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POSITIVE SOLUTIONS OF SOME NONLINEAR ELLIPTIC PROBLEMS IN UNBOUNDED DOMAIN

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Abstract. We study the existence of positive solutions of the nonlinear elliptic equation $\Delta u + \varphi(\cdot, u) = 0$, in an unbounded domain D in $\mathbf{R}^n$, $n \geq 3$, with compact boundary. Our purpose is to give some existence results for the above equation with some boundary values, where the nonlinear term $\varphi(t,x)$ satisfies some appropriate conditions related to a certain Kato class $K^\infty(D)$. We give also some estimates on the solution u.

1. Introduction

Numerous results are obtained for elliptic equation of the form

$$\Delta u + \varphi(\,\cdot\,, u) = 0 \qquad \text{in } D, \tag{1.1}$$

which characterize asymptotic properties of solutions of this equation for both bounded and unbounded domain $D \subset \mathbf{R}^n$, $n \geq 3$ (see for example [1], [2], [4], [6], [9], [12] and the references therein). Specially, existence and asymptotic behaviour of solutions for exterior boundary value problems have been widely studied.

In the simplest case where $\varphi(x,u) = q(x)u^\alpha$, $\alpha > 0$ and $D = \mathbf{R}^n$, the equation (1.1) has been extensively studied for both superlinear case (i.e. $\alpha > 1$) and sublinear case (i.e. $0 < \alpha < 1$). In [6], Lin proved the existence of a family of positive solutions for the equation (1.1) under the conditions $\alpha > 1$ and

$$|q(x)| \leq \frac{\phi(|x|)}{|x|^2} \text{ at infinity with } \int^\infty \frac{\phi(r)}{r}\,dr < \infty. \tag{1.2}$$

He also proved that each of these solutions tends to some positive limit at infinity.

In [2], Brezis and Kamin considered $0 < \alpha < 1$ and proved the existence of a unique positive solution of (1.1) satisfying $\liminf_{|x|\to\infty} u(x) = 0$ provided that q is locally bounded such that Vq is bounded ($V = \Delta^{-1}$).

As a tool of studying global solutions of semilinear elliptic equations, Zhao introduced the class of Green tight functions $K_n^\infty(D)$, where D is an unbounded domain in $\mathbf{R}^n$, $n \geq 3$, with compact nonempty boundary ∂D (see [12]). More precisely, he considered the case $\varphi(x,u) = q(x)f(u)$, where $q \in K_n^\infty(D)$ and there

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is no restriction on the sign of f but it is superlinear at 0. He showed that for a small constant $\lambda > 0$, the equation (1.1) has a positive solution $u \in C(\overline{D})$ satisfying

$$u|_{\partial D} = 0 \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = \lambda.$$

Note that if q satisfies (1.2) then $q \in K_n^\infty(D)$.

More recently, in the case where D is an unbounded domain in $\mathbf{R}^n$, $n \geq 3$, with compact nonempty boundary, Bachar et al. introduced in [1] a new Kato class $K^\infty(D)$ (see Definition 1), and they established interesting properties pertaining to this class, which contains properly the classical Kato class $K_n^\infty(D)$. It is also shown in [1], that if φ is nonincreasing with respect to the second variable such that $\varphi(\,\cdot\,, c) \in K^\infty(D)$, for every $c > 0$, then (1.1) has a unique positive solution $u \in C(\overline{D})$ satisfying

$$u|_{\partial D} = 0 \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0.$$

In the present paper, our purpose is to give existence and asymptotic behaviour of solutions for the equation (1.1) with some boundary values (see problems (1.3) and (1.4) below), where D is an unbounded domain in $\mathbf{R}^n$, $n \geq 3$, with compact nonempty boundary ∂D and the nonlinear term $\varphi(t, x)$ satisfies some appropriate conditions related to the class $K^\infty(D)$.

Our paper is organized as follows. In the next section, we collect a number of preliminary results about the Green function G_D of the laplacian in D and the class $K^\infty(D)$. Our existence results are given in Sections 3 and 4.

In Section 3, we use a potential theory approach to investigate the existence of continuous solutions in $\overline{D}$ of the following nonlinear elliptic problem

$$\text{(1.3)} \qquad \begin{cases} \Delta u - uf(\,\cdot\,, u) = 0 \text{ in } D, \\ u > 0 \text{ in } D, \\ u|_{\partial D} = g, \\ \lim_{|x|\to\infty} u(x) = \lambda, \end{cases}$$

where $\lambda > 0$ and $g \in C^+(\partial D)$. We assume that $f\colon D \times [0, \infty) \to [0, \infty)$ is measurable and satisfies the following assumptions:

(H_1): For all $x \in D$, the map $t \longrightarrow tf(x, t)$ is nonnegative continuous on $[0, \infty)$.

