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MAPPINGS OF FINITE DISTORTION: REMOVABILITY OF CANTOR SETS

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Abstract. Let f be a mapping of finite distortion omitting a set of positive conformal modulus. We show that if the distortion of f satisfies a certain subexponential integrability condition, then small regular Cantor sets are removable.

1. Introduction

We call a mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ *a mapping of finite distortion* if it satisfies

$$|Df(x)|^n \le K(x,f)J(x,f) \qquad \text{a.e.,} \tag{1.1}$$

where $K(x,f) < \infty$ and if also $J(\,\cdot\,,f) \in L^1_{\mathrm{loc}}(\Omega)$. Here $\Omega \subseteq \mathbf{R}^n$ is a domain i.e. an open and connected set. When $K(\,\cdot\,,f) \in L^\infty(\Omega)$, f is called a mapping of bounded distortion, or a quasiregular mapping. Quasiregular mappings are by now well understood, see the monographs [25] and [28]. Many basic results of quasiregular mappings have recently been generalized for mappings of finite distortion under some integrability assumptions on the distortion function K. Let us describe an assumption that has turned out to be sharp in many respects. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a strictly increasing, differentiable function. We call such functions Orlicz functions and we make the following two assumptions:

(Φ-1) $\displaystyle\int_1^\infty \frac{\Phi'(t)}{t}\,dt = \infty$,

(Φ-2) $t\,\Phi'(t)$ increases to infinity when $t \to \infty$.

We shall consider mappings of finite distortion f for which there exists a Φ, satisfying conditions (Φ-1) and (Φ-2), such that

$$\exp\bigl(\Phi\bigl(K(\,\cdot\,,f)\bigr)\bigr) \in L^1_{\mathrm{loc}}(\Omega). \tag{1.2}$$

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It has been shown that mappings of finite distortion satisfying this assumption are continuous, open and discrete, and that they have the Lusin property, i.e. they map sets of measure zero to sets of measure zero, see [7], [9] [11], [12], [13] for these as well as other results.

The main methods in the study of quasiregular mappings are modulus inequalities and quasilinear partial differential equations. Until very recently, only the latter method has been available in the more general setting of mappings of finite distortion. In [15], Koskela and Onninen generalize the modulus inequalities to the class of mappings of finite distortion satisfying assumption (1.2). Their results open new possibilities in the development of this theory. In this paper we prove a certain removability theorem using these inequalities. See [2], [3], [4], [17], [21], [22], [24] and [30] for other results concerning modulus inequalities in cases where the distortion is unbounded.

Let us next recall some removability results related to the setting of this paper. Recall that the (conformal) n-modulus of a path family Γ is defined by

$$M(\Gamma)=\inf\left\{\int_{\mathbf{R}^n}\varrho^n(x)\,dx : \varrho\colon \mathbf{R}^n\to[0,\infty) \text{ is a Borel function such that } \int_\gamma \varrho\ge 1 \text{ for each locally rectifiable } \gamma\in\Gamma\right\}.$$

For a mapping of finite distortion f, define the K^{n-1}-modulus $M_{K^{n-1}(\,\cdot\,,f)}(\Gamma)$ by

$$M_{K^{n-1}(\,\cdot\,,f)}(\Gamma)=\inf\left\{\int_{\mathbf{R}^n}\varrho^n(x)K^{n-1}(x,f)\,dx : \varrho\colon \mathbf{R}^n\to[0,\infty) \text{ is a Borel function such that } \int_\gamma \varrho\ge 1 \text{ for each locally rectifiable } \gamma\in\Gamma\right\}.$$

For a set $A\subseteq\mathbf{R}^n$, we denote $M_{K^{n-1}(\,\cdot\,,f)}(A)=M_{K^{n-1}(\,\cdot\,,f)}(\Gamma)$, where Γ is the family of all non-constant curves starting at A. Note that since $K(x,f)\ge 1$ for all $x\in\Omega$, the inequality $M(\Gamma)\le M_{K^{n-1}(\,\cdot\,,f)}(\Gamma)$ always holds. If E and F are compact sets contained in a domain $D\subseteq\mathbf{R}^n$, we then denote by $\Delta(E,F,D)$ the family of all curves joining E and F in D.

