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SHARP ESTIMATES FOR HYPERBOLIC METRICS AND COVERING THEOREMS OF LANDAU TYPE

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Abstract. One of the main covering results asserts that if a holomorphic function f in the unit disk satisfies $|f'(0)| \geq A|f(0)|$ with $A > 4$, then f covers an annulus of the form $r < |w| < Kr$ for some $r > 0$, where K is a certain function of A. Extremals are furnished by universal covering maps onto complements of certin discrete sets. The covering theorems are proved by solving minimum problems for hyperbolic metrics.

1. Introduction

In this paper we prove sharp covering theorems for nonconstant holomorphic functions f in the unit disk $\mathbf{U}$. Theorem 1 asserts that if $|f'(0)| \geq A|f(0)|$, where A is a given number larger than 4, then f covers some annulus of the form $r < |w| < Kr$, where $K = K(A)$ is a number depending on A. The theorem is sharp; extremals are furnished by universal covering maps from $\mathbf{U}$ onto the plane minus a doubly-infinite geometric sequence with ratio K along a ray through the origin. The covering theorem is proved by solving a minimum problem for hyperbolic metrics. The crucial step is to prove that among all domains Ω of the form $\mathbf{C}\backslash(S\times 2\pi\mathbf{Z})$, where S is a closed subset of $\mathbf{R}$ which intersects every interval of length $\log K$, the hyperbolic density $\lambda_\Omega(z)$ is smallest when S consists of all integer multiples of $\log K$, and $z = \frac{1}{2}\log K + i\pi$. A second covering theorem, Theorem 2, gives the precise value for a "real Landau constant" about covering intervals on the real axis when $f(0)$ is real. The covering and minimum problems occupy Sections 2–7 of the paper. In Sections 8–11 we study some properties of the function $K(A)$.

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2. Covering theorems

For $K > 1$, consider the region $D_K = \mathbf{C}^*\backslash\{-K^{n+1/2} : n \in \mathbf{Z}\}$, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$, and let $F_K\colon \mathbf{U} \to D_K$ be the universal covering of D_K by the unit disc such that $F_K(0) = 1$ and $F_K'(0) > 0$. Put $A(K) = F_K'(0)$. We shall see that $A(K)$ is a strictly increasing continuous function on $(1,\infty)$, that

$$\lim_{K\to 1+} A(K) = 4, \quad \text{and that} \quad \lim_{K\to+\infty} A(K) = +\infty. \tag{1}$$

So the inverse function $K(A)$ is defined for $4 < A < \infty$.

Theorem 1. *Let f be a nonconstant holomorphic function in the unit disc $\mathbf{U}$ satisfying $|f'(0)| \geq A|f(0)|$ where $A > 4$. Then the image $f(\mathbf{U})$ contains an annulus of the form $r < |w| < Kr$ for some $r > 0$, where $K = K(A)$ was defined above.*

Moreover, $f(\mathbf{U})$ contains a closed annulus $r \leq |w| \leq Kr$ for some $r > 0$, unless $f(z) = cF_K(e^{i\alpha}z)$ for some $c \in \mathbf{C}^*$ and $\alpha \in \mathbf{R}$.

The function F_K shows that the estimate of the "thickness" of the annulus in this theorem is best possible.

Set

$$F_1(z) = \left(\frac{1+z}{1-z}\right)^2.$$

Then ${F_1}'(0) = 4$ and $F_1(\mathbf{U}) = \mathbf{C}\backslash(-\infty,0]$ contains no annulus centered at the origin. Thus, there is no theorem like Theorem 1 when $A \leq 4$.

The historical background begins with work of Valiron [31]:

Theorem A. *For every non-constant entire function f and every number $N > 0$, there exists a branch of the inverse f^{-1} which has an analytic continuation to a region of the form $\{w : r < |w| < Nr,\ |\mathrm{Arg}\, w| < N\}$ (on the Riemann surface of the $\log w$) for some $r > 0$.*

A. J. MacIntyre [25] proved a result of this type about covering of slit annuli. We are grateful to David Minda for calling this paper to our attention.

A corollary to Theorem A is that there are branches of the inverse f^{-1} in discs of arbitrarily large radii. A. Bloch [8] extended this corollary to functions holomorphic in the unit disc:

Theorem B. *For every holomorphic function f in $\mathbf{U}$ there exists a branch of the inverse f^{-1} in some disc of radius $b|f'(0)|$, where b is an absolute constant.*

The following corollary of Theorem B is sometimes called Landau's Theorem:

Theorem C. *For every holomorphic function f in $\mathbf{U}$, the image $f(\mathbf{U})$ contains a disc of radius $l|f'(0)|$, where l is an absolute constant.*

Landau [24] stated extremal problems related to Theorems B and C. To state the problem corresponding to Theorem C, we define $l(f)$ to be the inradius of $f(\mathbf{U})$, that is, the least upper bound of radii of discs contained in $f(\mathbf{U})$. Then the extremal problem is to find

$$\mathscr{L} = \inf\{l(f) : |f'(0)| = 1\}.$$

The constant $\mathscr{L}$ is called Landau's constant. The solution of the analogous extremal problem for Theorem B, denoted $\mathscr{B}$, is called Bloch's constant. Conjectures for the precise values of $\mathscr{B}$ and $\mathscr{L}$, and for the corresponding extremal domains, are given in the papers [3] and [28], respectively. Both conjectures remain open. The best bounds known to us are

$$.43\ldots = \frac{\sqrt{3}}{4} + 2\cdot 10^{-4} < \mathscr{B} \le \frac{\Gamma(1/3)\Gamma(11/12)}{\Gamma(1/4)\big(1+\sqrt{3}\big)^{1/2}} = .47\ldots,$$
$$\frac{1}{2} + 10^{-335} < \mathscr{L} \le \frac{\Gamma(1/3)\Gamma(5/6)}{\Gamma(1/6)} = .54\ldots.$$

These inequalities imply that $\mathscr{L} > \mathscr{B}$. The upper bounds are the conjectural correct values. The lower bound for $\mathscr{B}$ is due to Chen and Gauthier [11], the lower bound for $\mathscr{L}$ to Yanagihara [35].

We consider Theorem 1 to be a theorem of Landau type which also bears some resemblance to the original result of Valiron. The main point of Theorem 1 is that we have succeeded in solving the corresponding extremal problem.

Here is another sharp covering theorem. Its proof is essentially the same as that of Theorem 1. For f holomorphic in $\mathbf{U}$, define $l_0(f)$ to be the least upper bound of lengths of subintervals of $\mathbf{R}$ contained in $f(\mathbf{U})$, and define

$$\mathscr{L}_0 = \inf\{l_0(f) : f(0) \in \mathbf{R},\ |f'(0)| = 1\}.$$

One could call $\mathscr{L}_0$ the "real Landau constant".