(H_2): For all $c > 0$, there exists a positive function $q_c \in K^\infty(D)$ such that for each $x \in D$ and $0 \leq s < t \leq c$, we have

$$\frac{tf(x, t) - sf(x, s)}{t - s} \leq q_c(x).$$

Under these conditions, we also prove that if $g \equiv 0$, then the solution u of (1.3) satisfies the following asymptotic behaviour

$$\frac{1}{C}\frac{\delta_D(x)}{\delta_D(x)+1} \le u(x) \le C\frac{\delta_D(x)}{\delta_D(x)+1} \qquad \text{for } x \in D,$$

where C is a positive constant and $\delta_D(x)$ denotes the Euclidean distance between x and ∂D.

Note that in this section, our techniques are inspired by [9].

In Section 4, we establish an existence result for the following nonlinear problem:

$$\begin{cases} \Delta u + \varphi(\,\cdot\,, u) = 0 \text{ in } D, \\ u > 0 \text{ in } D, \; u|_{\partial D} = 0, \\ u \in C_0(D), \end{cases} \tag{1.4}$$

where $C_0(D)$ denotes the set of continuous functions u in $\overline{D}$ vanishing at ∂D and satisfying $u(x) \to 0$ as $|x| \to \infty$ within D.

Here the function φ is required to satisfy the following hypotheses:

(A_1): φ is a nonnegative measurable function on $D \times [0,\infty)$, continuous with respect to the second variable.

(A_2): There exist a nontrivial nonnegative function $p \in L^1_{\mathrm{loc}}(D)$ and a nonnegative function $q \in K^\infty(D)$ such that for $x \in D$ and $t > 0$,

$$p(x)f(t) \le \varphi(x,t) \le q(x)g(t), \tag{1.5}$$

where f and g: $(0,\infty) \to [0,\infty)$ are two measurable nondecreasing functions satisfying

$$\lim_{t\to 0}\frac{f(t)}{t} = +\infty \quad \text{and} \quad \lim_{t\to\infty}\frac{g(t)}{t} = 0.$$

Moreover, we give in this section some estimates on the solution u of (1.4), including the lower bound inequality

$$u(x) \ge \alpha\frac{\delta_D(x)}{|x|^{n-1}},$$

where $\alpha > 0$.

Similar conditions on φ have been adopted by Dalmasso in [4], where D is the unit ball. More precisely, he proved in [4] that if φ is nondecreasing with respect to the second variable and satisfies

$$\lim_{t\to 0}\left(\min_{x\in\overline{D}}\frac{\varphi(t,x)}{t}\right) = \infty \quad \text{and} \quad \lim_{t\to\infty}\left(\max_{x\in\overline{D}}\frac{\varphi(t,x)}{t}\right) = 0,$$

then the problem (1.4) has at least one positive solution $u \in C^2(\overline{D})$.

Note that throughout this paper, the nonlinear terms of the problems (1.3) and (1.4) are not necessarily monotone with respect to the second variable. Moreover, the solutions of (1.3) and (1.4) are understood as distributional solutions in D.

Notation and preliminaries. We start by introducing some notation, which will be useful throughout this paper.

(i) $s \wedge t = \min(t,s)$ and $s \vee t = \max(t,s)$, for $t, s \in \mathbf{R}$.

(ii) D is an unbounded domain in $\mathbf{R}^n$, $n \geq 3$, such that $\overline{D}^c = \bigcup_{j=1}^k D_j$, where D_j is a bounded $C^{1,1}$ domain and $\overline{D}_i \cap \overline{D}_j = \emptyset$, for $i \neq j$.

Moreover, for $x \in D$, we denote by

$\delta_D(x)$ the distance from x to ∂D,
$\varrho_D(x) = \delta_D(x)/(\delta_D(x)+1)$,
$\lambda_D(x) = \delta_D(x)\big(\delta_D(x)+1\big)$.

(iii) Let f and g be two positive functions on a set S.
We call $f \sim g$, if there is $c > 0$ such that

$$\frac{1}{c} g(x) \leq f(x) \leq c g(x) \qquad \text{for all } x \in S.$$

We call $f \preceq g$, if there is $c > 0$ such that

$$f(x) \leq c g(x) \qquad \text{for all } x \in S.$$

(iv) For a nonnegative measurable function f in D, we denote by Vf the potential of f defined in D by

$$Vf(x) = \int_D G_D(x,y) f(y)\, dy.$$

We recall that if $f \in L^1_{\mathrm{Loc}}(D)$ such that $Vf \in L^1_{\mathrm{Loc}}(D)$, we have in the distributional sense (see [3, p. 52])

$$\Delta(Vf) = -f \quad \text{in } D. \tag{1.6}$$

(v) Let $(X_t, t > 0)$ be the Brownian motion in $\mathbf{R}^n$ and P^x be the probability measure on the Brownian continuous paths starting at x. For a nonnegative measurable function q in D, we denote by V_q the kernel defined by

$$V_q f(x) = E^x\left(\int_0^{\tau_D} e^{-\int_0^t q(X_s)\, ds} f(X_t)\, dt\right),$$

where E^x is the expectation on P^x and $\tau_D = \inf\{t > 0 : X_t \notin D\}$.