Already in 1970 Martio, Rickman and Väisälä [20] proved that a set of zero conformal modulus is removable for a quasimeromorphic (see below) mapping omitting a set of positive conformal modulus. This theorem holds also for mappings of finite distortion satisfying assumption (1.2), if we assume the removable set to be of zero K^{n-1}-modulus; see [23] for discussion. Later Iwaniec and Martin [8] and Iwaniec [5] have shown that there exists a bound $m(K)>0$ so that sets with Hausdorff dimension smaller than m are removable for bounded K-quasiregular mappings. In dimension three, Rickman [29] has shown that for each $\lambda>0$ there exists a Cantor set E in $\mathbf{R}^3$ with Hausdorff dimension smaller than

λ, and a bounded $K(\lambda)$-quasiregular mapping $f : \mathbf{R}^3 \setminus E \to \mathbf{R}^3$ that does not extend continuously to any point of E.

In [10] Järvi and Vuorinen showed that even when the omitted set consists only of a sufficiently large finite number of points, then some self-similar Cantor sets are removable for quasiregular mappings. Their proof is based on Rickman's work on the generalizations of the Picard theorems and the corresponding continuity estimates near the omitted values; see [26] and [27]. In this paper we shall give a removability result similar to the theorem by Järvi and Vuorinen. Our proof uses the modulus inequlities of [15] and the methods in [10]. Since we do not have any decent Rickman–Picard theorems for mappings of finite distortion, we have to make a stronger assumption on the omitted set; we shall assume that the omitted set is of positive conformal modulus. Naturally, the removable Cantor sets in our case are also smaller than in the quasiregular case. We shall state our main theorem in Section 3. Since the statement is a bit technical, we give a simplified statement at the end of this introduction.

Our notation will be similar to [10]. We denote by $\overline{\mathbf{R}}^n$ the one-point compactification of $\mathbf{R}^n$, and give $\overline{\mathbf{R}}^n$ the chordal metric

$$\begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{(1+|x|^2)(1+|y|^2)}}, & x,y \neq \infty, \\ q(x,\infty) = \dfrac{1}{\sqrt{1+|x|^2}}. \end{cases}$$

We shall denote a chordal ball with center x and radius r by $Q(x,r)$, while the corresponding Euclidean notation is $B(x,r)$. Also, for a set A, $d(A)$ will denote the Euclidean diameter of A, and $q(A)$ the chordal diameter. For modulus, the notation introduced earlier will also be used for curve families in $\overline{\mathbf{R}}^n$.

We extend the definition of a mapping of finite distortion to $\overline{\mathbf{R}}^n$-valued mappings as follows: Let $f\colon \Omega \to \overline{\mathbf{R}}^n$ be a mapping, $\Omega \subseteq \mathbf{R}^n$. Then f is a mapping of finite distortion, if each $x \in \Omega$ has a neighborhood $U \subseteq \Omega$ so that $g \circ f_U\colon U \to \mathbf{R}^n$ is a mapping of finite distortion for some Möbius transformation $g\colon \overline{\mathbf{R}}^n \setminus \{p\} \to \mathbf{R}^n$, where f_U is the restriction of f to U. Note that assumption (1.2) is well-defined for $\overline{\mathbf{R}}^n$-valued mappings also, since the distortion does not depend on the Möbius transformation g. In the case of quasiregular mappings, the $\overline{\mathbf{R}}^n$-valued generalizations are often called quasimeromorphic mappings.

Let us state a special case of Theorem 3.7 below.

Theorem 1.1. *Let $F \subseteq \overline{\mathbf{R}}^n$ be the image of the one-third Cantor set under the stereographic projection, and let E be a regular (as defined in Section 2) Cantor set. Moreover, let $f\colon B(0,1) \setminus E \to \overline{\mathbf{R}}^n \setminus F$ be a mapping of finite distortion for which*

$$I = \int_{B(0,1)} \exp(\gamma K) < \infty, \qquad \text{where } \gamma > 0. \tag{1.3}$$

Then there exists a constant $C = C(I, \gamma, n) > 0$ *such that if*

$$\mathscr{H}_{|\log t|^{-C}}(E) = 0,$$

then f *extends to mapping of finite distortion* $\tilde{f}\colon B(0,1) \to \overline{\mathbf{R}}^n$ *satisfying* (1.3). *Moreover,* $C(I, \gamma, n) \to \infty$ *as* $\gamma \to \infty$, *and thus* E *may have positive conformal modulus when* γ *is large enough* (*compared to the other data*).