Theorem 2. *The real Landau constant* $\mathscr{L}_0$ *satisfies*

$$\mathscr{L}_0 = \frac{4\pi^3}{\Gamma(1/4)^4} = .718\ldots.$$

Theorem 1 is a consequence of Theorem 3, about hyperbolic metrics, which will be stated in Section 4 and proved in Section 5. The main inequality underlying Theorem 3 also produces an inequality for distribution functions of hyperbolic densities on intervals. This is stated as Theorem 4, in Section 6. The proof of Theorem 2, along with associated extremal problems for hyperbolic metrics, is discussed in Section 7. Our principal tools are Ahlfors's method of ultrahyperbolic metrics and Weitsman's theorem on symmetrization. Some of our proofs are near neighbors to those found in [9] and [26]. The papers [10] and [30] also contain some related results.

3. The function $A(K)$

The domains D_K, universal covering maps F_K, and numbers $A(K) = F_K'(0)$ were defined at the beginning of Section 2. Using a theorem of Hejhal, (Theorem 1 and the remark following its proof in [18]), one can prove that if $K_0 \in (1, \infty)$, then F_K converges to F_{K_0} locally uniformly in $\mathbf{U}$ when $K \to K_0$. Thus, $A(K)$ is continuous on $(1, \infty)$. Similarly, as $K \to 1+$, F_K converges locally uniformly in $\mathbf{U}$ to the conformal map

$$F_1(z) = \left(\frac{1+z}{1-z}\right)^2$$

of $\mathbf{U}$ onto the plane minus the negative real axis. Thus,

$$\lim_{K \to 1+} A(K) = F_1'(0) = 4.$$

Let G_K be the universal cover from $\mathbf{U}$ onto the annulus $K^{-1/2} < |w| < K^{1/2}$ with $G_K(0) = 1$, $G_K'(0) > 0$. Then, by the principle of subordination, $A(K) > G_K'(0)$ for $K > 1$. Now $G_K = \exp\bigl((\log K)H\bigr)$, where H is the conformal map of $\mathbf{U}$ onto the strip $|\operatorname{Re}\zeta| < \frac{1}{2}$ with $H(0) = 0$, $H'(0) > 0$. Thus, $\lim_{K\to\infty} G_K'(0) = \infty$ and hence

$$\lim_{K \to \infty} A(K) = \infty.$$

We will derive the strict monotonicity of $A(K)$ from Theorem 3 in the next section.

4. Hyperbolic metrics

Each plane domain D for which $\mathbf{C}\backslash D$ contains at least two points has a hyperbolic metric, that is, a complete conformal Riemannian metric of curvature -4. Such a metric is unique and its length element will be denoted by $\lambda_D(z)\,|dz|$. Thus, for example, $\lambda_{\mathbf{U}}(z) = 1/(1-|z|^2)$. Whenever we speak of λ_D we shall be implicitly assuming that D has a hyperbolic metric. For $z \in \mathbf{C}\backslash D$ set $\lambda_D(z) = +\infty$. Then λ_D is defined in all of $\mathbf{C}$.

Let f be holomorphic in $\mathbf{U}$, and consider the region $D = f(\mathbf{U})$. Then

$$|f'(0)| \le \frac{1}{\lambda_D\bigl(f(0)\bigr)} \tag{2}$$

by the invariant form of the Schwarz Lemma, with equality if and only if f is a universal covering map from $\mathbf{U}$ onto D. See, for example, [2, p. 13]. In particular, $\lambda_{D_K}(1) = 1/A(K)$. Theorem 1 is consequence of

Theorem 3. *If D is a region in the plane which contains no annuli of the form $r \le |w| \le Kr$ for $r > 0$, then for $w \in D$,*

$$|w|\lambda_D(w) \ge \lambda_{D_K}(1),$$

with equality if and only if $D = cD_K$ and $w = c$ for some $c \in \mathbf{C}^*$.

Corollary 1. *The function $K \mapsto A(K) = 1/\lambda_{D_K}(1)$ is strictly increasing for $1 < K < \infty$.*

Concerning the equality statement in Theorem 3, note that $K^n D_K = D_K$, from which it follows that $\lambda_{D_K}(K^n\zeta) = K^{-n}\lambda_{D_K}(\zeta)$ for all $\zeta \in \mathbf{C}$. Thus an equivalent statement of the equality condition is: Equality holds for a given D and w if and only if there exists $c \in \mathbf{C}^*$ and $n \in \mathbf{Z}$ such that $D = cD_K$ and $w = cK^n$.

Assuming now that Theorem 3 is true, we shall show that it implies Theorem 1. Take a function f satisfying the hypotheses of Theorem 1 for some $A > 4$. Let $K = K(A)$. If $f(0) = 0$ then $f(\mathbf{U})$ contains a neighborhood of the origin, hence an annulus of the form $r \le |w| \le Kr$, so that the conclusion of Theorem 1 is fulfilled. Suppose that $|f(0)| > 0$, and that f covers no annulus $r \le |w| \le Kr$. Let $D = f(\mathbf{U})$ and $w_0 = f(0)$. Then

$$A|w_0| \le |f'(0)| \le \frac{1}{\lambda_D(w_0)} \le |w_0|A.$$

The first inequality is from the hypothesis of Theorem 1, the second from (2), and the third from Theorem 3, together with the identity $A(K) = 1/\lambda_{D_K}(1)$. We conclude that equality holds in all three inequalities. The equality statements in (2) and Theorem 3 imply existence of $c \in \mathbf{C}^*$ such that f is a universal covering map of $\mathbf{U}$ onto cD_K with $f(0) = c$. This is the conclusion of Theorem 1. Thus, Theorem 1 is proved, modulo Theorem 3.

The proof of Theorem 3 will be accomplished through several applications of the Ahlfors–Schwarz Lemma [1], [2]. Recall that if λ_D is the density of the hyperbolic metric in a domain D and we put $u = \log \lambda_D$, then u satisfies in D the Liouville equation

$$\Delta u = 4e^{2u}. \tag{3}$$

Let v be a function in D. A C^2 function u_a defined in some neighborhood N of a point $a \in D$ is called a *support function* for v at a if $u_a(a) = v(a)$, $u_a(z) \le v(z)$ for all $z \in N$, and u_a is a classical subsolution of (3), that is, $\Delta u_a \ge 4e^{2u_a}$ in N. We shall call a function v in D *ultrahyperbolic* in D if it is upper semicontinuous and has a support function at each point of D. The

Ahlfors–Schwarz Lemma states that every ultrahyperbolic function v in a domain D satisfies

$$v \leq \log \lambda_D.$$

Our use of the term "ultrahyperbolic" differs slightly from that in [2], where "ultrahyperbolic metric" would refer to the Riemannian metric with length element $e^{v(z)}\,|dz|$, with v as above.