Furthermore, if q satisfies $Vq < \infty$, then we have the following resolvent equation (see [7] or [10]):

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{1.7}$$

So for each measurable bounded function u on D, we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.))u = u. \tag{1.8}$$

(vii) Let $a \in \mathbf{R}^n \backslash \overline{D}$ and $r > 0$ such that $\overline{B(a,r)} \subset \mathbf{R}^n \backslash \overline{D}$. Then we remark that

$$G_D(x,y) = r^{2-n} G_{(D-a)/r}\left(\frac{x-a}{r}, \frac{y-a}{r}\right) \quad \text{for } x, y \in D,$$

and

$$\delta_D(x) = r\delta_{(D-a)/r}\left(\frac{x-a}{r}\right) \quad \text{for } x \in D.$$

So without loss of generality, we may assume throughout this paper that $\overline{B(0,1)} \subset \mathbf{R}^n \backslash \overline{D}$.

(viii) For $x \in D$, we put $x^* = x/|x|^2$ and $D^* = \{x^* \in B(0,1) : x \in D \cup \{\infty\}\}$. Then we have the following properties, for $x, y \in D$ (see [1])

$$G_D(x,y) = |x|^{2-n} |y|^{2-n} G_{D^*}(x^*, y^*). \tag{1.9}$$

$$1 + \delta_D(x) \sim |x| \tag{1.10}$$

$$\delta_{D^*}(x^*) \sim \varrho_D(x) = \frac{\delta_D(x)}{\delta_D(x) + 1}. \tag{1.11}$$

2. Properties of the Green function and the class $K^\infty(D)$

First, we briefly recall some related results on the Green function G_D which are stated in [1].

Proposition 1. *On D^2 (that is $x, y \in D$), we have*

$$G_D(x,y) \sim \frac{1}{|x-y|^{n-2}} \min\left(1, \frac{\lambda_D(x)\lambda_D(y)}{|x-y|^2}\right).$$

Proposition 2. *On D^2, we have*

$$C\frac{\delta_D(x)\delta_D(y)}{|x|^{n-1}|y|^{n-1}} \leq G_D(x,y), \tag{2.1}$$

where C is a positive constant.

Theorem 1 (3G-Theorem). *There exists a constant $C_0 > 0$ depending only on D such that for all x, y and z in D*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leq C_0\left(\frac{\varrho_D(y)}{\varrho_D(x)}G_D(x,y) + \frac{\varrho_D(y)}{\varrho_D(z)}G_D(y,z)\right).$$

Definition 1. A Borel-measurable function q in D belongs to the class $K^\infty(D)$ if q satisfies

$$\lim_{\alpha\to 0}\left(\sup_{x\in D}\int_{(|x-y|\leq\alpha)\cap D}\frac{\varrho_D(y)}{\varrho_D(x)}G_D(x,y)|q(y)|\,dy\right) = 0$$

and

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{(|y|\geq M)\cap D}\frac{\varrho_D(y)}{\varrho_D(x)}G_D(x,y)|q(y)|\,dy\right) = 0.$$

In the sequel, we put

$$\|q\|_D = \sup_{x\in D}\int_D \frac{\varrho_D(y)}{\varrho_D(x)}G_D(x,y)|q(y)|\,dy.$$

Then by [1], for each $q \in K^\infty(D)$, we have $\|q\|_D < \infty$.

To state our main results, we recall some fundamental properties of the functions belonging to $K^\infty(D)$, which are established in [1].

Proposition 3. *Let q be a function in $K^\infty(D)$ and h be a positive superharmonic function in D. Then for each $x \in D$, we have*

$$\int_D G_D(x,y)|q(y)|h(y)\,dy \leq 2C_0\|q\|_D h(x), \tag{2.2}$$

where C_0 is the constant given in Theorem 1.

Corollary 1. *Let q be a nonnegative function in $K^\infty(D)$ then the potential Vq is bounded. Moreover, the function*

$$x \to \frac{\delta_D(x)}{|x|^{n-1}}q(x)$$

is in $L^1(D)$.