Here $\mathscr{H}_\Lambda$ is the Hausdorff measure with respect to a gauge function Λ. In our main theorem, Theorem 3.7, the function $\Phi(t) = \gamma t$ (see (1.2)) is replaced by a general Orlicz function. Also, general omitted sets are considered, and the size of the removable Cantor set will depend on the modulus of the omitted set. Estimates for the size of the removable Cantor sets are given in Section 4.

Earlier removability results for mappings of finite distortion have been proved in [1], [6], [16] and [23]. In [1] it is proved that if

$$f\colon \mathbf{D}(0,1) \setminus E \to \mathbf{C} \tag{1.4}$$

is a bounded mapping of finite distortion satisfying (1.3) for a large enough constant γ, and if $E \subseteq \mathbf{D}(0,1)$ is any compact set with $\mathscr{H}_{|\log t|^{-3/2}}(E) = 0$, then E is removable for f. It follows, in particular, that in this case all sets of zero conformal modulus are removable. They also show that if γ is small, then there are regular Cantor sets E of zero conformal modulus that are not removable for mappings f as in (1.4). Thus Theorem 1.1 can be viewed as an extension of their result, and their example shows that our theorem is in a sense sharp. In fact they have given, using Rickman's nonremovability result mentioned above, also a similar example in dimension three.

2. Cantor sets

We shall construct our Cantor set inductively. First, remove an open interval of length $1 - t_1$ from the middle of $[0,1]$. Denote $E_1 = \bigl[0, \frac{1}{2}t_1\bigr] \cup \bigl[1 - \frac{1}{2}t_1, 1\bigr]$. In the second step remove similarly an open interval of length $\frac{1}{2}t_1(1-t_2)$ from the middle of each of the two intervals in E_1. The resulting set E_2 then consists of four intervals of equal length. Continuing similarly, we have in the pth step a set E_p consisting of 2^p intervals of length

$$\frac{\prod_{i=1}^p t_i}{2^p} = \frac{T_p}{2^p}. \tag{2.1}$$

Let

$$\widetilde{E}_p = E_p \times E_p \times \cdots \times E_p \subset \mathbf{R}^n.$$

We obtain the Cantor set E by setting

$$E = \bigcap_{p=1}^{\infty} \widetilde{E}_p.$$

We shall give a decomposition of the complement of E, which will be useful in proving Theorem 3.7. The set $\widetilde{E}_p$ consists of 2^{np} cubes. By (2.1), each cube is contained in a ball with the same center and of radius $r_p = \sqrt{n}\,T_p/2^{p+1}$, and such balls are disjoint even if we enlarge the radii to

$$R_p = \frac{r_p}{\sqrt{n}}\left(\frac{2}{t_p} - 1\right).$$

From now on we assume that $R_p > r_p$, which is implied by $t_p < 2/\bigl(1+2\sqrt{n}\,\bigr)$. Let the centers of these balls be $z(p,i)$, $i = 1,\dots,2^{np}$, and let

$$S_{p,i} := S^{n-1}\bigl(z_{p,i}, \alpha(r_p,R_p)\bigr), \qquad i = 1,\dots,2^{np}, \tag{2.2}$$

where $\alpha(r_p,R_p) \in (r_p,R_p)$ is suitably chosen, see Theorem 3.7. Each $S_{p,i}$ encloses 2^n of the spheres $S_{p+1,j}$ of the next generation, and we denote by $G_{p,i}$ the domain bounded by $S_{p,i}$ and the spheres of the next generation. Then we have the decomposition

$$\bigcup_{i=1}^{2^n} (\overline{B}_{1,i} \setminus E) = \bigcup_{p=1}^{\infty} \bigcup_{i=1}^{2^{np}} \overline{G}_{p,i},$$

where $B_{1,i} = B\bigl(z_{1,i}, \alpha(r_1,R_1)\bigr)$, $i = 1,\dots,2^n$.

3. Modulus of continuity on annuli

We will use the following modulus inequality, which is a counterpart for the Poletsky inequality of quasiregular mappings. This is a special case of Theorem 4.1 in [15].

