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ON FACTORIZATIONS OF ENTIRE FUNCTIONS OF BOUNDED TYPE

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Abstract. We prove that if f is a transcendental entire function and the set of all finite singularities of its inverse function f^{-1} is bounded, then $f(z)+P(z)$ is prime for any nonconstant polynomial $P(z)$, unless $f(z)$ and $P(z)$ has a nonlinear common right factor. Particularly, it is shown that $f(z)+az$ is prime for any constant $a\neq 0$.

1. Introduction

A transcendental meromorphic function F is said to be prime (pseudo-prime) if, and only if, whenever $F=f(g)$ for some meromorphic functions f and g, either f or g must be bilinear (rational); F is called left-prime (right-prime) if every factorization of F implies that f is bilinear whenever g is transcendental (g is linear if f is transcendental). It is easily seen F is prime if and only if F is left-prime as well as right-prime. We refer the readers to [3] or [4] for an introduction to the factorization theory of entire and meromorphic functions.

A point a is called a singularity of f^{-1} (the inverse function of f), if a is either a critical value or asymptotic value of f. We denote by $\operatorname{sing}(f^{-1})$ the set of all finite singularities of f^{-1}, i.e.

$$\operatorname{sing}(f^{-1})=\{z\in\mathbf{C}: z \text{ is a singularity of } f^{-1}\}.$$

We denote by B the class of all entire functions f such that $\operatorname{sing}(f^{-1})$ is bounded and by S the class of all entire functions f such that $\operatorname{sing}(f^{-1})$ is finite. If $f\in B$ ($f\in S$), we say f is of bounded (finite) type.

In 1981, Noda [8] proved the following result.

Theorem A. *Let $f(z)$ be a transcendental entire function. Then the set*

$$NP(f)=\{a \mid a\in\mathbf{C},\ f(z)+az \text{ is not prime}\}$$

is at most countable

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As a further study on the cardinality of $NP(f)$, which is denoted by $|NP(f)|$, Ozawa and Sawada [9] posed the following interesting question:

Question. *Is there any f for which the exceptional set $NP(f)$ in Theorem A is really infinitely countable? Or what is the maximal cardinal number of the exceptional set $NP(f)$?*

Theorem B (Ozawa and Sawada [9]). *Let $G(w)$ be an entire function satisfying*

$$M\big(R, G(w)\big) \le \exp(KR)$$

for $R \ge R_0 > 0$ and for some constant $K > 0$. Then either $G(e^z) + az$ or $G(e^z) + bz$ is prime if $ab(a-b) \neq 0$.

This shows that the cardinality of $NP(G(e^z))$ is at most 2 if $M(R, G(w)) \le \exp(KR)$ for $R \ge R_0 > 0$ and for some constant $K > 0$. As a study of the above question, Liao–Yang [6] proved the following result.

Theorem C. *Let f be a transcendental entire function of finite order in S. Then for any constant $a \neq 0$, $f(z) + az$ is prime, i.e. $|NP(f)| \le 1$.*

Recently, Wang–Yang [13] proved the following theorem.

Theorem D. *Let P, Q be nonconstant polynomials, $\alpha \in B$, h a periodic entire function of order one and mean type, $G(z) = P \circ h \circ \alpha(z)$. If $F(z) = G^n(z) + Q(z)$ has a factorization $F(z) = f\big(g(z)\big)$, then $g(z)$ must be a common right factor of $\alpha(z)$ and $Q(z)$.*

Remark 1. The original statement of Theorem D only requires that h is of order one. Here we would like to point out that h should be at most order one of mean type, as it is needed in the proof of Theorem D, Lemma 5 in [13]. However, f in Lemma 5 should be an entire function of exponential type, i.e. f has order less than one or order one and mean type; see p. 27 in [4].

Remark 2. Let G be defined in Theorem D. Then $G^n(z) + az$ is prime for any constant $a \neq 0$.

As a continuation of the study of our previous work [6], we are able to extend Theorem C to a large class of functions, namely, functions of bounded type. The following is our main result.

Theorem. *Let f be a transcendental entire function in B, then for any nonconstant polynomial $P(z)$, $f(z) + P(z)$ is prime unless $f(z)$ and $P(z)$ has a nonlinear common right factor.*

2. Some lemmas

Lemma 1 (Rippon and Stallard [11]). *Let f be a meromorphic function with a bounded set of all finite critical and asymptotic values. Then there exists $K > 0$ such that if $|z| > K$ and $|f(z)| > K$, then*

$$|f'(z)| \geq \frac{|f(z)| \log |f(z)|}{16\pi |z|}.$$

Lemma 2 ([5]). *Let f be a transcendental entire function, and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r,f)\nu(r,f)^{-(1/4)+\delta} \tag{1}$$

holds. Then there exists a set F in R^+ and of finite logarithmic measure, i.e.,

$$\int_F \frac{dt}{t} < +\infty$$

such that

$$f^{(m)}(z) = \left(\frac{\nu(r,f)}{z}\right)^m \big(1 + o(1)\big) f(z) \tag{2}$$

holds whenever m is a fixed nonnegative integer and $r \notin F$.