Corollary 2. *Let q be a nonnegative function in $K^\infty(D)$ and h be a superharmonic function in D, then for $x \in D$ such that $0 < h(x) < \infty$, we have*

$$\exp(-2C_0\|q\|_D)h(x) \leq \big(h - V_q(qh)\big)(x) \leq h(x),$$

where C_0 is the constant given in Theorem 1.

Proof. By ([11, Theorem 2.1, p. 164]), there exists a sequence of positive measurable functions (f_k) in D such that $h = \sup_{k\in\mathbf{N}} Vf_k$.

Let $x \in D$ such that $0 < h(x) < \infty$. Then there exists $k_0 \in \mathbf{N}$ such that $0 < Vf_k(x) < \infty$ for all $k \geq k_0$.

Let $k \geq k_0$. We consider $\gamma(t) = V_{t.q} f_k(x)$ for $t \geq 0$. Then the function γ is completely monotone on $[0,\infty)$ and so $\operatorname{Log}\gamma$ is convex on $[0,\infty)$. This implies that

$$\gamma(0) \leq \gamma(1)\exp\left(-\frac{\gamma'(0)}{\gamma(0)}\right).$$

That is

$$Vf_k(x) \leq V_q f_k(x)\exp\left(\frac{V(qVf_k)(x)}{Vf_k(x)}\right).$$

As Vf_k is superharmonic in D, it follows from (2.2) that

$$Vf_k(x) \leq V_q f_k(x)\exp(2C_0\|q\|_D).$$

Hence, using (1.7), we obtain

$$\exp(-2C_0\|q\|_D)Vf_k(x) \leq Vf_k(x) - V_q(qVf_k)(x) \leq Vf_k(x).$$

So, the result holds by letting $k \to \infty$. □

Remark 1. Let $\mu, \nu \in \mathbf{R}$ and θ be the function defined in D by

$$\theta(x) = \frac{1}{|x|^{\mu-\nu}\big(\delta_D(x)\big)^{\nu}}.$$

Then by [1], $\theta \in K^{\infty}(D)$ if and only if $\nu < 2 < \mu$. Moreover, by Corollary 1, for $\nu < 2 < \mu$, the potential $V\theta$ is bounded. In fact, we give in the next proposition more precise estimates on $V\theta$, for $\nu < 2 < \mu$ and $\mu > n$.

Proposition 4 (see [1]). *On D, we have*

(i) $$\frac{\delta_D(x)}{|x|^{n-1}} \preceq V\theta(x) \preceq \frac{(\delta_D(x))^{2-\nu}}{|x|^{n-\nu}} \quad \text{for } \mu > n \text{ and } 1 < \nu < 2.$$

(ii) $$\frac{\delta_D(x)}{|x|^{n-1}} \preceq V\theta(x) \preceq \frac{\delta_D(x)}{|x|^{n-1}}\operatorname{Log}\left(\frac{4|x|}{\delta_D(x)}\right) \quad \text{for } \mu > n \text{ and } \nu = 1.$$

(iii) $$\frac{\delta_D(x)}{|x|^{n-1}} \preceq V\theta(x) \preceq \frac{\delta_D(x)}{|x|^{n-1}} \quad \text{for } \mu > n \text{ and } \nu < 1.$$

Remark 2. Note that similar estimates have been established in [1] for $\nu < 2 < \mu$ and $\mu \leq n$.

3. First existence result

In this section we will give an existence result for the nonlinear elliptic problem (1.3). To this end let us denote by $H_D 1$ the solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } D, \\ w = 1 & \text{on } \partial D, \\ w(x) \to 0, & \text{as } |x| \to \infty. \end{cases}$$

Let h be the function defined on D by $h(x) = 1 - H_D 1(x)$. Then h is a positive harmonic function in D satisfying

$$\lim_{x \to z \in \partial D} h(x) = 0 \quad \text{and} \quad \lim_{|x| \to \infty} h(x) = 1.$$

Moreover, we have the following proposition.

Proposition 5. *We have for each $x \in D$,*

$$h(x) = c_n |x^*|^{n-2} G_{D^*}(x^*, 0) = c_n \lim_{|y| \to \infty} |y|^{n-2} G_D(x, y), \text{ where } c_n = \frac{4\pi^{n/2}}{\Gamma\left(\frac{1}{2}n - 1\right)}.$$

In particular, we have on D

$$h(x) \sim \frac{\delta_D(x)}{\delta_D(x) + 1}. \tag{3.1}$$

Proof. Let $H_{D^*} f$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } D^*, \\ w = f \in C^+(\partial D^*) & \text{on } \partial D^*. \end{cases}$$

Then, using ([8], Theorem 1, p. 473), we have

$$h(x) = 1 - \frac{1}{|x|^{n-2}} H_{D^*}\left(\frac{1}{|\cdot|^{n-2}}\right)(x^*) = |x|^{2-n}\big(|x^*|^{2-n} - H_{D^*}(|\cdot|^{2-n})(x^*)\big).$$

