. We denote by $\sigma(\beta)$ the sum of the weights of the geodesics in $\mathscr{L}$ intersecting β. In [9], Thurston defines the *norm* of σ to be

$$\|\sigma\|_{\mathrm{Th}} = \sup_{l(\beta)\le 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta), \tag{14}$$

where β is a closed geodesic segment transversal to $\mathscr{L}$ and $l(\beta)$ denotes the hyperbolic length of β. As the same as defined in the introduction, we let

$$V_\sigma^+(x) = \iint E_{ab}(x)\, d\sigma(a,b) \quad \text{and} \quad V_\sigma^-(x) = -\iint E_{ab}(x)\, d\sigma(a,b). \tag{15}$$

Let V be a Zygmund bounded vector field on $\mathbf{S}^1$. One can introduce a cross-ratio norm to measure the Zygmund bound of V. Given a quadruple $Q =$

$\{a,b,c,d\}$ consisting of four points a, b, c, d on the unit circle $\mathbf{S}^1$ arranged in counterclockwise order, we denote by

$$\operatorname{cr}(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} \tag{16}$$

and

$$V[Q] = \frac{V(b)-V(a)}{b-a} - \frac{V(c)-V(b)}{c-b} + \frac{V(d)-V(c)}{d-c} - \frac{V(d)-V(a)}{d-a}. \tag{17}$$

Then the *cross-ratio norm* $\|V\|_{\mathrm{cr}}$ is defined to be

$$\|V\|_{\mathrm{cr}} = \sup_{\operatorname{cr}(Q)=1} |V[Q]|. \tag{18}$$

It is easy to see that $\|V_\sigma^+\|_{\mathrm{cr}} = \|V_\sigma^-\|_{\mathrm{cr}}$. Therefore, we simply use V_σ to denote either V_σ^+ or V_σ^-.

By Theorem 4.1 in [1] or Theorem 4 in Section 18.4 in [3], $\|V_\sigma\|_{\mathrm{cr}} \le C\|\sigma\|_{\mathrm{Th}}$ for a constant $C>0$. In [4] and [5], it is shown that the Thurston norm of σ is equivalent to the cross-ratio norm of V_σ.

Norm-equivalence theorem. *There exists a universal constant $C>0$ such that for any finite earthquake measure σ,*

$$\frac{1}{C}\|\sigma\|_{\mathrm{Th}} \le \|V_\sigma\|_{\mathrm{cr}} \le C\|\sigma\|_{\mathrm{Th}}.$$

Now we begin the proof of the necessity part of Theorem 1.

Proof. Let V be a Zygmund bounded vector field on $\mathbf{S}^1$ and $V_n = V|_{S_n}$. Let σ_n^+, σ_n^-, L_n^+ and L_n^- be the same as defined in the introduction. By the same method in [5] used to show $\|\sigma\|_{\mathrm{Th}} \le C\|V_\sigma\|_{\mathrm{cr}}$ in the norm-equivalence theorem, one can show that there exists a universal constant $C>0$ such that for all n,

$$\|\sigma_n^+\|_{\mathrm{Th}} \le C\|V\|_{\mathrm{cr}} \quad \text{and} \quad \|\sigma_n^-\|_{\mathrm{Th}} \le C\|V\|_{\mathrm{cr}}.$$

Then the norm-equivalence theorem implies

$$\|V^+_{\sigma_n^+}\|_{\mathrm{cr}} \le C\|\sigma_n^+\|_{\mathrm{Th}} \le C'\|V\|_{\mathrm{cr}}.$$

Similarly, we have

$$\|V^-_{\sigma_n^-}\|_{\mathrm{cr}} \le C''\|V\|_{\mathrm{cr}}.$$

Therefore both $V^+_{\sigma_n^+}$ and $V^-_{\sigma_n^-}$, and hence $V^+_{\sigma_n^+} + L_n^+$ and $V^-_{\sigma_n^-} + L_n^-$, are Zygmund bounded with their bounds only depending on the Zygmund constant of V. The proof of necessity follows by applying Lemma 1 to the difference $(V^+_{\sigma_n^+} + L_n^+) - (V^-_{\sigma_n^-} + L_n^-)$. □

3. Proof of sufficiency

Again let $S_n = \{e^{2\pi ik/2^n} : k = 0, 1, 2, \dots, 2^n - 1\}$.

Proposition 1. *A continuous vector field V on $\mathbf{S}^1$ is Zygmund bounded if and only if there exists a constant $C > 0$ such that for any $n \in \mathbf{N}$ and any triple $\{x - t, x, x + t\}$ of three consecutive points in S_n,*

$$\left| V(x) - \frac{V(x-t) + V(x+t)}{2} \right| \le C|t|.$$

Proof. The necessity of the statement is obvious. To prove the sufficiency, we must show that if there exists a constant $C > 0$ such that for any $n \in \mathbf{N}$ and any triple $\{x - t, x, x + t\}$ of three consecutive points in S_n,

$$\left| V(x) - \frac{V(x-t) + V(x+t)}{2} \right| \le C|t|, \tag{19}$$

then V is Zygmund bounded.

Let $S = \bigcup_{n=1}^{\infty} S_n$. Since S is a dense subset of $\mathbf{S}^1$, by the continuity of V it is sufficient to show for any symmetric triple $\{x - t, x, x + t\}$ of three points in S, inequality (19) holds.

Let $f(x) = V(e^{2\pi ix})/ie^{2\pi ix}$. Then f is a periodic continuous function on $\mathbf{R}$ with period 1. Let $B_n = \{k/2^n : k = 0, 1, 2, \dots, 2^n\}$ and $B = \bigcup_{n=1}^{\infty} B_n$. It is sufficient to show that for any symmetric triple $\{x - t, x, x + t\}$ contained in B, f satisfies

$$\left| \frac{f(x+t) - f(x)}{t} - \frac{f(x) - f(x-t)}{t} \right| < C_1 \tag{20}$$

for a constant $C_1 > 0$.

Let n be the smallest integer such that the interval $[x-t, x+t]$ contains three consecutive points in B_n, and denote them by $x_n - t_n$, x_n and $x_n + t_n$. Without loss of generality, we may assume $x - t < x_n - t_n < x < x_n < x_n + t_n < x + t$. Then there exist integers $n < n_1 < n_2 < \cdots < n_l$, $n < m_1 < m_2 < \cdots m_j$ and $n \le k_1 < k_2 < \cdots < k_i$ such that

$$x + t = x_n + t_n + \sum_{s=1}^{l} \frac{1}{2^{n_s}}, \quad x = x_n - \sum_{s=1}^{j} \frac{1}{2^{m_s}} \quad \text{and} \quad x - t = x_n - t_n - \sum_{s=1}^{i} \frac{1}{2^{k_s}}.$$

Let $x_{n_0} = x_n + t_n$ and

$$x_{n_s} = x_n + t_n + \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \cdots + \frac{1}{2^{n_s}}$$

for each $1 \le s \le l$. Denote by

$$r_R = r_{n_0} = \frac{f(x_n + t_n) - f(x_n)}{t_n} \quad \text{and} \quad r_{n_s} = \frac{f(x_{n_s}) - f(x_{n_{s-1}})}{x_{n_s} - x_{n_{s+1}}}$$

for each $1 \le s \le l$.

Sublemma. *Let* $x = -1/2^m$, $y = 0$ *and* $z = 1/2^n$ *with* $m < n$. *Then*

$$\left|\frac{f(z)-f(y)}{z-y} - \frac{f(y)-f(x)}{y-x}\right| \leq (n-m+2)C,$$

where C *is the constant satisfying the inequality* (19).