Lemma 3.1. *Let $f\colon \Omega \to \mathbf{R}^n$ be a mapping of finite distortion satisfying (1.2), with an Orlicz function Φ for which (Φ-1) and (Φ-2) hold. Let Γ be a path family in Ω. Then*

$$M(f\Gamma) \le M_{K^{n-1}(\,\cdot\,,f)}(\Gamma). \tag{3.1}$$

For spherical rings we have the following upper bound for the K^{n-1}-modulus, see [15, Theorem 5.3].

Lemma 3.2. *Suppose that $I = \int_{B(0,1)} \exp\bigl(\Phi(K)\bigr) < \infty$. Let $0 < 4r < R < 1$. Then there exist $C_1, C_2 > 0$ depending on n, Φ and I such that*

$$M_{K^{n-1}(\,\cdot\,,f)}(\Gamma) \le C_1 \left(\int_{2r}^{R/2} \frac{ds}{s\Phi^{-1}\bigl(\log(C_2 s^{-n})\bigr)} \right)^{1-n} =: \varphi(I,R,r), \tag{3.2}$$

where Γ is the family of all curves connecting $\overline{B}(0,r)$ and $\mathbf{R}^n \setminus B(0,R)$.

In what follows, φ will stand for the function defined in inequality (3.2). The next lemma gives a lower n-modulus bound, see [10, Lemma 2.12] for a proof.

Lemma 3.3. *Let $F \subset B(0,t)$ be a continuum. Then there exists a constant $\lambda > 0$ so that*

$$M(\Gamma) \geq \left(\log\left(\frac{\lambda t}{d(F)}\right)\right)^{1-n},$$

where Γ is the family of all curves connecting F to $S^{n-1}(0,t)$.

Now we have the following estimate on annuli.

Lemma 3.4. *Let $0 < a < b < \infty$, $s > 0$, and let*

$$f\colon B(0,b)\setminus \overline{B}(0,a) \to B(0,s)$$

be a mapping of finite distortion with

$$\int_{B(0,b)\setminus B(0,a)} \exp\bigl(\Phi(K)\bigr) \leq I.$$

If $\alpha(a,b) \in (a,b)$, then

$$d\bigl(fS^{n-1}\bigl(0,\alpha(a,b)\bigr)\bigr) \leq \lambda s \exp\Bigl(-\bigl(\varphi\bigl(I,b,\alpha(a,b)\bigr)+\varphi\bigl(I,\alpha(a,b),a\bigr)\bigr)^{1/(1-n)}\Bigr). \tag{3.3}$$

Proof. Fix $x,y \in S^{n-1}\bigl(0,\alpha(a,b)\bigr)$ with $|f(x)-f(y)| = d\bigl(fS^{n-1}\bigl(0,\alpha(a,b)\bigr)\bigr)$ and a continuum $C \subseteq S^{n-1}\bigl(0,\alpha(a,b)\bigr)$ joining x and y. Let Γ' be the family of all curves joining fC and $S^{n-1}(0,s)$ and let Γ be the family of the maximal liftings (see [28, Chapter II, Section 3]) of paths of Γ' starting at C. Then $\overline{|\gamma|} \cap \partial\bigl(B(0,b)\setminus B(0,a)\bigr) \neq \emptyset$ for all $\gamma \in \Gamma$. The K^{n-1}-modulus of the family Γ can now be estimated by the moduli of two spherical rings, and by Lemma 3.2

$$M_{K^{n-1}}(\Gamma) \leq \varphi\bigl(I,b,\alpha(a,b)\bigr) + \varphi\bigl(I,\alpha(a,b),a\bigr).$$

On the other hand, Lemma 3.3 implies that

$$M(\Gamma') \geq \left(\log\left(\frac{\lambda s}{d\bigl(fS^{n-1}\bigl(0,\alpha(a,b)\bigr)\bigr)}\right)\right)^{1-n},$$

and thus by Lemma 3.1,

$$\begin{aligned}\left(\log\left(\frac{\lambda s}{d\bigl(fS^{n-1}\bigl(0,\alpha(a,b)\bigr)\bigr)}\right)\right)^{1-n} &\leq M(\Gamma') \leq M_{K^{n-1}}(\Gamma)\\ &\leq \varphi\bigl(I,b,\alpha(a,b)\bigr) + \varphi\bigl(I,\alpha(a,b),a\bigr),\end{aligned}$$

which implies (3.3). $\square$

Lemma 3.4 is also locally valid for mappings taking values in $\overline{\mathbf{R}}^n$. We state this as a corollary, see [10, Corollary 2.14] for a proof of a similar statement.