Now, it is well known (see [3]) that for $x^*, y^* \in D^*$,

$$G_{D^*}(x^*, y^*) = G(x^*, y^*) - H_{D^*}\big(G(\cdot, y^*)\big)(x^*),$$

where $G(x^*, y^*) = c_n^{-1} |x^* - y^*|^{2-n}$. Taking $y^* = 0$, it follows that

$$h(x) = c_n |x^*|^{n-2} G_{D^*}(x^*, 0) \quad \text{for } x \in D.$$

Moreover, by (1.9) we obtain

$$h(x) = c_n \lim_{|y| \to \infty} |y|^{n-2} G_D(x, y) \quad \text{for } x \in D.$$

Let us prove (3.1). By Proposition 1 and (1.10), we have for $x \in D$,

$$h(x) \sim 1 \wedge \lambda_D(x) \sim 1 \wedge \delta_D(x).$$

So using that $a \wedge b \sim ab/(a+b)$, for $a, b \in \mathbf{R}^+$, we obtain (3.1). □

In the sequel, we consider

$$C_0(D)=\Big\{u\in C(D):\lim_{x\to z\in\partial D}u(x)=\lim_{|x|\to\infty}u(x)=0\Big\}$$

endowed with the uniform norm $\|u\|_\infty=\sup_{x\in D}|u(x)|$.

For a fixed nonnegative function $q\in K^\infty(D)$, we put

$$\Gamma_q=\{p\in K^\infty(D):|p|\le q\}.$$

Proposition 6 (see [1]). *Let q be a nonnegative function in $K^\infty(D)$ then the family of functions*

$$\mathscr{F}_q=\Big\{\int_D G_D(\,\cdot\,,y)p(y)\,dy:p\in\Gamma_q\Big\}$$

is relatively compact in $C_0(D)$.

Theorem 2. *Let $f\colon D\times[0,\infty)\to[0,\infty)$ be a measurable function satisfying (H_1) and (H_2). Then for each $\lambda>0$, the problem (1.3) has a positive solution $u\in C(\overline{D})$ satisfying for each $x\in D$*

$$C_\lambda\big(\lambda h(x)+H_Dg(x)\big)\le u(x)\le\lambda h(x)+H_Dg(x), \tag{3.2}$$

where C_λ is a positive constant. In particular, if $g\equiv0$, then we have on D

$$u(x)\sim\frac{\delta_D(x)}{\delta_D(x)+1}. \tag{3.3}$$

Proof. Given $\lambda>0$. For convenience, we denote $s(x)=\lambda h(x)+H_Dg(x)$, for $x\in D$. We aim to show an existence result for the equation

$$u+V\big(uf(\,\cdot\,,u)\big)=s. \tag{3.4}$$

Let $c:=\lambda+\|g\|_\infty$ and q_c be the function in $K^\infty(D)$ given by (H_2). For simplicity we write q for q_c. We consider the closed convex set Λ given by

$$\Lambda=\big\{u\in B_b(D):\exp(-2C_0\|q\|_D)s\le u\le s\big\}.$$

We define the operator T on Λ by

$$Tu=\big(s-V_q(qs)\big)+V_q\big[\big(q-f(\,\cdot\,,u)\big)u\big].$$

Since $\|s\|_\infty\le\lambda+\|g\|_\infty=c$, then (H_2) implies that

$$0\le f(\,\cdot\,,u)\le q\quad\text{for any } u\in\Lambda. \tag{3.5}$$

So the operator T is well defined.

First, we claim that $T\Lambda \subset \Lambda$. Let $u \in \Lambda$, then from (3.5) and Corollary 2, it follows that

$$Tu \geq s - V_q(qs) \geq \exp(-2C_0\|q\|_D)s.$$

Moreover, we have

$$Tu \leq \big(s - V_q(qs)\big) + V_q(qs) = s.$$

So $T\Lambda \subset \Lambda$. Next, we prove that T is nondecreasing on Λ. Let $u, v \in \Lambda$ such that $u \leq v$. Then we have

$$Tu - Tv = V_q\big[\big(q - f(\,\cdot\,,u)\big)u - \big(q - f(\,\cdot\,,v)\big)v\big].$$

By (H_2) it follows that $t \to t\big(q(x) - f(x,t)\big)$ is a nondecreasing function on $[0,c]$ for $x \in D$. This implies that $Tu \leq Tv$.