Returning now to our extremal domain D_K, let us make a conformal change of variable $w = -e^{\zeta}$ and put

$$L = \tfrac{1}{2}\log K.$$

Then to $D_K \subset \mathbf{C}_w$ corresponds the domain

$$\Omega_L = \mathbf{C}\backslash S(L) \subset \mathbf{C}_\zeta, \quad \text{where } S(L) = \{(2m+1)L + 2\pi i n : m, n \in \mathbf{Z}\}.$$

The hyperbolic densities of these two domains are related by the equation

$$\lambda_{\Omega_L}(\zeta) = \lambda_{D_K}(-e^{\zeta})|e^{\zeta}|. \tag{4}$$

From the uniqueness of the hyperbolic metric it follows that λ_{Ω_L} is doubly periodic with periods $2L$ and $2\pi i$, and enjoys the symmetry properties

$$\lambda_{\Omega_L}(\zeta) = \lambda_{\Omega_L}(\pm\bar{\zeta}).$$

One expects that λ_{Ω_L} should have monotonicity properties along horizontal and vertical lines. The lemma below confirms this. For fixed $y \in \mathbf{R}$ define $\phi(x) = \lambda_{\Omega_L}(x+iy)$, and for fixed $x \in \mathbf{R}$ define $\psi(y) = \lambda_{\Omega_L}(x+iy)$.

Lemma 1. *The function $\phi(x)$ is strictly increasing on $[0, L]$ for every y, and the function $\psi(y)$ is strictly decreasing on $[0, \pi]$ for every x.*

From symmetry and periodicity properties of λ_{Ω_L} it follows that ϕ and ψ are even functions, that ϕ has period $2L$, and that ψ has period 2π. Together with Lemma 1, these show that the minima of $\log \lambda_{\Omega_L}$ along horizontal lines are achieved at $x = 0$ and the minima along vertical lines at $y = \pi$. Consequently,

$$\inf_{\mathbf{C}} \lambda_{\Omega_L} = \lambda_{\Omega_L}(\pi i).$$

Proof of Lemma 1. Weitsman [33] proved that for circularly symmetric domains D, $\lambda_D(re^{i\theta})$ is an even function of θ which is nondecreasing for $0 \leq \theta \leq \pi$. Along with (4), this ensures that ψ is nonincreasing on $[0, \pi]$. A slightly different change of variable shows that ϕ is nondecreasing on $[0, L]$. The same changes of variables show that the monotonicities, this time strict, follow also from a theorem of Minda, [26, Theorem 4(ii)], or, in a more general context, from a polarization theorem of Solynin, [29, Theorem 13].

Here is yet another proof of monotonicity, different from those of Weitsman and Minda. For fixed $L \in (0,\infty)$ set $u = \log \lambda_{\Omega_L}$. Define a function u^* in Ω_L as follows: For $z = x + iy \in \Omega_L$ with $0 \le x \le L$, set

$$u^*(x+iy) = \max\{u(t+iy) : 0 \le t \le x\}.$$

Extend u^* to Ω_L by setting $u^*(z) = u(z+2L)$, $u^*(x+iy) = u^*(-x+iy)$. Then u^* is continuous on Ω_L.

Clearly, $u \le u^*$ in Ω_L. We shall apply the Ahlfors–Schwarz Lemma to prove the opposite inequality. Take $a = x_0 + iy_0 \in \Omega_L$ with $0 \le x_0 \le L$. There exists $x_1 \in [0, x_0]$ such that $u^*(a) = u(x_1 + iy_0)$. If $x_1 = x_0$ take $u_a = u$. Then $u_a(a) = u(a) = u^*(a)$ and $u_a(z) = u(z) \le u^*(z)$ for all z, so that u_a is a support function for u^* at a. If $0 \le x_1 < x_0$, set $b = x_1 + iy_0$ and define

$$u_a(z) = u(z + b - a) = u(z + x_1 - x_0).$$

Take $\delta > 0$ so small that the disks $|z-a| < \delta$ and $|z-b| < \delta$ are disjoint. Then again $u_a(a) = u^*(a)$, while the definition and symmetry properties of u^* imply $u_a \le u^*$ in $|z-a| < \delta$. Thus, u_a is again a support function for u^*. We have shown that u^* has a support function at each point of Ω_L with $0 \le x \le L$. Using symmetry and periodicities, we can construct a support function at each point of Ω_L. Thus, the Ahlfors–Schwarz Lemma implies that $u \ge u^*$ in Ω_L. The opposite inequality was already noted. We deduce that $u = u^*$ in Ω_L, which implies that the function $\phi(x)$ is nondecreasing in $[0, L]$. A similar argument shows that $\psi(y)$ is nonincreasing on $[0, \pi]$.

To prove strict monotonicity, let $g = \partial u/\partial x$ and $p(z) = 8e^{u(z)}$. Differentiation of the Liouville equation shows that g satisfies in Ω_L the Schrödinger equation

$$\Delta g = pg.$$

Since ϕ is nondecreasing on $(0, L)$, we have $g \ge 0$ in the strip

$$\Pi_L = \{z : 0 < \operatorname{Re} z < L\}.$$

The potential $p(z)$ is positive. If g were zero at a point $z_0 \in \Pi_L$, the Hopf strong maximum principle applied to the operator $L = \Delta - p$, see, for example [14, p. 35], would imply that g is identically zero in each compact subdomain of Π_L which contains z_0, and hence g would be identically zero in Π_L. This is impossible, since $u(z) \to \infty$ as $z \to L$. We conclude that g is strictly positive in Π_L, so that $\phi(x)$ is strictly increasing in $[0, L]$. In the same way, one proves that ψ is strictly decreasing in $[0, \pi]$. □

5. Proof of Theorem 3

Theorem 3 asserts that

$$\lambda_{D_K}(1) \le |w|\lambda_D(w), \quad w \in D, \tag{5}$$

when D is a domain in the plane which contains no annuli of the form $r \le |w| \le Kr$.