Proof. Let $z_k = 2^k z$ for each $1 \leq k \leq n-m$. Since y, z and z_1 satisfy (19),

$$|f(y)+f(z_1)-2f(z)| \leq 2C(z-y).$$

Hence

$$\left|\frac{f(z_1)-f(y)}{2(z-y)} - \frac{f(z)-f(y)}{z-y}\right| \leq C,$$

that is,

$$\left|\frac{f(z)-f(y)}{z-y} - \frac{f(z_1)-f(y)}{z_1-y}\right| \leq C.$$

By the same reasoning, for each $1 \leq k \leq n-m-1$,

$$\left|\frac{f(z_k)-f(y)}{z_k-y} - \frac{f(z_{k+1})-f(y)}{z_{k+1}-y}\right| \leq C.$$

By the triangle inequality,

$$\left|\frac{f(z)-f(y)}{z-y} - \frac{f(z_{n-m})-f(y)}{z_{n-m}-y}\right| \leq (n-m)C.$$

Clearly,

$$\left|\frac{f(z_{n-m})-f(y)}{z_{n-m}-y} - \frac{f(y)-f(x)}{y-x}\right| \leq 2C.$$

The last two inequalities imply the sublemma. □

Let $C_3 = 3C$. By the previous sublemma, for each $1 \leq s \leq l$,

$$|r_{n_s} - r_{n_{s-1}}| \leq (n_s - n_{s-1} + 2)C \leq 3(n_s - n_{s-1})C = C_3(n_s - n_{s-1}),$$

and hence

$$|r_{n_s} - r_R| = |r_{n_s} - r_{n_0}| \leq C_3(n_s - n_0) = C_3(n_s - n),$$

where $n_0 = n$.

Similarly, let $x_{m_0} = x_n$ and

$$x_{m_s} = x_n - \left(\frac{1}{2^{m_1}} + \frac{1}{2^{m_2}} + \cdots + \frac{1}{2^{m_s}}\right)$$

for each $1 \le s \le j$. Denote by

$$r'_{m_0} = r_R \quad \text{and} \quad r'_{m_s} = \frac{f(x_{m_{s-1}}) - f(x_{m_s})}{x_{m_{s-1}} - x_{m_s}}$$

for each $1 \le s \le j$. By the same reasoning, we have for each $1 \le s \le j$,

$$|r'_{m_s} - r'_{m_{s-1}}| \le C_3(m_s - m_{s-1}),$$

and hence

$$|r'_{m_s} - r_R| = |r'_{m_s} - r'_{m_0}| \le C_3(m_s - m_0) = C_3(m_s - n),$$

where $m_0 = n$.

Clearly, $x_{m_j} = x$ and $x_{n_l} = x + t$. Then

$$\begin{aligned} f(x+t) - f(x) &= \sum_{s=1}^{j} [f(x_{m_{s-1}}) - f(x_{m_s})] \\ &\quad + [f(x_n + t_n) - f(x_n)] + \sum_{s=1}^{l} [f(x_{n_s}) - f(x_{n_{s-1}})] \\ &= \sum_{s=1}^{j} r'_{m_s}(x_{m_{s-1}} - x_{m_s}) + r_R t_n + \sum_{s=1}^{l} r_{m_s}(x_{n_s} - x_{n_{s+1}}) \\ &= \sum_{s=1}^{j} (r'_{m_s} - r_R)(x_{m_{s-1}} - x_{m_s}) \\ &\quad + \sum_{s=1}^{l} (r_{m_s} - r_R)(x_{n_s} - x_{n_{s+1}}) + r_R t, \end{aligned}$$

where

$$t = t_n + \sum_{s=1}^{j} (x_{m_{s-1}} - x_{m_s}) + \sum_{s=1}^{l} (x_s - x_{s+1}) = \frac{1}{2^n} + \sum_{s=1}^{j} \frac{1}{2^{m_s}} + \sum_{s=1}^{l} \frac{1}{2^{n_s}}.$$

Therefore

$$\begin{aligned} &\left| \frac{f(x+t) - f(x)}{t} - r_R \right| \\ &\le \frac{\sum_{s=1}^{j} |r'_{m_s} - r_R|(x_{m_{s-1}} - x_{m_s}) + \sum_{s=1}^{l} |r_{m_s} - r_R|(x_s - x_{s+1})}{t} \\ &\le C_3 \frac{\sum_{s=1}^{j} \dfrac{m_s - n}{2^{m_s}} + \sum_{s=1}^{l} \dfrac{n_s - n}{2^{n_s}}}{\dfrac{1}{2^n} + \sum_{s=1}^{j} \dfrac{1}{2^{m_s}} + \sum_{s=1}^{l} \dfrac{1}{2^{n_s}}} \\ &= C_3 \frac{\sum_{s=1}^{j} \dfrac{m_s - n}{2^{m_s - n}} + \sum_{s=1}^{l} \dfrac{n_s - n}{2^{n_s - n}}}{1 + \sum_{s=1}^{j} \dfrac{1}{2^{m_s - n}} + \sum_{s=1}^{l} \dfrac{1}{2^{n_s - n}}}. \end{aligned}$$

Since the series $\sum_{n=1}^{\infty} n/2^n$ converges, the numerator of the above quotient is bounded. Therefore there exists a constant $C_4 > 0$ such that

$$\left| \frac{f(x+t) - f(x)}{t} - r_R \right| \le C_4.$$

Similarly, we can also show

$$\left| \frac{f(x) - f(x-t)}{t} - r_L \right| \le C_5$$

for a constant $C_5 > 0$, where $r_L = \big(f(x_n) - f(x_n - t_n)\big)/(t_n)$. Since $|r_R - r_L| \le C_1$, we obtain

$$\left| \frac{f(x+t) - f(x)}{t} - \frac{f(x) - f(x-t)}{t} \right| \le C_4 + C_5 + C_1,$$

which completes the proof. □

Now we briefly recall the procedure in [1] for constructing the infinitesimal left earthquake V_σ associated to a vector field V defined on a finite subset A of $\mathbf{S}^1$. For convenience, we work with the real line $\mathbf{R}$ and the upper half plane $\mathbf{H}$ instead of the circle $\mathbf{S}^1$ and the unit open disk $\mathbf{D}$. Assume that A is a subset of $\mathbf{R}$ and V takes values only at the points in A. This procedure yields a finite lamination $\mathscr{L}$ consisting of non-intersecting hyperbolic geodesics l in the upper half plane $\mathbf{H}$ together with nonnegative weights ϱ_l associated to each geodesic l in $\mathscr{L}$.