We consider the sequence (u_k) defined by

$$u_0 = \exp(-2C_0\|q\|_D)s \quad \text{and} \quad u_{k+1} = Tu_k \text{ for } k \in \mathbf{N}.$$

From the monotonicity of T we obtain

$$u_0 \leq u_1 \leq \cdots \leq u_k \leq u_{k+1} \leq s.$$

So using (H_1), it follows from the dominated convergence theorem that the sequence (u_k) converges to a function $u \in \Lambda$ which is a fixed point of T. That is, u satisfies

$$u = \big(s - V_q(qs)\big) + V_q\big[\big(q - f(\,\cdot\,,u)\big)u\big].$$

This implies that

$$\big(I - V_q(q.)\big)u + V_q\big(uf(\,\cdot\,,u)\big) = \big(I - V_q(q.)\big)s. \tag{3.6}$$

So applying $\big(I + V(q.)\big)$ on both sides of (3.6), we deduce from (1.7) and (1.8) that u is a solution of the equation (3.4).

Now, since $q \in K^\infty(D)$, it follows from Corollary 1 that $q \in L^1_{\mathrm{loc}}(D)$. Moreover, from (3.5) we have

$$uf(\,\cdot\,,u) \leq cq.$$

This shows that $uf(\,\cdot\,,u) \in L^1_{\mathrm{loc}}(D)$ and $V\big(uf(\,\cdot\,,u)\big) \in \mathscr{F}_{cq}$. So by Proposition 6, $V\big(uf(\,\cdot\,,u)\big) \in C_0(D) \subset L^1_{\mathrm{loc}}(D)$. Thus, applying Δ on both sides of (3.4), we conclude by (1.6), that u is a solution of $\Delta u - uf(\,\cdot\,,u) = 0$ in D (in the distributional sense). Now, as $V\big(uf(\,\cdot\,,u)\big) \in C_0(D)$, it follows from (3.4) that

$$\lim_{x \to z \in \partial D} u(x) = \lim_{x \to z \in \partial D} H_D g(x) = g(z)$$

and

$$\lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} \lambda h(x) = \lambda.$$

So u is a solution of (1.3), which is continuous in $\overline{D} \cup \{\infty\}$. Finally, since $u \in \Lambda$ then u satisfies (3.2). In addition, if $g \equiv 0$, then combining (3.2) and (3.1), we conclude that

$$u(x) \sim h(x) \sim \frac{\delta_D(x)}{\delta_D(x)+1} \qquad \text{for } x \in D. \ \square$$

Remark 3. If (H_1) and (H_2) hold and we assume further that for all $x \in D$, the map $t \to tf(x,t)$ is nondecreasing on $[0,\infty)$, then for each $\lambda > 0$, the problem (1.3) has a unique positive solution $u \in C(\overline{D})$.

Indeed, let $u, v \in C(\overline{D})$ be two positive solutions of (1.3). Suppose that there exists $x_0 \in D$ such that $u(x_0) < v(x_0)$. Let $w = v - u$, then $w \in C_0(D)$ and satisfies

$$\Delta w + uf(\cdot\,,u) - vf(\cdot\,,v) = 0 \quad \text{in } D \text{ (in the distributional sense).}$$

Consider $\Omega = \{x \in D,\ w(x) > 0\}$. Then Ω is a nonempty open set. So, since $t \to tf(x,t)$ is nondecreasing on $[0,\infty)$, we deduce that

$$\begin{cases} \Delta w \geq 0, & \text{in } \Omega, \\ w|_{\partial\Omega} = 0, & \\ \lim_{|x|\to\infty,\, x\in\Omega} w(x) = 0, & \text{whenever } \Omega \text{ is unbounded.} \end{cases}$$

Hence, the maximum principle ([5]) implies that $w \leq 0$ in Ω, which gives a contradiction.

Example 1. Let $B^c = \{x \in \mathbf{R}^n : |x| > 1\}$, $p > 1$ and V be a nonnegative measurable function in B^c. Suppose that there exists a nonnegative function k on $(1,\infty)$ such that

$$V(x) \leq k(|x|) \qquad \text{for } x \in B^c$$

and

$$\int_1^\infty (r-1)k(r)\,dr < \infty. \tag{3.7}$$

Then for each $\lambda > 0$, the problem

$$\begin{cases} \Delta u(x) - k(x)u^p(x) = 0, & x \in B^c, \\ u > 0 & \text{in } B^c,\ u|_{\partial B} = 0, \\ \lim_{|x|\to\infty} u(x) = \lambda, & \end{cases}$$

has a unique solution $u \in C(\overline{B^c})$ satisfying for each $x \in B^c$,

$$u(x) \sim \frac{|x|-1}{|x|}.$$

Indeed, for $D = B^c$, we have $h(x) = 1 - 1/|x|^{n-2} \sim 1 - 1/|x|$ for $x \in B^c$. Moreover, by [1], if q is a radial nonnegative function in B^c then $q \in K(B^c)$ if and only if

$$\int_1^\infty (r-1)q(r)\,dr < \infty.$$

So (3.7) implies that $x \to k(|x|) \in K(B^c)$. Thus (H_2) is satisfied.