Let $E \subset (-\infty, 0]$ be the set of all numbers $-r$ for which the circle $|w| = r$ is not contained in D. Let $D' = \mathbf{C}\backslash E$, and let D^* be the circular symmetrization of D. A theorem of Weitsman [34] asserts that $\lambda_D(w) \ge \lambda_{D^*}(|w|)$. Solynin [29] proved that equality in Weitsman's theorem is possible only if $D = D^*$ and $w = |w|$. Moreover, $D^* \subset D'$. From Theorem 7.1 of Heins's paper [16] with $F = D^*$, it follows that $\lambda_{D^*} \ge \lambda_{D'}$ everywhere in D^*, with equality at some point if and only if $D^* = D'$. Thus:

To prove Theorem 3, it suffices to prove (5) when D has the form $\mathbf{C}\backslash E$, with E a closed subset of $(-\infty, 0]$ which, for fixed K, intersects every interval of the form $[-r, -Kr]$, and to show that equality in (5) is possible only for $D = cD_K$ and $w = c$, where $c \in \mathbf{R}^+$.

To obtain (5) for this special class of D, we shall prove some stronger inequalities for the corresponding domains $\Omega = \{\zeta \in \mathbf{C} : -e^\zeta \in D\}$. Let

$$S = \{x \in \mathbf{R} : -e^x \in E\} = \{\log r : r \in -E\}.$$

Then

$$\Omega = \mathbf{C}\backslash(S \times 2\pi\mathbf{Z}) \tag{6}$$

where S is a closed subset of $\mathbf{R}$ which intersects every closed interval of length $2L = \log K$. Let us call a set S of this type an L-*set* and a domain Ω of this type an L-*domain.* The domains Ω_L of Section 4 are L-domains. They will furnish extremals for the problems we study.

For an L-set S and $x \in \mathbf{R}$, let $d(x)$ be the distance from x to S. Then $0 \le d(x) \le L$.

Our main inequality for L-domains is embodied in the following lemma.

Lemma 2. *For an L-domain Ω and $z = x + iy \in \Omega$,*

$$\lambda_\Omega(x + iy) \ge \lambda_{\Omega_L}\bigl(L - d(x) + iy\bigr), \tag{7}$$

with equality for some $z \in \Omega$ if and only if Ω is a translation of Ω_L.

Proof. For $z = x + iy \in \Omega$, define $v(z) = \log \lambda_{\Omega_L}\big(L - d(x) + iy\big)$. Then v is finite and continuous in Ω. Take $a = x_0 + iy_0 \in \Omega$. There are two mutually exclusive possibilities for x_0:

(a) There is a unique point of S closest to x_0. Then we put $u_a(z) = v(z)$ for z in a small neighborhood of a, and it is evident that u_a is a support function for v at a.

(b) There are exactly two points of S closest to x_0. Then we set

$$u_a(z) = \log \lambda_{\Omega_L}\big(L - d(x_0) + z - x_0\big).$$

Evidently u_a satisfies (3), and $u_a(a) = v(a)$. Now notice that

$$L - d(x) \geq L - d(x_0) + x - x_0$$

in a neighborhood of x_0, and thus Lemma 1 implies that $v(z) \geq u_a(z)$ in a neighborhood of a. So u_a is a support function for v at a. Thus, v is ultrahyperbolic in Ω, and (7) holds by the Ahlfors–Schwarz lemma.

Suppose that equality in (7) holds for some $z \in \Omega$. Then by Heins's Theorem 7.1 in [16], equality holds for all $z \in \Omega$. Let I be a component interval of $\mathbf{R} \setminus S$. Then $|I| \leq 2L$. If $|I|$ were strictly less than $2L$, then the strict monotonicity statement of Lemma 1 would imply that for each y the right-hand side of (7) is not differentiable in x at the midpoint of I. But λ_Ω is real analytic in Ω. This contradiction shows that the complement of S in $\mathbf{R}$ is the union of open intervals each having length $2L$.

Suppose now that S contains some nondegenerate interval I. Then $d(x) = 0$ on I. From equality in (7) and real analyticity of λ_Ω, it follows that λ_Ω is constant on all vertical lines between the lines $y = 0$ and $y = 2\pi$. Hence, λ_Ω is constant on $\mathbf{R} \setminus S$. But this is impossible, since $\lambda_\Omega(x) \to \infty$ as $x \to S$ from within $\Omega \setminus S$.

We have shown that if equality holds in (7) for some z then S contains no nondegenerate interval, and that each complementary interval of S has length $2L$. Such an S must be a translate of $S(L)$. Hence, Ω is translate of Ω_L. This completes the proof of Lemma 2. □

Returning now to the proof of Theorem 3, when Ω is an *L-domain* which is not a translate of Ω_L, Lemmas 2 and 1 imply that

$$\lambda_\Omega(x+iy) > \lambda_{\Omega_L}\big(L - d(x) + iy\big) \geq \lambda_{\Omega_L}(iy) \geq \lambda_{\Omega_L}(i\pi). \tag{8}$$

When Ω is a translate of Ω_L and $z = x + iy \in \Omega_L$ then Lemma 1 implies $\lambda_\Omega(z) > \lambda_{\Omega_L}(i\pi)$ unless x is a midpoint of a complementary interval of S and $y = \pi$. Inequality (5) and the accompanying equality statement now follow from (8) and the relations

$$\lambda_\Omega(x+iy) = \lambda_D(-e^{x+iy})e^x, \quad \lambda_{\Omega_L}(i\pi) = \lambda_{D_K}(1), \quad \big(L = \tfrac{1}{2}\log K\big).$$

The proof of Theorem 3 is complete.

6. Distribution inequalities for hyperbolic metrics

Let Ω be an L-domain, as defined in Section 5. We just saw that from (7) and (8), it follows that for each $x \in \mathbf{R}$ and $y \in \mathbf{R}$,

$$\lambda_\Omega(x+iy) \ge \lambda_{\Omega_L}(iy) = \min_{x\in\mathbf{R}} \lambda_{\Omega_L}(x+iy).$$

In this section, we show that (7) implies a more general sharp inequality for the *distribution function* of λ_Ω on sub-intervals of horizontal lines. For a measurable real valued function f on an interval $I \subset \mathbf{R}$ of finite Lebesgue measure $|I|$, the distribution function α_f of f on I is defined to be

$$\alpha_f(t) = |\{x \in I : f(x) > t\}|, \quad t \in \mathbf{R}.$$

We shall denote by $f^\#$ the symmetric decreasing rearrangement of f. Its domain is the interval $|x| \le \frac{1}{2}|I|$ and it satisfies $\alpha_f = \alpha_{f^\#}$. If f and g are two such functions, defined on possibly different intervals I_1 and I_2 with the same measure $2T$, we have the following well-known lemma.