Corollary 3.5. *Let* $0 < a < b < \infty$, $0 < s < 1/\sqrt{2}$, *and let*

$$f\colon B(0,b)\setminus \overline{B}(0,a) \to Q(0,s)$$

be a mapping of finite distortion with

$$\int_{B(0,b)\setminus B(0,a)} \exp\bigl(\Phi(K)\bigr) \le I.$$

Then

(3.4)

$$q\bigl(fS^{n-1}\bigl(0,\alpha(a,b)\bigr)\bigr) \le \sqrt{2}\,\lambda s \exp\Bigl(-\bigl(\varphi\bigl(I,b,\alpha(a,b)\bigr)+\varphi\bigl(I,\alpha(a,b),a\bigr)\bigr)^{1/(1-n)}\Bigr).$$

The following lemma will give continuity estimates for $\overline{\mathbf{R}}^n$-valued mappings omitting sets of positive conformal modulus. See [28, Lemma III 2.6] for the proof.

Lemma 3.6. *Let* F *be a compact subset of* $\mathbf{R}^n$ *such that* $M(F) > 0$. *Moreover, let* $0 < L < 1$ *and choose a ball* $B(x,r)$ *containing* F, *with the property* $q\bigl(\overline{\mathbf{R}}^n \setminus B(x,r)\bigr) < \frac{1}{2}L$. *Assume that* C *is a continuum in* $\overline{\mathbf{R}}^n$ *with* $q(C) \ge L$. *Then*

$$M(\Gamma) \ge 3^{-n}\min\bigl\{C(n), \bigl(\log(2\lambda r/L)\bigr)^{1-n}, M(\Gamma_{x,L})\bigr\} =: \eta(L), \tag{3.5}$$

where Γ *is the family of all curves joining* F *and* C, λ *is as in Lemma 3.3 and* $\Gamma_{x,L} = \Delta\bigl(F, S^{n-1}(x,2r), B(x,2r)\bigr)$.

We are finally ready to state our main theorem. The proof will be given in Section 5.

Theorem 3.7. *Assume that* E *is a Cantor set defined as in Section 2. Moreover, assume that* $F \subseteq \overline{\mathbf{R}}^n$ *is a closed set of positive conformal modulus. Let* $f\colon B(0,1)\setminus E \to \overline{\mathbf{R}}^n \setminus F$ *be a mapping of finite distortion satisfying*

$$I = \int_{B(0,1)} \exp\bigl(\Phi(K)\bigr) < \infty \tag{3.6}$$

with an Orlicz function Φ *for which* (Φ-1) *and* (Φ-2) *hold. Moreover, assume that in the construction of Section 2, for each* $p \in \mathbf{N}$ *there exists* $\alpha(r_p,R_p) \in (r_p,R_p)$ *such that*

$$\begin{aligned}\varphi\bigl(I,R_p,\alpha(r_p,R_p)\bigr) &+ \varphi\bigl(I,\alpha(r_p,R_p),r_p\bigr)\\ &\le \min\bigl\{(3n+\log\lambda)^{1-n}, \eta\bigl(\bigl(2\sqrt{2}\,(2^n+1)\bigr)^{-1}\bigr)\bigr\} =: w,\end{aligned} \tag{3.7}$$

where λ *is as in Lemma 3.3 and* η *as in* (3.5). *Then* f *extends to mapping of finite distortion* $\tilde f\colon B(0,1) \to \overline{\mathbf{R}}^n$ *satisfying* (3.6).