To describe this procedure we refer to the example illustrated in Figure 1, which treats a case where $A \cup \{\infty\}$ consists of 9 points. We will need the formulae for the parabolic bump function $E_{ab}(x)$ given similarly in (9) as

$$E_{ab}(x) = \begin{cases} \dfrac{(x-a)(x-b)}{a-b} & \text{for } a \le x \le b, \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

and the special cases

$$E_{a\infty}(x) = \begin{cases} x - a & \text{for } a \le x \le \infty, \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

and

$$E_{-\infty b}(x) = \begin{cases} -(x - b) & \text{for } -\infty \le x \le b, \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

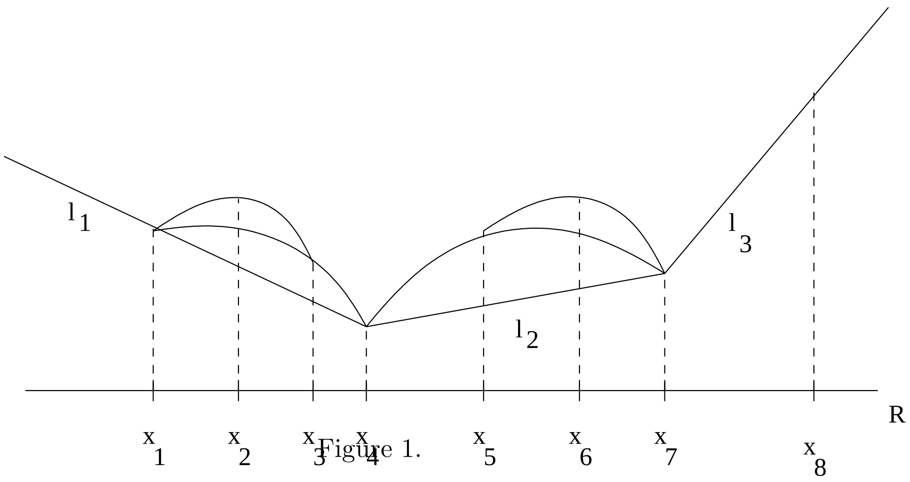

Figure 1.

The procedure consists of the following three steps.

Step 1. Draw a line l_1 which meets the graph of V over A at the point furthest to the left, which meets the graph of V at at least one more point p and which lies on or below the graph. In Figure 1, this is the line l_1 and $p = x_4$.

Step 2. If p is the furthest right point on the graph of V on A, then stop. Otherwise draw a line l_2 which meets the graph of V at the point $\bigl(p, V(p)\bigr)$, meets the graph of V at at least one more point q and which lies on or below the graph of V. Replace p by q, continue this step inductively until p is the furthest right point on the graph of V on A. Then we obtain a finite sequence of the line segments $l_1, l_2, \dots, l_k$. In Figure 1 there are three such line segments, l_1, l_2 and l_3.

Step 3. Let $R(x)$ be the piecewise linear function whose graph consists of these line segments. Over each linear piece of the graph of $R(x)$ add a function $\varrho_{ab}E_{ab}(x)$ where a and b are the left and right endpoints of the linear piece, and $\varrho_{ab} \geq 0$ and is as large as possible so that the graph of $R(x) + \varrho_{ab}E_{ab}(x)$ lies on or below the graph of V.

Continue inductively to add parabolic bump functions $\varrho_{a_j b_j}E_{a_j b_j}(x)$ in the above way until the graph of the resulting function passes through all points on the graph of V. Now we obtain a function $\widetilde{V}(x)$ of the form

$$\widetilde{V}(x) = \sum_{a_j b_j} \varrho_{a_j b_j} E_{a_j b_j}(x) + B(x) \tag{24}$$

where (a_j, b_j)'s are pairs of points in $A \cup \{\infty\}$, the hyperbolic goedesics l_j with endpoints a_j and b_j do not intersect, and $B(x)$ is affine. Moreover, it satisfies

(i) $\widetilde{V}(x_j) = V(x_j)$ for each x_j in A and

(ii) the graph of each parabolic or linear segment of $\widetilde{V}$, if extended, lies entirely on or below the graph of V.

In the example depicted in Figure 1, the weight assigned to the line connecting x_4 to ∞ is equal to the slope of l_2 minus the slope of l_1 and the weight assigned to the line connecting x_7 to ∞ is equal to the slope of l_3 minus the slope of l_2. All weights are given by the numbers $\varrho_{a_j b_j}$ in the sum (24). For example, the weight assigned to the geodesic connecting x_4 to x_7 is the coefficient of $E_{x_4 x_7}$ in the sum (24).

Figure 2 is the translation of Figure 1 from the upper half plane to the unit disk.

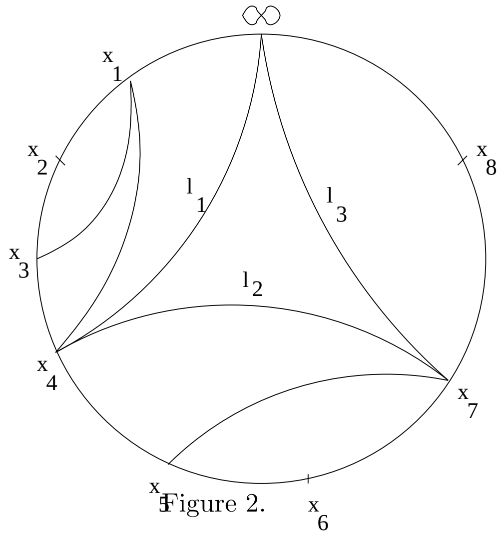

Figure 2.

We obtain the following proposition.

Proposition 2. *Assume that f is a function defined on a set A consisting of finitely many points on the real line $\mathbf{R}$. Suppose that E is the unique finite infinitesimal left (resp. right) earthquake on $\mathbf{H}$ which coincides with f on A. For any two adjacent points X and Y on the graph of f, let E' be the collection of the summands of E which contribute values at X and/or Y, and Q be the sum of the summands in E'. Then the graph of Q passes through X and Y, and all other points on the graph of f are not below (resp. above) the graph of Q.*

In order to prove the sufficiency of the condition in Theorem 1, we also need the following elementary lemmas and corollaries.

Lemma 2. *Given $t > 0$ and $h > 0$, let $X = (-t, 0)$, $Y = (0, h)$ and $Z = (t, 0)$. Suppose that $f(x) = ax^2 + bx + c$ is a quadratic polynomial function with $a < 0$ such that X and Y are on the graph of f and Z is not below the graph. Let $L(x)$ be the piecewise linear function obtained by connecting X to Y and Y to Z. Then*

$$(f - L)\left(-\tfrac{1}{2}t\right) \geq \tfrac{1}{4}h.$$

Proof. Clearly, $L(x) = (h/t)x + h$ for $x \leq 0$ and

$$f(x) = ax(x + t) + \frac{h}{t}x + h.$$

Since $f(t) = 2at^2 + 2h \le 0$, $-at^2 \ge h$. Then

$$(f - L)\big(\tfrac{1}{2}(-t)\big) = -\tfrac{1}{2}at\big(\tfrac{1}{2}(-t) + t\big) = -\tfrac{1}{4}at^2 \ge \tfrac{1}{4}h. \ \square$$

Corollary 1. *Suppose that $t > 0$ and $y - \frac{1}{2}(x+z) = h > 0$, and let $X = (-t, x)$, $Y = (0, y)$ and $Z = (t, z)$. Under the same notation and assumptions as in the previous lemma 2, we also have*

$$(f - L)\big(-\tfrac{1}{2}t\big) \ge \tfrac{1}{4}h.$$

Similarly, we obtain

Lemma 3. *Given $t > 0$ and $h < 0$, let $X = (-t, 0)$, $Y = (0, h)$ and $Z = (t, 0)$. Suppose that $f(x) = ax^2 + bx + c$ is a quadratic polynomial function with $a > 0$ such that X and Y are on the graph of f and Z is not above the graph. Let $L(x)$ be the piecewise linear function obtained by connecting X to Y and Y to Z. Then*

$$(L - f)\big(-\tfrac{1}{2}t\big) \ge -\tfrac{1}{4}h.$$

Corollary 2. *Suppose that $t > 0$ and $y - \frac{1}{2}(x+z) = h < 0$, and let $X = (-t, x)$, $Y = (0, y)$ and $Z = (t, z)$. Under the same notation and assumptions in the previous lemma 3, we also have*

$$(L - f)\big(-\tfrac{1}{2}t\big) \ge -\tfrac{1}{4}h.$$

To show the sufficiency of the condition in Theorem 1, we proceed by making a contradiction.