Lemma 3. *For f and g as above, the following are equivalent.*
(a) $\alpha_f(t) \ge \alpha_g(t)$, *for all* $t \in \mathbf{R}$.
(b) $f^\#(x) \ge g^\#(x)$, *for all* $|x| \le T$.
(c) $\int_{I_1} \Phi\big(f(x)\big)\,dx \ge \int_{I_2} \Phi\big(g(x)\big)\,dx$
for every nondecreasing function $\Phi\colon \mathbf{R} \to \mathbf{R}$ *for which the integrals exist.*

For information about distribution functions and symmetric decreasing rearrangements, the reader may consult, for example, [15].

Here now is our distribution inequality for hyperbolic densities.

Theorem 4. *Let $L > 0$, Ω be an L-domain, and $I \subset \mathbf{R}$ be a closed interval of length $2T$, where $T \le L$. Then for each $y \in \mathbf{R}$ and $t \in \mathbf{R}$, we have*

$$|\{x \in I : \lambda_\Omega(x+iy) > t\}| \ge |\{x \in [-T,T] : \lambda_{\Omega_L}(x+iy) > t\}|. \tag{9}$$

From Lemma 3, we see that an equivalent conclusion is

$$\int_I \Phi\big(\lambda_\Omega(x+iy)\big)\,dx \ge \int_{-T}^{T} \Phi\big(\lambda_{\Omega_L}(x+iy)\big)\,dx \tag{10}$$

for all nondecreasing Φ.

The methods in [5] and [13] involving "star-functions" can be adapted to prove that (10) holds for all nondecreasing $\Phi(r)$ which are also concave functions of $\log r$. (The arguments in [5] and [13] show that for horizontal $*$-functions, $(-\log\lambda_\Omega)^* \le (-\log\lambda_{\Omega_L})^*$ holds in the strip $|\operatorname{Re} z| < L$.) Thus, for the problems considered in this paper, ultrahyperbolic metrics furnish a more powerful tool than star-functions.

Uniqueness statements associated with Theorem 4 exist, but we shall leave their formulations and proofs to the interested reader.

Proof of Theorem 4. Fix $y \in \mathbf{R}$. Write

$$I = [a,b], \quad f(x) = \lambda_{\Omega_L}(L - x + iy), \quad g(x) = f\big(d(x)\big).$$

Then f is an even $2L$-periodic function on $\mathbf{R}$ which, by Lemma 1, strictly decreases on $[0,L]$. Here $d(x)$ denotes the distance from x to S. Moreover, f is continuous as a function into $(0,\infty]$, and f and $\lambda_{\Omega_L}(x+iy)$ have the same distribution on $[-L,L]$. According to Lemma 2,

$$\lambda_\Omega(x+iy) \ge g(x).$$

Assume for now that $|I| = b - a = 2L$. Then it suffices to prove (9) with $g(x)$ in place of $\lambda_\Omega(x+iy)$ and $f(x)$ in place of $\lambda_{\Omega_L}(x+iy)$. If $t \ge \max_{[0,L]} f$ then both sides of (9) are zero, and if $t \le \min_{[0,L]} f$ both sides are $2L$, so assume $\min_{[0,L]} f < t < \max_{[0,L]} f$. Let $\sigma \in (0,L)$ be the unique solution of $f(\sigma) = t$, and let $E = \{x \in \mathbf{R} : d(x) < \sigma\}$. Then $g(x) > t$ if and only if $x \in E$. So to prove (9) when $T = L$, it will suffice to prove

$$|E \cap I| \ge 2\sigma. \tag{11}$$

Since S is a L-set, the intersection $S \cap I$ is nonempty. At least one of the following three cases occurs: (a) $[a+\sigma, b-\sigma] \cap S$ is nonempty. (b) $[a+\sigma, b-\sigma] \cap S$ is empty, but $[a, a+\sigma) \cap S$ is nonempty. (c) $[a+\sigma, b-\sigma] \cap S$ is empty but $(b-\sigma, b] \cap S$ is nonempty.

Suppose case (a) occurs. Take $x_0 \in [a+\sigma, b-\sigma] \cap S$. Then $d(x) \le |x - x_0| < \sigma$ for $x \in (x_0 - \sigma, x_0 + \sigma)$, so $(x_0 - \sigma, x_0 + \sigma) \subset E \cap I$. Thus, (11) holds in case (a).

Cases (b) and (c) can be handled symmetrically; we will treat (b). Let x_1 be the largest element of $[a, a+\sigma) \cap S$. Then $x_1 < a + \sigma$. Take $x_2 \in (x_1, x_1 + 2L] \cap S$. Then $x_2 > b - \sigma$ and $x_2 - \sigma \le x_1 + 2L - \sigma < a + 2L = b$. Thus, $b \in (x_2 - \sigma, x_2 + \sigma)$, and hence

$$(a, x_1 + \sigma) \cup (x_2 - \sigma, b) \subset E \cap I.$$

The measure of the set on the left is the smaller of $b - a$ and

$$(x_1 + \sigma - a) + (b - x_2 + \sigma) = 2\sigma + (b-a) - (x_2 - x_1).$$

Either way, the measure of the set on the left is at least as large as 2σ. Inequality (11) is proved for case (b), and the proof of Theorem 4 when $|I| = 2L$ is finished.

Suppose that $|I| = 2T < 2L$. Let J be the open interval with the same center as I and length $2L$. Set $\delta = L - T$. If $f(\delta) \le t$, then the right side of (9) is zero, so we assume $f(\delta) > t$. Then

$$|\{x \in (-L,L) : f(x) > t\}| = 2|\{x \in (\delta, L) : f(x) > t\}| + 2\delta,$$

while

$$|\{x \in J : g(x) > t\}| \le |\{x \in I : g(x) > t\}| + 2\delta.$$

From $|\{x \in (-L,L) : f(x) > t\}| \le |\{x \in J : g(x) > t\}|$, it follows that

$$|\{x \in I : g(x) > t\}| \ge 2|\{x \in (\delta, L) : f(x) > t\}|,$$

and this implies (9). □

Numerous comparison theorems involving minima of hyperbolic metrics have appeared in the literature. But except for the trivial case when one domain contains the other, Theorem 4 is the first result we know of in which the distribution function of the hyperbolic density of some domain is proved to be everywhere larger than that of another domain. For related comparison theorems involving various p.d.e.'s, the reader may consult [6].

7. More covering and distribution inequalities

For $L>0$ let $S\subset \mathbf{R}$ be an L-set as defined in Section 5. Thus S is closed, and S intersects every closed interval of length $2L$. In this section we set

$$S(L)=(2\mathbf{Z}+1)L.$$

This set $S(L)$ is different from the $S(L)$ introduced in Section 4.