4. Estimates for the Cantor set

In this section we will estimate the Hausdorff dimensions of the removable sets in Theorem 3.7 with respect to suitable gauge functions. We first note that

$$\int_0 \frac{ds}{s\Phi^{-1}\bigl(\log(C_2 s^{-n})\bigr)} = \frac{1}{n}\int^{\infty} \frac{\Phi'(t)}{t}\,dt = \infty$$

as soon as assumptions (Φ-1) and (Φ-2) hold for Φ, and thus for each $I, R > 0$, $\varphi(I,R,r) \to 0$ as $r \to 0$. So, given the distortion function and the omitted set in Theorem 3.7, there exists a sequence (t_p) so that if the construction of the Cantor set is carried out by using this sequence, then the assumptions of Theorem 3.7 are satisfied. One may for example choose $\alpha(a,b) = \sqrt{ab}$, which is the most convenient choice in the case of quasiregular mappings. In principle it is also possible to estimate the size of the largest possible set satisfying these assumptions. However, if we do not know the Orlicz function Φ, these estimates are quite implicit. For this reason we will only consider the case where the Orlicz function is of form $\Phi(t) = \gamma t$, $\gamma > 0$. In particular, we will establish the bound given in Theorem 1.1.

First we want to choose the numbers $\alpha(r_p, R_p)$ so that we have the smallest possible upper bound for $\varphi\bigl(I, R_p, \alpha(r_p,R_p)\bigr) + \varphi\bigl(I, \alpha(r_p,R_p), r_p\bigr)$. From the equation $\varphi\bigl(I, R_p, \alpha(r_p,R_p)\bigr) = \varphi\bigl(I, \alpha(r_p,R_p), r_p\bigr)$ we have

$$\alpha(r_p, R_p) = \exp\Bigl(-\sqrt{|\log r_p|\,|\log R_p|}\,\Bigr).$$

From the inequality

$$2\varphi\bigl(I, \alpha(r_p,R_p), r_p\bigr) \le w,$$

where w is as in Theorem 3.7, we then have the essentially sharp condition

$$r_p \le R_p^b, \qquad \text{where } b = \exp\bigl(3C_1^{1/(n-1)} n w^{1/(1-n)}/\gamma\bigr). \tag{4.1}$$

Since

$$R_p = \frac{r_p}{\sqrt{n}}\left(\frac{2}{t_p} - 1\right) = \frac{T_p}{2^{p+1}}\left(\frac{2}{t_p} - 1\right) \ge \frac{T_{p-1}}{2^{p+1}} = \frac{r_{p-1}}{2\sqrt{n}},$$

the radii r_p can be defined by the requirement

$$r_p \le \left(\frac{r_{p-1}}{2\sqrt{n}}\right)^b.$$

So if we set

$$r_p = C(n)^{-(2b)^p}$$

for a large enough constant $C(n) > 1$, condition (4.1) is then satisfied for all $p \in \mathbf{N}$. The gauge function Λ for the Hausdorff measure $\mathscr{H}_\Lambda$ has to be defined so that the sequence

$$\bigl(2^{np}\Lambda\bigl(C(n)^{-(2b)^p}\bigr)\bigr)_p$$

behaves essentially like a sequence converging to a positive number (recall that in the pth step there are 2^{np} balls). So the correct function is of type

$$\Lambda(t) = |\log t|^{-n\log_b 2}. \tag{4.2}$$

Since our Cantor construction is regular and the gauge function Λ is nice, this 'obvious upper bound' for the gauge function is also a lower bound. Because sets with finite $\mathscr{H}_{|\log t|^{1-n-\varepsilon}}$-measure have positive conformal modulus (see cf. [1]), it especially follows that when γ is large enough compared to the other data so that $n\log_b 2 > n-1$, the removable set may have positive conformal modulus.

5. Proof of Theorem 3.7

Let the Cantor set E and the decomposition be as above. By composing f with a Möbius transformation, we may assume that $\infty \notin E$. So there is no problem in using Lemma 3.6. We shall first show that there exists a continuous extension $\tilde f$ of f, defined in $B(0,1)$. To prove this, we show that for each $p = 0, 1, \dots$ the following estimates hold for every $r = p,\ p+1, \dots$ and $i = 1, \dots, 2^{nr}$:

$$\begin{cases} q(fG_{r+1,i}) \le \dfrac{1}{2\sqrt2}(Ah)^p, \\ q(fS_{r+2,i}) \le \lambda A^p h^{p+1}, \qquad A = \sqrt2\,2(2^n+1)\lambda, \end{cases} \tag{5.1}$$

where λ is as in Lemma 3.3. Here we require $Ah < 2^{-n}$, i.e.