Proof. Let V be a continuous vector field on $\mathbf{S}^1$ satisfying the condition (11) in our Theorem 1. Suppose that V is not Zygmund bounded. By Proposition 1, there exists an infinite sequence $\{T_n\}_{n=1}^{\infty}$ of symmetric triples $T_{n-k} = \{x_n - t_n, x_n, x_n + t_n\}$ in S such that $t_n \to 0$ and

$$\left| \left(V(x_n) - \frac{V(x_n - t_n) + V(x_n + t_n)}{2} \right) \Big/ \, t_n \right| \to +\infty$$

as $n \to \infty$. Therefore, by passing to a subsequence if necessary, we may assume

$$C_n = \left(V(x_n) - \frac{V(x_n - t_n) + V(x_n + t_n)}{2} \right) \Big/ \, t_n \to +\infty$$

as $n \to \infty$. We may also assume that $C_n > 0$ for all n.

For the convenience in estimates, we first convert the vector field V to the vector fields $\widetilde{V}_n$ on the real line $\mathbf{R}$ as follows.

Let $B_n(z) = x_n(i-z)/(i+z)$. Then B_n is an orientation-preserving Möbius transformation mapping $\mathbf{H}$ onto $\mathbf{D}$, i to the center of $\mathbf{D}$, the origin of $\mathbf{H}$ to x_n and $|B_n'(0)| = 2$. Denote by

$$\widetilde{V}_n(x) = \frac{V\big(B_n(x)\big)}{B_n'(x)}\Big|_{B_n^{-1}(S_n)}. \tag{25}$$

Now let $\tilde{\sigma}_n^+ = (B_n^{-1})^*\sigma_n^+$ (resp. $\tilde{\sigma}_n^-$) be the pushforward of σ_n^+ (resp. σ_n^-) by B_n^{-1}. Denote by

$$V_{\tilde{\sigma}_n}^+(x) = \iint E_{ab}(x)\,d\tilde{\sigma}_n^+(a,b) \quad \text{and} \quad V_{\tilde{\sigma}_n}^-(x) = -\iint E_{ab}(x)\,d\tilde{\sigma}_n^-(a,b), \tag{26}$$

where E_{ab} is defined by (21). It is proved in Lemma 1 in [5] that

$$V_{\tilde{\sigma}_n}^+(x) = \frac{V_{\sigma_n}^+\big(B_n(x)\big)}{B_n'(x)}. \tag{27}$$

Similarly,

$$V_{\tilde{\sigma}_n}^-(x) = \frac{V_{\sigma_n}^-\big(B_n(x)\big)}{B_n'(x)}. \tag{28}$$

Corresponding to a trivial vector field on $\mathbf{S}^1$, a trivial vector field on $\mathbf{R}$ is just an affine map. Then by the finite infinitesimal earthquake theorem in [1] (see the introduction), there are affine maps $\tilde{L}_n^+$ and $\tilde{L}_n^-$ such that

$$\widetilde{V}_n = (V_{\tilde{\sigma}_n}^+ + \tilde{L}_n^+)|_{B_n^{-1}(S_n)} = (V_{\tilde{\sigma}_n}^- + \tilde{L}_n^-)|_{B_n^{-1}(S_n)}.$$

Clearly, B_n^{-1} maps a symmetric triple T_n to a triple $\widetilde{T}_n = \{-\tilde{t}_n, 0, \tilde{t}_n\}$. Since $|(B_n^{-1})'(x_n)| = 1/|B_n'(0)| = \frac{1}{2}$, all (B_n^{-1})'s are almost linear maps in small neighborhoods of x_n with the same asymptotic rescaling. Therefore as $n \to +\infty$,

$$\widetilde{C}_n = \left[\widetilde{V}_n(0) - \frac{\widetilde{V}_n(-\tilde{t}_n) + \widetilde{V}_n(\tilde{t}_n)}{2}\right] \Big/ \tilde{t}_n \to +\infty.$$

Let $X_n = \big(-\tilde{t}_n, \widetilde{V}_n(-\tilde{t}_n)\big)$, $Y_n = \big(0, \widetilde{V}_n(0)\big)$ and $Z_n = \big(\tilde{t}_n, \widetilde{V}_n(\tilde{t}_n)\big)$. Let E_n be the collection of all quadratic polynomials comprising $V_{\tilde{\sigma}_n}^+$ and E_n' the collection of the quadratic polynomials in E_n which contribute values at X_n and/or Y_n. Now let Q_n^+ be the sum of all polynomials in E_n', then by Proposition 2, Z_n is not below the graph of Q_n^+. Assume that L_n is the linear approximation of $V_{\tilde{\sigma}_n}^+$ on $\widetilde{T}_n$. By Corollary 1,

$$(Q_n^+ - L_n)\big(-\tfrac{1}{2}\tilde{t}_n\big) \ge \tfrac{1}{4}\tilde{C}_n\tilde{t}_n > 0.$$

On the interval $[-\tilde{t}_n, 0]$, $Q^+ = V^+_{\tilde{\sigma}_n}$ and then

$$\big[(V^+_{\tilde{\sigma}_n} + \tilde{L}^+_n) - (L_n + \tilde{L}^+_n)\big]\big(-\tfrac{1}{2}\tilde{t}_n\big) = (Q^+_n - L_n)\big(-\tfrac{1}{2}\tilde{t}_n\big) \geq \tfrac{1}{4}\tilde{C}_n\tilde{t}_n.$$

On the other hand, the graph of the right finite infinitesimal earthquake $V^-_{\tilde{\sigma}_n} + \tilde{L}^-_n$ is not above the graph of $L_n + \tilde{L}^+_n$ on the interval $[-\tilde{t}_n, 0]$ since $L_n + \tilde{L}^+_n$ is equal to the linear approximation of $\widetilde{V}_n$ on this interval. Hence

$$\big[(L_n + \tilde{L}^+_n) - (V^-_{\tilde{\sigma}_n} + \tilde{L}^-_n)\big]\big(-\tfrac{1}{2}\tilde{t}_n\big) \geq 0.$$

Therefore

$$\big[(V^+_{\tilde{\sigma}_n} + \tilde{L}^+_n) - (V^-_{\tilde{\sigma}_n} + \tilde{L}^-_n)\big]\big(-\tfrac{1}{2}\tilde{t}_n\big) \geq \tfrac{1}{4}\tilde{C}_n\tilde{t}_n.$$

Since all B_n's are almost linear maps in small neighborhoods of the origin with the same asymptotic rescaling,

$$\lim_{n\to\infty} \frac{\tilde{t}_n}{t_n} \to 2.$$

By (27) and (28),

$$\big[(V^+_{\sigma_n}+L^+_n)-V^-_{\sigma_n}+L^-_n)\big]\big(B_n\big(-\tfrac{1}{2}\tilde{t}_n\big)\big) = B'_n\big(-\tfrac{1}{2}\tilde{t}_n\big)\big[(V^+_{\tilde{\sigma}_n}+\tilde{L}^+_n)-(V^-_{\tilde{\sigma}_n}+\tilde{L}^-_n)\big]\big(-\tfrac{1}{2}\tilde{t}_n\big).$$

As $n \to \infty$,

$$\big|B'_n\big(-\tfrac{1}{2}\tilde{t}_n\big)\big| \to 2 \quad \text{and} \quad \widetilde{C}_n \to +\infty,$$

and therefore

$$\frac{\|(V^+_{\sigma_n} + L^+_n) - V^-_{\sigma_n} + L^-_n)\|_\infty}{t_n} \geq \frac{1}{4}\left|B'_n\left(-\frac{\tilde{t}_n}{2}\right)\right|\frac{\tilde{t}_n}{t_n}\widetilde{C}_n \to +\infty,$$

which contradicts with the condition (11). This completes the proof of sufficiency. □

4. Counterexamples

In this section we first give an example of a Zygmund unbounded vector field V on $\mathbf{S}^1$ which can be well approximated by its linear approximations on binary points. Then we show that neither the condition (12) nor the condition (13) on the approximations of V is sufficient by itself to imply that V is Zygmund bounded. For simplicity, we will construct such examples of V on the real line.