Let λ_m be the hyperbolic metric of $\mathbf{C}\backslash(S\times m\mathbf{Z})$. Then, by Hejhal's theorem as in Section 3,

$$\lim_{m\to\infty}\lambda_m(z)=\lambda_{\mathbf{C}\backslash S}(z),\quad z\in\mathbf{C}.$$

Similarly, $\lambda_{\mathbf{C}\backslash S(L)}$ is the pointwise limit of hyperbolic metrics for $\mathbf{C}\backslash(S(L)\times m\mathbf{Z})$. The results we proved earlier are valid for each λ_m and $\lambda_{\mathbf{C}\backslash(S(L)\times m\mathbf{Z})}$. We conclude that $\lambda_{\mathbf{C}\backslash S(L)}$ is even, $2L$-periodic, nondecreasing on horizontal segments $[iy,L+iy]$, and nonincreasing on vertical half-lines $[x,x+i\infty]$. Using Hopf's maximum principle, as in the proof of Lemma 1, we see that the monotonicities are strict. Since $e^{\pi i S(1)}=\{-1\}$, the monotonicities in fact already follow from the theorem of Hempel [17] about monotonicity of $\lambda_{\mathbf{C}\backslash\{0,1\}}$ on circles and rays through the origin.

Next, we note that inequality (7) is valid when Ω is replaced by $\mathbf{C}\backslash S$ and Ω_L by $\mathbf{C}\backslash S(L)$. Arguing as in Sections 5–6, we obtain

Theorem 5. *Let S be an L-set on the real line. Then*

(a) $\lambda_{\mathbf{C}\backslash S}(x+iy)\ge\lambda_{\mathbf{C}\backslash S(L)}(iy)$, $x\in\mathbf{R}$, *and this inequality is strict unless S is a translate of $S(L)$.*

(b) *For each closed interval $I\subset\mathbf{R}$ with $|I|=2T\le 2L$ and each $y,t\in\mathbf{R}$ we have*

$$|\{x\in I:\lambda_{\mathbf{C}\backslash S}(x+iy)>t\}|\ge|\{x\in(-T,T):\lambda_{\mathbf{C}\backslash S(L)}(x+iy)>t\}|.$$

To prove the strictness assertion, one first establishes the $\ge$ statement in (a), then invokes Heins's theorem.

Now let n be a positive integer, and let E be a closed subset of the unit circle which intersects every closed arc of length $2\pi/n$. Put $E_n=\{e^{2\pi ik/n}:k=0,\dots,n-1\}$. The mapping $w=e^{i\zeta}$, which is the universal covering from $\mathbf{C}$ onto $\mathbf{C}^*$, takes horizontal lines in the ζ-plane to circles $|w|=r$. Theorem 5(b) with $L=T=\pi/n$, and Lemma 3, imply

Corollary 2. *Let E and n be as above. Then for each arc I of the unit circle with $|I| = 2\pi/n$, each $r > 0$, and each nondecreasing function Φ for which the integrals exist, we have*

$$\int_I \Phi\big(\lambda_{\mathbf{C}^*\setminus E}(re^{i\theta})\big)\,d\theta \geq \int_I \Phi\big(\lambda_{\mathbf{C}^*\setminus E_n}(re^{i\theta})\big)\,d\theta.$$

Since $\lambda_{\mathbf{C}^*\setminus E_n}(re^{i\theta})$ has period $2\pi/n$ as a function of θ, the integral on the right side has the same value over every interval of length $2\pi/n$. The inequality remains true if I has length $2\pi k/n$ for some $k \in \{0,\dots,n-1\}$, in particular if $I = [-\pi,\pi]$.

One can prove an analog of Corollary 2 with $\mathbf{C}^*$ replaced by $\mathbf{C}$ or by $\mathbf{C}\cup\{\infty\}$. In the first case the additional condition $n \geq 2$ is needed and in the second case $n \geq 3$. To establish these results, one can repeat the arguments proving Theorems 3–5, with minor modifications.

Theorem 5(a) implies various covering theorems, of which the following is the simplest to state. Let G_L be the universal covering from $\mathbf{U}$ onto $\mathbf{C}\setminus S(L)$ with $G_L(0) = 0$ and $G_L'(0) > 0$. Then $G_L = LG_1$.

Corollary 3. *For $B \in (0,\infty)$ let f be a holomorphic function in $\mathbf{U}$ with $f(0) \in \mathbf{R}$ and $|f'(0)| \geq B$. Then $f(\mathbf{U}) \cap \mathbf{R}$ contains an open interval of length $2B/{G'}_1(0)$. Moreover, $f(\mathbf{U}) \cap \mathbf{R}$ contains a closed interval of length $2B/{G'}_1(0)$ unless $f = G_L + c$, where $c \in \mathbf{R}$ and $L = B/G_1'(0)$.*

An equivalent statement of Corollary 3 is $\mathscr{L}_0 = 2/{G'}_1(0)$, where $\mathscr{L}_0$ is the real Landau constant defined at the end of Section 2. Since $e^{\pi i G_1}$ is the universal cover from $\mathbf{U}$ onto $\mathbf{C}\setminus\{-1,0\}$, we deduce

$${G'}_1(0) = 1/\big(\pi\lambda_{\mathbf{C}\setminus\{-1,0\}}(1)\big).$$

When λ has curvature -4, then, see [15, p. 707],

$$\lambda_{\mathbf{C}\setminus\{-1,0\}}(1) = \lambda_{\mathbf{C}\setminus\{0,1\}}(-1) = \frac{2\pi^2}{\Gamma(1/4)^4}. \tag{12}$$

Thus,

$$\mathscr{L}_0 = 2\pi\lambda_{\mathbf{C}\setminus\{0,1\}}(-1) = \frac{4\pi^3}{\Gamma(1/4)^4}.$$

Hence, Corollary 3 coincides with Theorem 2 in Section 2, and thus Theorem 2 is proved.

8. The function $A(K)$

In the rest of the paper we study the function $A(K) = F'_K(0)$, defined in Section 2. We consider rectangular lattices $\{2m\omega + 2n\omega'\}$ where $\omega = L = (\ln K)/2$, $K > 1$, and $\omega' = \pi i$. Let f be a universal cover of the complement of the lattice by the unit disc such that $f(0) = \omega + \omega'$, the center of the lattice, and $f'(0) > 0$. Then $F_K = -K^{-1/2}e^f$. We are interested in the quantity $A(K) = f'(0)$ as a function of $\ln K$.