$$h < \frac{1}{2^{n+3/2}(2^n+1)\lambda}.$$

For $p = 0$ and fixed r, i, let Γ be the family of all paths connecting $fS_{r+2,i}$ and F. Moreover, denote by Γ' the family of all maximal liftings of paths in Γ starting at $S_{r+2,i}$. Since all paths in Γ' intersect $S(z_{r+2,i}, R_p) \cup S(z_{r+2,i}, r_p)$, Lemmas 3.1 and 3.2, together with assumption (3.7) imply

$$M(\Gamma) \le M_{K^{n-1}}(\Gamma') \le M_{K^{n-1}}(\Gamma^*) \le w,$$

where Γ^* is the family of all paths joining $S_{r+2,i}$ and the complement of $B(z_{r+2,i}, R_p)\setminus \overline{B}(z_{r+2,i}, r_p)$. Thus by Lemma 3.6,

$$q(fS_{r+2,i}) \le \frac{1}{2\sqrt2\,(2^n+1)}, \tag{5.2}$$

and this holds for the spheres $S_{1,i}$ also. From inequality (5.2) and the openness of f, it follows that the chordal diameter of a set $fG_{k,i}$ is bounded by the sum of the chordal diameters of the images of the spheres forming the boundary of $G_{k,i}$:

$$q(fG_{r+1,i}) \le q(fS_{r+1,i}) + \sum_{j=1}^{2^n} q(fS_{r+2,j}) \le \frac{2^n+1}{2\sqrt{2}\,(2^n+1)} = \frac{1}{2\sqrt{2}}.$$

This is the first inequality in (5.1). For the second inequality we notice that $S_{r+2,i}$ is on the common boundary of G_{r+1,i_1} and G_{r+2,i_2} for some i_1, i_2. Thus by the first inequality of (5.1), f maps the annulus $B(z_{p,i}, R_{r+3}) \setminus \overline{B}(z_{p,i}, r_{r+3})$ into a ball of chordal diameter $1/\sqrt{2}$, and Corollary 3.5 combined with assumption (3.7) gives

$$q(fS_{r+2,i}) \le \lambda h.$$

Note that assumption (3.7) suffices for this, since

$$\varphi\big(I, R_p, \alpha(r_p, R_p)\big) + \varphi\big(I, \alpha(r_p, R_p), r_p\big) \le (3n + \log\lambda)^{1-n}$$

implies

$$\exp\Big(-\Big(\varphi\big(I, R_p, \alpha(r_p, R_p)\big) + \varphi\big(I, \alpha(r_p, R_p), r_p\big)\Big)^{1/(1-n)}\Big) < \frac{1}{2^{n+3/2}(2^n+1)\lambda}.$$

Thus both inequalities in (5.1) hold for $p = 0$. We then assume that (5.1) holds for a fixed p. Using the second inequality in (5.1) and the openness of f as above, we then have

$$q(fG_{r+2,i}) \le \lambda(2^n+1)h^{p+1} = \frac{1}{2\sqrt{2}}\,(Ah)^{p+1},$$

which is the first inequlity in (5.1) for $p+1$. For the second inequality, notice again that $S_{r+3,i}$ is on the common boundary of G_{r+2,i_1} and G_{r+3,i_2} for some i_1, i_2. Thus by the first inequality of (5.1), f maps the annulus $B(z_{p,i}, R_{r+3}) \setminus \overline{B}(z_{p,i}, r_{r+3})$ into a ball of chordal diameter $A^{p+1}h^{p+1}/\sqrt{2}$, and Lemma 3.4 and the openness of f give

$$q(fS_{r+3,i}) \le \lambda A^{p+1} h^{p+2}.$$

This proves (5.1) for all p.

Now let $B_{s,i}$ be the ball whose boundary is $S_{s,i}$. Then there are $2^{(m-s)n}$ balls $B_{m,j}$, $m > s$, in $B_{s,i}$ and by (5.1),

$$\begin{aligned} q(\tilde{f}(B_{s,i} \setminus E)) &\le q\bigg(f\bigg(\bigcup_{m=s}^{\infty} \overline{G}_{m,j}\bigg)\bigg) \le \frac{1}{2\sqrt{2}} \sum_{m=s}^{\infty} 2^{(m-s)n} A^{m-1} h^{m-1} \\ &= \frac{1}{\sqrt{2}}\, 2^{-sn-1} A^{-1} h^{-1} \sum_{m=s}^{\infty} (2^n A h)^m \to 0 \end{aligned} \tag{5.3}$$

as $s \to \infty$, whenever $2^n Ah < 1$. Thus f has a limit at every $x \in E$.