Proposition 3. *There exists a Zygmund unbounded continuous function on* $\mathbf{R}$ *which can be approximated as well as in* (12) *by its piecewise linear approximations on binary points.*

Proof. For each $n \in \mathbf{N}$, define

$$f_n(x) = \begin{cases} \dfrac{1}{2^n} - \left|x - \dfrac{1}{2^n}\right| & \text{for } \left|x - \dfrac{1}{2^n}\right| \le \dfrac{1}{2^n}, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Let $V = \sum_{n=1}^{\infty} f_n$, $S_n = \{k/2^n : k \in \mathbf{Z}\}$ and $S = \bigcup_{n=0}^{\infty} S_n$. Suppose that V_n is the piecewise linear approximation of V on S_n. Then

$$\|V - V_n\|_\infty \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}.$$

This means that V can be approximated by V_n's as well as in (12). On the other hand,

$$\frac{\left|V(0) - \frac{1}{2}\big(V(-t_n) + V(t_n)\big)\right|}{t_n} = \frac{V(t_n)}{t_n} = n \to \infty$$

as $n \to \infty$, where $t_n = 1/2^n$. Hence V is not Zygmund bounded. □

Proposition 4. *There is a Zygmund unbounded continuous function V on $\mathbf{R}$ and an infinite sequence $\{(\sigma_n, \mathscr{L}_n)\}_{n=1}^{\infty}$ of earthquake measures σ_n supported on the laminations $\mathscr{L}_n$ consisting of finitely many geodesics such that*

(i) *for each n and each geodesic $\overline{ab} \in \mathscr{L}_n$, $|a - b| \ge 1/2^n$;*
(ii) *for each n, $\mathscr{L}_n \subset \mathscr{L}_{n+1}$; and*
(iii) *$\|V - V_{\sigma_n}\|_\infty \le C/2^n$ for a constant $C > 0$ independent of n.*

Proof. Let $\mathscr{L}$ be the lamination consisting of all geodesics l_n connecting the origin to $1/2^n$ for all $n \in \{0\} \cup \mathbf{N}$ and assume the earthquake measure σ has the weight 1 on each geodesic in $\mathscr{L}$.

For each $n \in \mathbf{N}$, let $\mathscr{L}_n$ be the lamination consisting of all geodesics $\overline{ab} \in \mathscr{L}$ with $|a - b| \ge 1/2^n$ and σ_n be the restriction of σ on $\mathscr{L}_n$.

Let $V = V_\sigma^+$ and $V_n = V_{\sigma_n}^+$. By the norm-equivalence theorem, V_n is a Zygmund bounded vector field on $\overline{\mathbf{R}}$ for each $n \in \mathbf{N}$ since $\|\sigma_n\|_{\mathrm{Th}} < n$. It is easy to see that $V(x)$ is continuous at any $x \ne 0$; and the continuity of $V(x)$ at $x = 0$ is implied by the following that

$$0 < V\left(\frac{1}{2^n}\right) \le \frac{n}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Now we estimate $\|V - V_n\|_\infty$. Clearly, for each $k \in \{0\} \cup \mathbf{N}$, $\|V_{\sigma|_{l_k}}\|_\infty = \frac{1}{4}(1/2^k)$. Then

$$|V(x) - V_n(x)| = \left|\sum_{k=n+1}^{\infty} V_{\sigma|_{l_k}}(x)\right| \le \sum_{k=n+1}^{\infty} \|V_{\sigma|_{l_k}}\|_\infty = \frac{1}{4}\sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{4}\frac{1}{2^n}.$$

Hence

$$\|V - V_n\|_\infty \le \frac{C}{2^n} \tag{30}$$

for a constant $C > 0$ independent of n.

It remains to check that V is not Zygmund bounded. This is true because of the norm-equivalence theorem for general earthquakes in the next section, or we can prove it as follows.

By the norm-equivalence theorem for finite earthquakes, $\|V_n\|_{\mathrm{cr}} \to \infty$ since $\|\sigma_n\|_{\mathrm{Th}} = n \to \infty$ as $n \to \infty$. From (30), we can show $\|V\|_{\mathrm{cr}} = \infty$. □

The same example constructed in the proof of the previous proposition implies

Proposition 5. *There exists a Zygmund unbounded continuous function V on $\mathbf{R}$ and an infinite sequence $\{(\sigma_n, \mathscr{L}_n)\}_{n=1}^\infty$ of earthquake measures supported on the laminations $\mathscr{L}_n$ consisting of finitely many geodesics such that*

(1) *for each n and each geodesic $\overline{ab} \in \mathscr{L}_n$, $|a - b| \ge 1/n$;*
(2) *for each n, $\mathscr{L}_n \subset \mathscr{L}_{n+1}$; and*
(3) *$\|V - V_{\sigma_n}\|_\infty \le C/n$ for a constant $C > 0$ independent of n.*

Proof. We use the same notation in the proof of the previous proposition except we let $\mathscr{L}_n$ be the lamination consisting of all geodesics $\overline{ab} \in \mathscr{L}$ with $|a - b| \ge 1/n$ and σ_n be the restriction of σ on $\mathscr{L}_n$. We only need to check

$$\|V - V_n\|_\infty \le \frac{C}{n}$$

for a constant $C > 0$ independent of n.

For each $n \in \mathbf{N}$, there exists $m \in \mathbf{N}$ such that $2^{m-1} \le n < 2^m$. Then

$$|V(x) - V_n(x)| = \left|\sum_{k=m}^\infty V_{\sigma|_{l_k}}(x)\right| \le \sum_{k=m}^\infty \|V_{\sigma|_{l_k}}\|_\infty = \frac{1}{4}\sum_{k=m}^\infty \frac{1}{2^k} = \frac{1}{2^{m+1}} < \frac{1}{2}\frac{1}{n}.$$

This implies the previous inequality. □

5. Nested laminations

In our main theorem the supporting geodesics of σ_m^+ (resp. σ_m^-) have no relation with the supporting geodesics of any other σ_n^+ (resp. σ_n^-). In this section we introduce a method to approximate uniformly Zygmund bounded vector fields on $\mathbf{S}^1$ by finite sums of parabolic bump functions in such a way that the supports of the measures are increasing sequences of nested sets.

A *geodesic lamination* $\mathscr{L}$ in the hyperbolic plane $\mathbf{D}$ is a collection of geodesics which foliate a closed subset L of $\mathbf{D}$. Here L is called *the locus* of $\mathscr{L}$, the geodesics are called the *leaves* of $\mathscr{L}$, the connected components of $\mathbf{D}\setminus L$ are called the *gaps*, and the gaps and the leaves of $\mathscr{L}$ are called the *strata* of $\mathscr{L}$.