Example 2. Let $p > 1$ and $\nu < 2 < \mu$. Then for each $\lambda > 0$, the problem

$$\begin{cases} \Delta u(x) - \dfrac{u^p(x)\exp\big(-u(x)\big)}{|x|^{\mu-\nu}\big(\delta_D(x)\big)^\nu} = 0, & x \in D, \\ u > 0 & \text{in } D,\ u|_{\partial D} = g, \\ \lim_{|x|\to\infty} u(x) = \lambda, \end{cases}$$

has a solution $u \in C(\overline{D})$ satisfying (3.3).

4. Second existence result

In this section, we are interested in the existence of continuous solutions for the problem (1.4).

Theorem 3. *Let φ: $D \times [0,\infty) \to [0,\infty)$ be a measurable function satisfying (A_1) and (A_2). Then the problem (1.4) has a positive solution u satisfying, for each $x \in D$,*

$$\frac{\delta_D(x)}{|x|^{n-1}} \preceq u(x) \preceq Vq(x), \tag{4.1}$$

where q is given in (1.5).

Proof. Let p be the nonnegative function in $L^1_{\mathrm{loc}}(D)$ given by (A_2) and K be a compact of D such that

$$0 < a := \int_K p(x)\,dx < \infty.$$

We put $b := \min\{\delta_D(x)/|x|^{n-1} : x \in K\}$. Since $\lim_{t\to 0} f(t)/t = \infty$, then there exists $\alpha > 0$ such that

$$\frac{f(b\alpha)}{\alpha} \geq \frac{1}{aCb},$$

where C is given by (2.1). On the other hand, let q be the function in $K(D)$ given by (A_2). Then using Corollary 1, we have $\|Vq\|_\infty < \infty$. So as $\lim_{t\to\infty} g(t)/t = 0$, there exists $\beta > 0$ such that

$$\|Vq\|_\infty \frac{g(\beta)}{\beta} \leq 1.$$

Here we want to use the Schauder fixed point theorem. To this end we consider the closed convex set

$$S = \left\{ u \in C_0(D) : \alpha \frac{\delta_D(x)}{|x|^{n-1}} \leq u(x) \leq \beta \text{ for all } x \in D \right\}$$

and we define the integral operator T on S by

$$Tu(x) = V\big(\varphi(\,\cdot\,, u)\big)(x) = \int_D G_D(x,y)\varphi\big(y, u(y)\big)\, dy \quad \text{ for all } x \in D.$$

We start by proving that $TS \subset S$. Let $u \in S$, so from (1.5) and the monotonicity of g, we have for $x \in D$,

$$Tu(x) \leq \int_D G_D(x,y)q(y)g\big(u(y)\big)\, dy \leq g(\beta)\|Vq\|_\infty \leq \beta.$$

Moreover, using (2.1), (1.5) and the monotonicity of f, we have for $x \in D$,

$$\begin{aligned} Tu(x) &\geq C\frac{\delta_D(x)}{|x|^{n-1}} \int_D \frac{\delta_D(y)}{|y|^{n-1}} p(y) f\left(\alpha \frac{\delta_D(x)}{|x|^{n-1}}\right) dy \\ &\geq C\frac{\delta_D(x)}{|x|^{n-1}} f(\alpha b) \int_K \frac{\delta_D(y)}{|y|^{n-1}} p(y)\, dy \geq Cab\frac{\delta_D(x)}{|x|^{n-1}} f(\alpha b) \geq \alpha \frac{\delta_D(x)}{|x|^{n-1}}. \end{aligned}$$

Now, since for all $u \in S$,

$$\varphi(\,\cdot\,, u) \leq g(\beta)q. \tag{4.2}$$

Hence Proposition 6 implies that $Tu \in \mathscr{F}_{g(\beta)q} \subset C_0(D)$ for all $u \in S$. So $TS \subset S$.