In our use of the standard notation of the theory of elliptic functions we follow [20] (see also [4]): $\tau = \omega'/\omega$, $h = e^{\pi i \tau}$; $\theta_j(\zeta)$ is the jth theta-function, $\theta_j = \theta_j(0)$; and Jacobi's Modular Function is denoted by κ^2.

We denote

$$k = \frac{\ln K}{2\pi} = \frac{L}{\pi} = \frac{i}{\tau}, \quad a(k) = A(e^{2\pi k}) = f'(0) = A(K). \tag{13}$$

We may assume that f maps a circular quadrilateral Q (having zero angles, inscribed in the unit circle, symmetric with respect to the reflections in the coordinate axes) onto a fundamental rectangle R of the lattice, such that $f(0)$ is the center of R. Then a simple symmetry and rescaling argument gives the functional equation

$$a(k^{-1}) = k^{-1}a(k), \quad k > 0.$$

Using Hejhal's theorem as in Section 3, it is easy to see that in the limit when $k \to 0$, f maps the unit disc onto the strip $0 < \operatorname{Im} w < 2\pi$, and we obtain

$$a(0) = 4. \tag{14}$$

Together with the functional equation this implies

$$a(k) \sim 4k, \quad k \to +\infty,$$

that is

$$A(K) \sim \frac{2\ln K}{\pi}, \quad K \to +\infty.$$

Differentiating the functional equation we obtain

$$a'(1) = \tfrac{1}{2}a(1).$$

In the next two sections we find $a(1)$ and $a'(0)$.

There is no closed form expression for $a(k)$, and even its numerical computation is a non-trivial task. Finding the density of the hyperbolic metric in the complement of a rectangular lattice is equivalent to finding a conformal map from a rectangle onto a hyperbolic quadrilateral with zero angles. This classical problem was investigated by Hilbert [19] and Klein [23], and in modern times in [22]. The mapping satisfies a Schwarz differential equation related to a Lamé equation. The problem of finding this mapping for a given circular quadrilateral (not necessarily inscribed in a circle) requires determination of the so-called accessory parameter which is a solution of a transcendental equation involving Hill's determinants. See [12], [27], [32] for results in this direction. The only paper we know of where the accessory parameter was actually computed is [22], but this paper does not contain a rigorous analysis of convergence of the algorithm. The authors say on p. 217: "It should be emphasized that our remarks about the implicit equations are purely heuristic and that the actual computation proceeded, as it were, fortuitously without any a priori justification."

9. Finding $a(1)$

We use the upper half-plane rather than the unit disc. Let T be the open triangle in the upper half-plane with zero angles and vertices 0, 1, ∞. Let the quadrilateral Q_1 be the union of T with its reflection about its left vertical side and with the positive imaginary axis. Take the "center" to be the point $\tau = i$.

Let f be the mapping of Section 8 when $k = 1$. Then R is a square. Define f_1 on Q_1 by

$$f_1(\tau) = f(\zeta)e^{-i\pi/4},$$

where $\tau(\zeta)$ is a map from the unit disc onto the upper halfplane with $\tau(0) = i$. Then f_1 maps Q_1 onto the square R_1 of side 2π which has a diagonal along the positive real axis from 0 to $2\pi\sqrt{2}$. To construct f_1 on Q_1, it suffices to find a map f_1 of T onto the upper half of R_1 which carries the positive imaginary axis to the real interval $\left(0, 2\pi\sqrt{2}\right)$, then reflect.

We define f_1 as a composition of two functions:

$$f_1 = g \circ \kappa^2,$$

where κ^2 is the Modular Function of Jacobi. (In [4] this function has a double notation, sometimes λ, sometimes κ^2.) κ^2 maps T onto the upper half-plane and sends $(\infty, 0, 1)$ to $(0, 1, \infty)$. It satisfies $\kappa^2(i) = \frac{1}{2}$.

Our second component is a Schwarz–Christoffel map

$$g(w) = C\int_0^w \zeta^{-3/4}(1-\zeta)^{-3/4}\,d\zeta,$$

with

$$C = \frac{2\pi\sqrt{2}}{B\left(\frac{1}{4}, \frac{1}{4}\right)} \approx 1.981,$$

where B is Euler's Beta-function. This g maps the upper half-plane onto the right triangle constituting the upper half of R_1, and sends $w=\frac{1}{2}$ to the middle of the hypotenuse $\left(0, 2\pi\sqrt{2}\right)$. We have

$$g'\left(\tfrac{1}{2}\right) = C \cdot 2^{3/2} = \frac{8\pi}{B\left(\frac{1}{4},\frac{1}{4}\right)} = 3.887.$$

Now κ^2 is the restriction to Q_1 of a universal cover from the upper half-plane onto $\mathbf{C}\backslash\{0,1\}$. Thus

$$|(\kappa^2)'(i)| = \frac{\lambda_H(i)}{\lambda_{\mathbf{C}\backslash\{0,1\}}\left(\frac{1}{2}\right)},$$

where H denotes the upper half plane.

The Möbius transformation $w=(z-1)/z$ maps $\mathbf{C}\backslash\{0,1\}$ onto itself and carries $\frac{1}{2}$ to -1. Using this with (12), we obtain

$$\lambda_{\mathbf{C}\backslash\{0,1\}}\left(\tfrac{1}{2}\right) = 4\lambda_{\mathbf{C}\backslash\{0,1\}}(-1) = \frac{8\pi^2}{\Gamma^4\left(\frac{1}{4}\right)}.$$

Since $\lambda_H(z) = 1/2y$ we have $\lambda_H(i) = \frac{1}{2}$. Thus,

$$|(\kappa^2)'(i)| = \frac{\Gamma^4\left(\frac{1}{4}\right)}{16\pi^2} = \frac{B^2\left(\frac{1}{4},\frac{1}{4}\right)}{16\pi}.$$

Finally, $|\tau'(0)| = 2$, so

$$a(1) = A(e^{2\pi}) = f'(0) = 2|(\kappa^2)'(i)|\left|g'\left(\tfrac{1}{2}\right)\right| = B\left(\tfrac{1}{4},\tfrac{1}{4}\right).$$

According to Matlab, the numerical value of $a(1) = B\left(\frac{1}{4},\frac{1}{4}\right)$ is ≈ 7.416.

10. Computation of $a'(0)$

Recall that for $k>0$ the function f maps a certain circular quadrilateral Q in the unit disk onto the rectangle R with vertices 0, $2\pi k$, $2\pi k + 2\pi i$, $2\pi i$, and that $a(k) = f'(0)$. Of course, Q and f are determined by k. We shall sketch a proof that

$$a(k) = 4 + \frac{8k\ln 4}{\pi} + o(k), \quad k \to 0. \tag{15}$$

Some details will be left to the reader.