Let $\mathbf{S}^1$ denote the boundary circle of $\mathbf{D}$, and $\mathbf{X}$ the space $\mathbf{S}^1 \times \mathbf{S}^1 \setminus \{$the diagonal$\}$ factorized by the equivalence relation $(a,b) \sim (b,a)$. A Borel measure σ defined on $\mathbf{X}$ is called *an earthquake measure* if there is a lamination $\mathscr{L}$ such that σ is supported on the pairs of the endpoints of the leaves in $\mathscr{L}$.

The Thurston norm of an earthquake measure σ can be defined as the same as the Thurston norm of finite earthquake measures in Section 2. More precisely, for any closed hyperbolic geodesic segment β in $\mathbf{D}$ of hyperbolic length ≤ 1, let l_1 and l_2 be the two geodesics which bound all geodesics in $\mathscr{L}$ intersecting β and suppose the stripe S bounded by l_1 and l_2 in the unit disk is of the form $[a,b] \times [c,d]$, where a, d and b, c are the endpoints of l_1 and l_2, respectively, and a, b, c, d are arranged on $\mathbf{S}^1$ in the counterclockwise order. Denote by

$$\sigma(\beta) = \sigma([a,b] \times [c,d]).$$

The Thurston norm of σ is defined to be

$$\|\sigma\|_{\mathrm{Th}} = \sup_{l(\beta) \leq 1} \sigma(\beta) = \sup_{l(\beta)=1} \sigma(\beta), \tag{31}$$

where β is a closed geodesic segment transversal to $\mathscr{L}$ and $l(\beta)$ denotes the hyperbolic length of β. Let $\mathscr{M}$ be the collection of all earthquake measures defined on $\mathbf{X}$. Any measure $\sigma \in \mathscr{M}$ is called *Thurston bounded* if it has finite Thurston norm.

Given $\sigma \in \mathscr{M}$, also denote by

$$V_\sigma(x) = E(\sigma)(x) = \iint E_{ab}(x)\, d\sigma(a,b), \tag{32}$$

where $E_{ab}(x) = \big((x-a)(x-b)\big)/(a-b)$ if $x \in [a,b]$, and otherwise $E_{ab}(x) = 0$. Then V_σ defines a tangent vector field on $\mathbf{S}^1$. The following two theorems hold in general.

Infinitesimal earthquake theorem ([1]). *If σ is Thurston bounded then V_σ is Zygmund bounded*; *conversely, for any Zygmund bounded tangent vector field V on $\mathbf{S}^1$, there exists a Thurston bounded earthquake measure σ such that*

$$V(x) = \pi \iint E_{ab}(x)\, d\sigma(a,b) \quad \textit{modulo a quadratic polynomial}; \tag{33}$$

and furthermore, if two V's differ by a quadratic polynomial then the corresponding σ's are the same.

Norm-equivalence theorem ([4] or [5]). *There exists a universal constant $C > 0$ such that for any earthquake measure σ,*

$$\frac{1}{C}\|\sigma\|_{\mathrm{Th}} \leq \|V_\sigma\|_{\mathrm{cr}} \leq C\|\sigma\|_{\mathrm{Th}}.$$

Remark. There is an analogue of the norm-equivalence theorem for earthquake measures and the boundary homeomorphisms of the corresponding earthquake maps. This work was initiated in [2] and completed in [6].

In what follows, we show the following theorem.

Theorem 2. *Let σ be a Thurston bounded earthquake measure. Then there exists an infinite sequence $\{(\sigma_n, \mathscr{L}_n)\}_{n=1}^{\infty}$ of earthquake measures satisfying*:

(i) *σ_n's are uniformly Thurston bounded*;

(ii) *for each n, $\mathscr{L}_n$ consists of finitely many geodesics in $\mathscr{L}$ and for each geodesic $\overline{ab}$ in $\mathscr{L}_n$, $|a-b| \ge 1/n$, where a and b denote the endpoints of $\overline{ab}$*;

(iii) $\mathscr{L}_1 \subset \mathscr{L}_2 \subset \mathscr{L}_3 \subset \cdots \subset \mathscr{L}_n \subset \cdots \subset \mathscr{L}$; *and*

(iv) *there exists a constant $C > 0$ such that for all n,*

$$\|V_\sigma - V_{\sigma_n}\|_\infty \le \frac{C}{n}.$$

Remark. The subsequence $\{\sigma_{2^k}, \mathscr{L}_{2^k}\}_{k=1}^{\infty}$ of $\{\sigma_n, \mathscr{L}_n\}_{n=1}^{\infty}$ in the previous theorem satisfies

$$\|V_\sigma - V_{\sigma_{2^k}}\|_\infty \le \frac{C}{2^k}$$

for any integer $k \ge 1$.

We divide the proof into two steps. In the first step, let $\mathscr{L}'$ be the lamination obtained from $\mathscr{L}$ by deleting the short geodesics (in the Euclidean metric) and σ' be the restriction of σ on $\mathscr{L}'$, we show that V_σ differs from $V_{\sigma'}$ by an amount commensurable to the maximal Euclidean length of the deleted geodesics. In the second step, we show that σ' can be approximated by measures σ'' supported on finitely many geodesics such that $V_{\sigma'}$ differs from $V_{\sigma''}$ by an error as small as required.

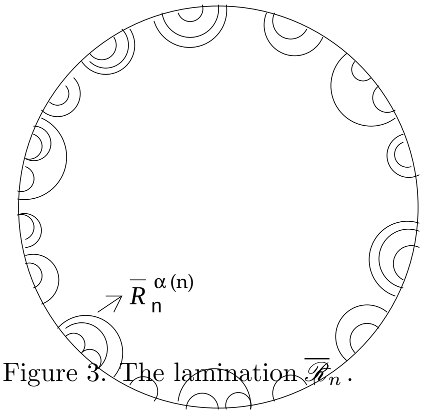

Figure 3. The lamination $\overline{\mathscr{R}}_n$.

For each $n \in \mathbf{N}$, let $\mathscr{R}_n$ be the collection of the geodesics $\overline{ab}$ in $\mathscr{L}$ with $|a-b| < 1/n$ and $\overline{\mathscr{R}}_n$ be the collection of the geodesics $\overline{ab}$ in $\mathscr{L}$ with $|a-b| \le 1/n$. For each n, $\overline{\mathscr{R}}_n$ can be written as a disjoint union of subcollections $\overline{\mathscr{R}}_n^{\alpha(n)}$ of $\overline{\mathscr{R}}_n$ satisfying that for each $\overline{\mathscr{R}}_n^{\alpha(n)}$ there exists a geodesic $\overline{ab} \in \overline{\mathscr{R}}_n^{\alpha(n)}$ such that any geodesic in $\overline{\mathscr{R}}_n$ connecting a point in $[a, b]$ to a point in $[a, b]$ belongs to $\overline{\mathscr{R}}_n^{\alpha(n)}$, where $[a,b]$ denote the shorter arc on $\mathbf{S}^1$ bounded by the endpoints of $\overline{ab}$. Assume $\mathscr{R}_n^{\alpha(n)} = \overline{\mathscr{R}}_n^{\alpha(n)} \cap \mathscr{R}_n$. Clearly,

$$\overline{\mathscr{R}}_n = \bigcup_{\alpha(n)} \overline{\mathscr{R}}_n^{\alpha(n)} \quad \text{and} \quad \mathscr{R}_n = \bigcup_{\alpha(n)} \mathscr{R}_n^{\alpha(n)}.$$

Let $\bar{\sigma}^{\alpha(n)}$ be the restriction of σ on $\overline{\mathscr{R}}_n^{\alpha(n)}$ and $\sigma^{\alpha(n)}$ be the restriction of σ on $\mathscr{R}_n^{\alpha(n)}$. Let $V_{\sigma^{\alpha(n)}} = E(\sigma^{\alpha(n)})$.