Denote the extended mapping by $\tilde{f}$. To show that $\tilde{f}$ is a mapping of finite distortion, we have to show that $\tilde{f}$ is absolutely continuous on all lines parallel to coordinate axis, that the weak partial derivatives of $\tilde{f}$ are locally integrable and that also the Jacobian of $\tilde{f}$ is locally integrable. We may assume that $\tilde{f}$ takes values in $\mathbf{R}^n$. First, since the Hausdorff dimension of E is less than $n-1$ and f has the ACL property, it follows that also $\tilde{f}$ has the ACL property, see cf. [31, Theorem 35.1]. Thus weak partial derivatives exist. Secondly, by using inequality (1.1), the subexponential integrability of K and Hölder's inequality, one sees that the local integrability of the Jacobian guarantees local integrability of the partial derivatives. Thus it suffices to show that the Jacobian of $\tilde{f}$ is locally integrable. For this we first show that $\tilde{f}$ satisfies the Lusin condition, which is implied by $|\tilde{f}E|=0$. Since the question is local, we may assume that $\tilde{f}E \subseteq B(0,1)$. By the construction of E, E can be covered by 2^{np} balls $B(z_{p,i}, r_p)$, $i = 1, \dots, 2^{np}$ for each $p = 1, 2, \dots$. Thus $\tilde{f}E$ can be covered by the sets $\tilde{f}B(z_{p,i}, r_p)$. As in inequality (5.3), we have

$$q\bigl(\tilde{f}B(z_{p,i}, r_p)\bigr) \le \frac{(Ah)^{p-1}}{1 - 2^n Ah} \qquad \text{for } p = 1, 2, \dots.$$

Denote

$$\lambda_p = \frac{(Ah)^{p-1}}{1 - 2^n Ah}.$$

Then we can estimate the θ-Hausdorff contents by

$$\mathscr{H}_\theta^{\lambda_p}(\tilde{f}E) \le \sum_{i=1}^{2^{np}} d\bigl(\tilde{f}B(z_{p,i}, r_p)\bigr)^\theta \le 2^{np} 2^\theta \lambda_p^\theta \to 0 \qquad \text{as } p \to 0,$$

whenever $2^n (Ah)^\theta < 1$. This holds for

$$\theta > \frac{n \log 2}{\log \dfrac{1}{Ah}}.$$

By our choice of h, it particularly follows that $|\tilde{f}E| = 0$. Having the Lusin property, we will prove the local integrability of the Jacobian by standard arguments using the topological degree, cf. [14, Lemma 3.3]. Let $x_0 \in E$. Since $E \cup \tilde{f}^{-1}\bigl(\tilde{f}(x_0)\bigr)$ is totally disconnected, there exists a sphere $S^{n-1}(x_0, \varepsilon)$ such that $S^{n-1}(x_0, \varepsilon) \cap \tilde{f}^{-1}\bigl(\tilde{f}(x_0)\bigr) = \emptyset$. Let V be the $\tilde{f}(x_0)$-component of $\overline{\mathbf{R}}^n \setminus \tilde{f}S^{n-1}(x_0, \varepsilon)$ and let U be the x_0-component of $\tilde{f}^{-1}V$. Then U is an open neighborhood of x_0. The topological degree $\mu\bigl(\tilde{f}(x_0), \tilde{f}, U\bigr)$ is now well-defined, and we have

$$N(y, \tilde{f}, U) \le \mu\bigl(\tilde{f}(x_0), \tilde{f}, U\bigr) = m < \infty,$$

see cf. [19]. Here $N(y,\tilde f,U)=|\{x\in U:\tilde f(x)=y\}|$. On the other hand by [18, Theorem 9.2], the area formula holds for weakly differentiable mappings satisfying the Lusin condition. Hence

$$\int_U J_{\tilde f}=\int_{\mathbf{R}^n} N(y,\tilde f,U)\le |V|\mu\big(\tilde f(x_0),\tilde f,U\big)<\infty.$$

Thus $J_{\tilde f}$ is locally integrable. The proof is complete.

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