Let $g_2(z) = 2\log(1+z)/(1-z)$. The map $z \to (1+z)/(1-z)$ maps geodesics of the unit disk symmetric with respect to the real axis onto semicircles of constant modulus in the right half plane. Thus, g_2 maps such geodesics onto vertical segments of length 2π which are symmetric with respect to the real axis. We assume throughout this section that k is small. Then g_2 maps the unit disk onto the horizontal strip $|\operatorname{Im} w| < \pi$ and maps Q onto a narrow quadrilateral Q_2 which is symmetric with respect to both coordinate axes and is bounded by two vertical segments each of length 2π and two small curves orthogonal to the boundary of the strip which are almost semicircles. We have

$$g_2{}'(0) = 4.$$

Suppose that Q_2 has width $\varepsilon > 0$. Then k and ε are functions of each other, each tending to zero when one of them does. Define f_2 by $f = f_2 \circ g_2$. Then f_2 maps Q_2 onto the rectangle R. By reflecting in vertical sides, we extend f_2 to a conformal map of a subdomain of $|\operatorname{Im} z| < \pi$ onto the strip $0 < \operatorname{Im} z < 2\pi$. As $k \to 0$, the maps converge locally uniformly to a conformal map of $|\operatorname{Im} z| < \pi$ onto $0 < \operatorname{Im} z < 2\pi$ which carries 0 to $i\pi$ and has positive derivative at the origin. This limit map must be $z \to z + i\pi$. In particular, the second derivatives of the f_2 converge locally uniformly to zero. It follows that

$$f_2(z) - f_2(0) = z f_2'(0) + o(z^2)$$

in a neighborhood of 0, where the error term is uniform in k. Taking $z = \frac{1}{2}\varepsilon$ and using $f_2\left(\frac{1}{2}\varepsilon\right) = 2\pi k + i\pi$, we deduce that

$$f_2'(0) = \frac{2\pi k}{\varepsilon} + o(\varepsilon), \quad \varepsilon \to 0.$$

Since $f_2'(0)$ converges to 1, it follows that $\varepsilon \sim 2\pi k$, and we can replace the equation above by

$$f_2'(0) = \frac{2\pi k}{\varepsilon} + o(k), \quad k \to 0. \tag{16}$$

Next, let us rescale both Q_2 and R to the width 1, denoting the rescaled quadrilaterals by Q_2' and R', and let G be the conformal map from Q_2' onto R' with $G'(0) > 0$. These quadrilaterals are tall; we are interested in the difference of their heights. If we cut Q_2' and R' by horizontal segments in the middle, the lower half of Q_2' will be conformally equivalent to the lower half of R' (as curvilinear quadrilaterals). Let us compare the restriction of G to these halves with the conformal map F of the triangle T (see Section 9) onto a vertical halfstrip with vertices 0, 1, ∞ (and right angles at 0 and 1). Such a map with the vertex correspondence $(0,1,\infty) \to (0,1,\infty)$ is given by

$$F(\tau) = \frac{1}{\pi} \arccos \frac{2 - \kappa^2(\tau)}{\kappa^2(\tau)}.$$

Using the notation from [4], [20], we have

$$\begin{aligned}
&\kappa^2 = (\theta_2/\theta_3)^4, \quad [20, \text{II}, 4, \text{Section } 5],\\
&\theta_2 = 2h^{1/4}(1 + h^2 + h^6 + h^{12} + \ldots),\\
&\theta_3 = 1 + 2h + 2h^4 + 2h^9 + \ldots, \quad (\text{see } [20, \text{II}, 2, \text{Section } 6]),
\end{aligned}$$

where

$$h = \exp(\pi i \tau).$$

It follows that $\kappa^2(h) = 16h - 128h^2 + O(h^3)$ as $h \to 0$ and that

$$F(\tau) = \tau - i\frac{\ln 4}{\pi} + o(1), \quad \tau \to +\infty,\ \tau \in T. \tag{17}$$

Moreover, using (17) together with extremal length, or by some other argument, one can show that uniformly for z in the bottom half of R',

$$G(z) = F\left(z + \frac{1}{2} + \frac{\pi i}{\varepsilon}\right) + o(1).$$

It follows that R' is shorter than Q_2' by

$$\frac{2}{\pi}\ln 4 + o(1).$$

The height of R' equals $1/k$ and the height of Q_2' equals $2\pi/\varepsilon$. Thus

$$\frac{1}{k} = \frac{2\pi}{\varepsilon} - \frac{2}{\pi}\ln 4 + o(1). \tag{18}$$

The desired relation (15) now follows from (16), (18) and $a(k) = f'(0) = 4f_2'(0)$. From (15) we recover the relation $a(0) = 4$ of (14) and also find that

$$a'(0) = \frac{8\ln 4}{\pi} = 3.5302\ldots.$$

Returning to our original notation, and using (14) we obtain

$$A(K) = 4 + \frac{4\ln 4}{\pi^2}(K-1) + o(K-1), \quad K \to 1+.$$

11. A conjecture

Rademacher's Conjecture for the exact value of Landau's constant may be formulated as follows: The ratio $|f'(0)|/l(f)$ is maximized over all holomorphic functions f in the unit disk $\mathbf{U}$ when f is a universal cover from the disk onto the complement of a hexagonal lattice and $f(0)$ is the barycenter of one of its complementary equilateral triangles. Here, as in Section 2, $l(f)$ denotes the inradius of the domain $f(\mathbf{U})$. See, for example, [7] for discussion.

This formulation suggests a corresponding problem for rectangular lattices: Maximize $|f'(0)|/l(f)$ when f runs through all universal covers from $\mathbf{U}$ onto the complement of a rectangular lattice.

Conjecture. *The maximum ratio is achieved when the lattice is square and $f(0)$ is the center of some fundamental complementary square.*

We may restrict attention to lattices of the form $\{2m\omega + 2n\omega'\}$ with ω equal to πk for some positive number k and $\omega' = \pi i$. Furthermore, from Lemma 1 in Section 4, it follows that we need only consider maps f which carry 0 to the center of a fundamental complementary rectangle R. We are now in the situation of Section 8, and have

$$|f'(0)| = a(k).$$

The inradius $l(f)$ equals half the length of the diagonal of R. Thus

$$l(f) = \pi(1 + k^2)^{1/2}.$$

The rectangle R is a square when $k = 1$. Thus, the conjecture above can be restated as

$$a(k) \le 2^{-1/2}(1 + k^2)^{1/2}a(1), \quad 0 < k < \infty.$$

Since $a(0) = 4$ and $a(1) = 7.416$, the conjecture is true for small values of k.

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