Lemma 4. *There exists a constant $C > 0$ only depending on $\|\sigma\|_{\mathrm{Th}}$ such that*

$$\|V_{\sigma^{\alpha(n)}}\|_\infty \le \frac{C}{n}.$$

Proof. Let $V_{\alpha(n)} = V_{\bar{\sigma}^{\alpha(n)}} = E(\bar{\sigma}^{\alpha(n)})$ and $\overline{ab}$ be the unique geodesic in $\overline{\mathscr{R}}_n^{\alpha(n)}$ such that any geodesic in $\overline{\mathscr{R}}_n^{\alpha(n)}$ connecting a point in $[a,b]$ to a point in $[a,b]$. Clearly,

$$\|V_{\sigma^{\alpha(n)}}\|_\infty \le \|V_{\bar{\sigma}^{\alpha(n)}}\|_\infty = \|V_{\alpha(n)}\|_\infty.$$

Therefore we only need to show

$$\|V_{\alpha(n)}\|_\infty \le \frac{C}{n}$$

for a constant $C > 0$.

For any $x \notin (a,b)$, $V_{\alpha(n)}(x) = 0$. For any $x \in (a,b)$, let y be the point on $\mathbf{S}^1$ such that y, a, x, b are on $\mathbf{S}^1$ in the counter-clockwise order and $\mathrm{cr}(\{y,a,x,b\}) = 1$. Let $Q = \{y,a,x,b\}$. Since $V_{\alpha(n)}(y) = V_{\alpha(n)}(a) = V_{\alpha(n)}(b) = 0$,

$$V_{\alpha(n)}[Q] = -\frac{V_{\alpha(n)}(x)}{x-a} - \frac{V_{\alpha(n)}(x)}{b-x} = -V_{\alpha(n)}(x)\frac{b-a}{(x-a)(b-x)}.$$

Hence

$$|V_{\alpha(n)}(x)| = |V_{\alpha(n)}[Q]|\frac{(x-a)(b-x)}{b-a} \le \|V_{\alpha(n)}\|_{\mathrm{cr}}\frac{b-a}{4}.$$

Since $\|\sigma^{\alpha(n)}\|_{\mathrm{Th}} \le \|\sigma\|_{\mathrm{Th}}$, by the norm-equivalence theorem,

$$\|V_{\alpha(n)}\|_\infty \le \|V_{\alpha(n)}\|_{\mathrm{cr}}\frac{b-a}{4} \le C\|\sigma\|_{\mathrm{Th}}\frac{b-a}{4} \le \frac{C\|\sigma\|_{\mathrm{Th}}}{4}\frac{1}{n}.$$

It completes the proof. □

Now let $\widetilde{\mathscr{L}}_n = \mathscr{L} \backslash \mathscr{R}_n$, that is the collection of the geodesics $l = \overline{ab}$ in $\mathscr{L}$ with $|a-b| \ge 1/n$, and let $\tilde{\sigma}_n$ be the restriction of σ on $\widetilde{\mathscr{L}}_n$. Assume $V_{\tilde{\sigma}_n} = E(\tilde{\sigma}_n)$.

Proposition 6. *We obtain the inequality*

$$\|V_\sigma - V_{\tilde{\sigma}_n}\|_\infty \leq \frac{C}{n}.$$

Proof. For each point $x \in \mathbf{S}^1$ and each $n \in \mathbf{N}$, there exists $\alpha(n)$ such that

$$V_\sigma(x) - V_{\tilde{\sigma}_n}(x) = V_{\sigma^{\alpha(n)}}(x).$$

By the previous lemma,

$$|V_{\sigma^{\alpha(n)}}(x)| \leq \|V_{\sigma^{\alpha(n)}}\|_\infty \leq \frac{C}{n}.$$

It implies the proposition. □

Now we start to do the second step.

Given a lamination $\mathscr{L}$, two geodesics l_1 and l_2 in $\mathscr{L}$ are said to be *simply parallel to each other* with respect to $\mathscr{L}$ if any other geodesic l of $\mathscr{L}$ contained in the stripe between l_1 and l_2 separates l_1 and l_2. Simply parallel geodesics have the reflexive, symmetric and transitive properties. Given each geodesic l in $\mathscr{L}$, there is a unique maximal collection $\mathscr{L}_l$ of the geodesics which are simply parallel to l with respect to $\mathscr{L}$. Since the locus of $\mathscr{L}$ is a closed subset in the hyperbolic plane, the locus of $\mathscr{L}_l$ is also so. Maximal collections of simply parallel geodesics with respect to a lamination are pairwise disjoint. Laminations without short geodesics in Euclidean metric have the following property.

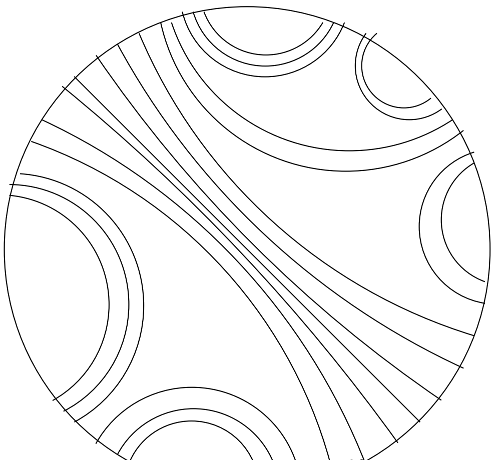

Figure 4. The lamination $\mathscr{L}_n$.

Lemma 5. *With respect to each lamination $\widetilde{\mathscr{L}}_n$, there are only finitely many maximal collections of simply parallel geodesics.*

Proof. For each n, the lamination $\widetilde{\mathscr{L}}_n$ has no geodesic $\overline{ab}$ with $|a-b|<1/n$. Arguing by contradiction, one can show that $\widetilde{\mathscr{L}}_n$ has only finitely many maximal collections of simply parallel geodesics. □

One easily obtains the following two elementary lemmas.

Lemma 6. *Let $a<a'<b'<b$ be four points on the real line. Then*

$$\|E_{ab}-E_{a'b'}\|_\infty \to 0$$

as $a\to a'$ and $b'\to b$.

Lemma 7. *Let $\mathscr{L}$ be a collection of simply parallel geodesics in the upper half plane and σ a measure supported on $\mathscr{L}$. Suppose that $a<a'<b'<b$ and all geodesics of $\mathscr{L}$ are contained in the strip between the geodesics connecting a to b and a' to b' respectively. Then for each x on* $\mathbf{R}$,

$$\sigma(\mathscr{L})E_{a'b'}(x)\le V_\sigma(x)\le \sigma(\mathscr{L})E_{ab}(x)$$

and then

$$\|V_\sigma-\sigma(\mathscr{L})E_{ab}\|_\infty \le \sigma(\mathscr{L})\|E_{ab}-E_{a'b'}\|_\infty.$$