Next, let us prove the continuity of T in S. Let (u_k) be a sequence in S, which converges uniformly to a function $u \in S$. Then we have for $x \in D$,

$$|Tu(x) - Tu_k(x)| \leq \int_D G_D(x,y)\big|\varphi\big(y,u(y)\big) - \varphi\big(y,u_k(y)\big)\big|\, dy.$$

Since

$$\big|\varphi\big(y,u(y)\big) - \varphi\big(y,u_k(y)\big)\big| \leq 2g(\beta)q(y),$$

and Vq is bounded, we deduce by (A_1) and the dominated convergence theorem, that for all $x \in D$

$$|Tu(x) - Tu_k(x)| \to 0, \quad \text{as } k \to \infty.$$

From (4.2), we have $TS \subset \mathscr{F}_{g(\beta)q}$. It follows from Proposition 6 that TS is relatively compact in $C_0(D)$, which implies that

$$\|Tu - Tu_k\|_\infty \to 0, \quad \text{as } k \to \infty.$$

Thus T is a compact mapping from S to itself. Hence by Shauder's fixed point theorem, there exists $u \in S$ such that

$$u = V\big(\varphi(\,\cdot\,, u)\big). \tag{4.3}$$

Now, since $q \in K^\infty(D)$, we deduce by (4.2) and Corollary 1, that $\varphi(\,\cdot\,, u) \in L^1_{\text{loc}}(D)$. Moreover, from (4.2) and Proposition 6, we have $V\big(\varphi(\,\cdot\,, u)\big) \in C_0(D) \subset L^1_{\text{loc}}(D)$. So applying Δ on both sides of the equality (4.3), we conclude, by (1.6), that u is a solution of (1.4).

On the other hand, since $u \in S$, u satisfies the lower estimates of (4.1). Finally, combining (4.2) and (4.3), we have

$$u(x) \le g(\beta)Vq(x) \quad \text{for } x \in D,$$

which implies (4.1). □

Example 3. Let $\alpha \le 0$, $0 < \mu < \lambda < 1$ and ψ be a nonnegative measurable function in B^c. Suppose that there exists a nonnegative measurable function k on $(1, \infty)$ such that $\psi(x) \le k(|x|)$ for $x \in B^c$, and

$$\int_1^\infty (r-1)k(r)\, dr < \infty.$$

Then the problem

$$\begin{cases} \Delta u + \dfrac{\big(u(x)\big)^\lambda}{|x|^\alpha + \big(u(x)\big)^\mu}\psi(x) = 0, & x \in B^c, \\ u|_{\partial B} = 0, \end{cases}$$

has a solution $u \in C_0(B^c)$ satisfying (4.1). Indeed, we take $f(t) = t^\lambda/(1+t^\mu)$, $g(t) = t^{\lambda-\mu}$, for $t \in (0, \infty)$ and $p(x) = q(x) = \psi(x)$ for $x \in B^c$. So we have obviously

$$\lim_{t\to 0} \frac{f(t)}{t} = \infty \quad \text{and} \quad \lim_{t\to\infty} \frac{g(t)}{t} = 0.$$

Moreover, if we suppose further that

$$\int_1^\infty r^{n-1}k(r)\,dr < \infty,$$

then

$$u(x) \sim \frac{|x|-1}{|x|^{n-1}} \qquad \text{for } x \in B^c.$$

Indeed, using (4.1), we have for $x \in B^c$

$$\begin{aligned}
u(x) \preceq Vk(|x|) &\preceq \int_1^\infty r^{n-1}(|x|\vee r)^{2-n}\big(1-(|x|\vee r)^{2-n}\big)k(r)\,dr\\
&\preceq \int_1^\infty r^{n-1}(|x|\vee r)^{2-n}\left(1-\frac{1}{|x|\vee r}\right)k(r)\,dr\\
&\preceq |x|^{2-n}\left(1-\frac{1}{|x|}\right)\int_1^\infty r^{n-1}k(r)\,dr\\
&\preceq \frac{|x|-1}{|x|^{n-1}}.
\end{aligned}$$

Example 4. Let $\alpha,\beta \geq 0$ such that $0 < \alpha+\beta < 1$ and $\nu < 2 < \mu$. Then the problem

$$\begin{cases}
\Delta u + \dfrac{\big(u(x)\big)^\alpha \operatorname{Log}\big(1+(u(x))^\beta\big)}{|x|^{\mu-\nu}\big(\delta_D(x)\big)^\nu} = 0, \quad x \in D,\\
u|_{\partial D} = 0,
\end{cases}$$

has a solution $u \in C_0(D)$. Moreover, using (4.1) and Proposition 4, we have for $x \in D$,

$$\frac{\delta_D(x)}{|x|^{n-1}} \preceq u(x) \preceq \begin{cases}
\dfrac{\big(\delta_D(x)\big)^{2-\nu}}{|x|^{n-\nu}} & \text{for } \mu > n \text{ and } 1 < \nu < 2,\\
\dfrac{\delta_D(x)}{|x|^{n-1}} \operatorname{Log}\left(\dfrac{4|x|}{\delta_D(x)}\right) & \text{for } \mu > n \text{ and } \nu = 1,\\
\dfrac{\delta_D(x)}{|x|^{n-1}} & \text{for } \mu > n \text{ and } \nu < 1.
\end{cases}$$

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