Lemmas 5, 6 and 7 imply

Proposition 7. *For each n and any $\varepsilon_n>0$, there exists an earthquake measure σ_n supported on a finite sublamination $\mathscr{L}_n$ of $\widetilde{\mathscr{L}}_n$ such that $\|\sigma_n\|_{\mathrm{Th}}\le 2\|\sigma_n\|_{\mathrm{Th}}$ and*

$$\|V_{\tilde\sigma_n}-V_{\sigma_n}\|_\infty<\varepsilon_n.$$

Proof. For each n, let $\widetilde{\mathscr{L}}_n^i$, $i=1,2,\dots,k(n)$, be the maximal collections of simply parallel goedesics with respect to $\widetilde{\mathscr{L}}_n$. Let $\tilde\sigma_n^i$ be the restriction of $\tilde\sigma_n$ on $\widetilde{\mathscr{L}}_n^i$ for each $1\le i\le k(n)$. Then

$$V_{\tilde\sigma_n}=\sum_{i=1}^{k(n)}V_{\tilde\sigma_n^i}.$$

For each i, we construct a finite earthquake measure $(\mathscr{L}_n^i,\sigma_n^i)$ as follows. Let l_a and l_b be the two geodesics in $\widetilde{\mathscr{L}}_n^i$ such that the strip between them contains all geodesics in $\widetilde{\mathscr{L}}_n^i$. Without loss of generality, we may assume $l_a\ne l_b$. Let $\mathscr{L}_n^i=\{l_1=l_a,l_2,l_2,\dots,l_m=l_b\}$ be a collection of finitely many geodesics contained in $\widetilde{\mathscr{L}}_n^i$ in order. We construct σ_n^i by defining $\sigma_n^i(l_k)$ to be $\tilde\sigma_n^i(l_k)$ plus the measure of the geodesics of $\widetilde{\mathscr{L}}_n^i$ contained in the open strip between l_k and l_{k+1} for $1\le k\le m-1$ and $\sigma_n^i(l_m)=\tilde\sigma_n^i(l_m)$. Since the total $\tilde\sigma_n^i$-measure of $\widetilde{\mathscr{L}}_n^i$

is finite, by Lemmas 6 and 7, for an arbitrary small $\varepsilon(i) > 0$, there exists a finite earthquake measure $(\mathscr{L}_n^i, \sigma_n^i)$ constructed as the same as the above such that

$$\|V_{\tilde{\sigma}_n^i} - V_{\sigma_n^i}\|_\infty < \varepsilon(i).$$

Let $\mathscr{L}_n = \bigcup_{i=1}^{k(n)} \mathscr{L}_n^i$ and $\sigma_n = \sum_{i=1}^{k(n)} \sigma_n^i$. It is easy to see that $\|\sigma_n\|_{\mathrm{Th}} \le 2\|\tilde{\sigma}_n\|_{\mathrm{Th}}$. Given any $\varepsilon_n > 0$, let $\varepsilon(i) < \varepsilon_n / k(n)$. Then

$$\|V_{\tilde{\sigma}_n} - V_{\sigma_n}\|_\infty = \sum_{i=1}^{k(n)} \|V_{\tilde{\sigma}_n^i} - V_{\sigma_n^i}\|_\infty < \varepsilon_n. \ \square$$

Furthermore, we obtain

Proposition 8. *There exists an infinite sequence $\{(\sigma_n, \mathscr{L}_n)\}_{n=1}^\infty$ of earthquake measures such that for each $n \in \mathbf{N}$,*

(i) *$\mathscr{L}_n$ is a finite subset of $\widetilde{\mathscr{L}}_n$ and $\|\sigma_n\|_{\mathrm{Th}} \le 2\|\tilde{\sigma}_n\|_{\mathrm{Th}}$,*
(ii) *$\mathscr{L}_n \subset \mathscr{L}_{n+1}$, and*
(iii) *$\|V_{\tilde{\sigma}_n} - V_{\sigma_n}\|_\infty \le 1/n$.*

Proof. The condition (i) is obviously satisfied, and the previous proposition implies the condition (iii) by letting $\varepsilon_n = 1/n$. It remains to show how one can construct $\mathscr{L}_{n+1}$ from $\mathscr{L}_n$ such that $\mathscr{L}_n \subset \mathscr{L}_{n+1}$.

We divide the maximal collections $\widetilde{\mathscr{L}}_{n+1}^j$ of simply parallel geodesics with respect to $\widetilde{\mathscr{L}}_{n+1}$ into two groups. The group I consists of the collections $\widetilde{\mathscr{L}}_{n+1}^{j(I)}$ which overlaps maximal collections $\widetilde{\mathscr{L}}_n^i$ of simply parallel geodesics with respect to $\widetilde{\mathscr{L}}_n$, and the group II consists of the collections $\widetilde{\mathscr{L}}_{n+1}^{j(II)}$ which do not overlap any maximal collection $\widetilde{\mathscr{L}}_n^i$ of simply parallel geodesics with respect to $\widetilde{\mathscr{L}}_n$. Clearly,

$$\bigcup_{i=1}^{k(n)} \widetilde{\mathscr{L}}_n^i \subset \bigcup_{j(I)} \widetilde{\mathscr{L}}_{n+1}^{j(I)}.$$

Following the procedure in the construction of $(\mathscr{L}_n, \sigma_n)$ in the previous proposition, for any small $\varepsilon(I) > 0$, we can have a finite earthquake measure σ_{n+1}^I supported on a finite lamination $\mathscr{L}_{n+1}^I$ which is a refinement of $\mathscr{L}_n$ such that

$$\|V_{\tilde{\sigma}_{n+1}^I} - V_{\sigma_{n+1}^I}\|_\infty < \varepsilon(I), \tag{34}$$

where $\tilde{\sigma}_{n+1}^I$ is the measure $\tilde{\sigma}_{n+1}$ restricted on $\bigcup_{j(I)} \widetilde{\mathscr{L}}_{n+1}^{j(I)}$.

Let $\tilde{\sigma}_{n+1}^{II}$ be the restriction of $\tilde{\sigma}_{n+1}$ on $\bigcup_{j(ii)} \widetilde{\mathscr{L}}_{n+1}^{j(II)}$. Repeating the same procedure in the construction of $(\mathscr{L}_n, \sigma_n)$ for $\left(\bigcup_{j(II)} \widetilde{\mathscr{L}}_{n+1}^{j(II)}, \tilde{\sigma}_{n+1}^{II}\right)$, for any $\varepsilon(II) > 0$, we can obtain a finite earthquake measure $(\mathscr{L}_{n+1}^{II}, \sigma_{n+1}^{II})$ such that

$$\|V_{\tilde{\sigma}_{n+1}^{II}} - V_{\sigma_{n+1}^{II}}\|_\infty < \varepsilon(II). \tag{35}$$

Let $\mathscr{L}_{n+1} = \mathscr{L}^{I}_{n+1} \cup \mathscr{L}^{II}_{n+1}$ and $\sigma_{n+1} = \sigma^{I}_{n+1} + \sigma^{II}_{n+1}$. Clearly, $\mathscr{L}_{n+1} \supset \mathscr{L}_n$. With the same reason to see $\|\sigma_n\|_{\mathrm{Th}} \le 2\|\tilde{\sigma}_n\|_{\mathrm{Th}}$, we also have $\|\sigma_{n+1}\|_{\mathrm{Th}} \le 2\|\tilde{\sigma}_{n+1}\|_{\mathrm{Th}}$. Given any $\varepsilon_{n+1} > 0$, let $\varepsilon(I) = \varepsilon(II) = \frac{1}{2}\varepsilon_{n+1}$. By (34) and (35), we obtain

$$\|V_{\tilde{\sigma}_{n+1}} - V_{\sigma_{n+1}}\|_\infty < \frac{1}{n}.$$

It completes the proof. □

Let $\varepsilon_n = 1/n$. Then Propositions 6, 7 and 8 imply our Theorem 2.

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