Lemma 3 (Baker and Singh [1], also see [2]). *Let f and g be two entire functions. Then*

$$\operatorname{sing}\big((f \circ g)^{-1}\big) \subset \operatorname{sing}(f^{-1}) \cup f\big(\operatorname{sing}(g^{-1})\big).$$

Lemma 4 (Polya [10]). *Let f and g be two transcendental entire functions. Then*

$$\lim_{r\to\infty} \frac{M(r, f \circ g)}{M(r,g)} = \infty.$$

Lemma 5. *Let f be a transcendental entire function. Then*

$$M(r,f') \leq M(r,f)^2$$

for a sufficiently large r.

Remark 3. This follows easily from a result of Valiron ([12]):

$$\lim_{r\to\infty} \frac{\log M(r,f')}{\log M(r,f)} = 1.$$

3. Proof of the theorem

Let $F(z) = f(z) + P(z)$, $P(z)$ is a nonconstant polynomial. We first prove that F is pseudo-prime. Assume that

$$F(z) = g\big(h(z)\big),$$

where g is a transcendental meromorphic function with at most one pole and h is a transcendental entire function. Thus

$$f(z) = g\big(h(z)\big) - P(z), \qquad f'(z) = g'\big(h(z)\big)h'(z) - P'(z). \tag{3}$$

First we consider the case that g is a transcendental entire function, and then we discuss two situations.

Case 1: g' has at least two zeros. Then there exists a zero c of g' such that $h(z) = c$ has infinitely many roots $\{z_k\}_{k=1}^{\infty}$. Thus we have

$$f(z_k) = -P(z_k) + g(c), \qquad f'(z_k) = -P'(z_k).$$

By Lemma 1, we would have

$$|P'(z_k)| \geq \frac{|P(z_k) - g(c)| \log |P(z_k) - g(c)|}{16\pi |z_k|},$$

which leads to a contradiction.

Case 2: g' has at most one zero. Thus

$$g'(w) = (w - w_0)^n e^{\alpha(w)}, \qquad f'(z) = \big(h(z) - w_0\big)^n e^{\alpha(h(z))} h'(z) - P'(z),$$

where n is a non-negative integer. Let $K(z) = e^{-\alpha(h(z))/(n+3)}$, and assume that Γ is a simple curve tending to infinity such that if $z \in \Gamma$ and $|z| = r$, then $|K(z)| = M(r, K)$. By Lemmas 4 and 5, we have, if $z \in \Gamma$ and $|z| = r$ is sufficiently large,

$$\begin{aligned} \big|g'\big(h(z)\big)h'(z)\big| &= \big|\big(h(z) - w_0\big)^n e^{\alpha(h(z))} h'(z)\big| \\ &= \frac{\big|\big(h(z) - w_0\big)^n h'(z)\big|}{M(r, K)^{n+3}} \leq \frac{1}{M(r, K)} \to 0. \end{aligned} \tag{4}$$

Let $L(z) = -\alpha(h(z))/(n+3)$ and $A(r, L) = \max_{|z|=r} \operatorname{Re} L(z)$. Thus if $z \in \Gamma$, $|K(z)| = M(r, K) = e^{A(r,L)}$, $\operatorname{Re} L(z) = A(r, L)$. By Hadamard's three-circle theorem, we have, for $r_1 < r_2 < r_3$,

$$A(r_2, L) \leq \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} A(r_3, L) + \frac{\log r_3 - \log r_2}{\log r_3 - \log r_1} A(r_1, L). \tag{5}$$

For $z_0 \in \Gamma$, we have

$$|L'(z_0)| = \lim_{z\to z_0, z\in\Gamma} \frac{|L(z)-L(z_0)|}{|z-z_0|} \geq \lim_{z\to z_0, z\in\Gamma} \frac{|\operatorname{Re} L(z) - \operatorname{Re} L(z_0)|}{|z-z_0|}. \tag{6}$$

Let $|z_0| = r_0$ and $|z| = r_0 + h$, $h > 0$, then as $z \to z_0$, $h \to 0$. Thus, by (5) and (6), we have, for sufficiently large r_0,

$$\begin{aligned}
|L'(z_0)| &\geq \lim_{z\to z_0, z\in\Gamma} \frac{A(r_0+h, L) - A(r_0, L)}{|z-z_0|} \\
&= \lim_{z\to z_0, z\in\Gamma} \frac{h}{|z-z_0|} \frac{A(r_0+h, L) - A(r_0, L)}{h} \\
&= \lim_{h\to 0} \frac{A(r_0+h, L) - A(r_0, L)}{h} \\
&\geq \lim_{h\to 0} \frac{\frac{\log(1+h/r_0)}{\log r_0}\big(A(r_0, L) - A(1, L)\big)}{h} \\
&= \frac{A(r_0, L) - A(1, L)}{r_0 \log r_o} > 1.
\end{aligned} \tag{7}$$

Let $w = G(z) = e^{\alpha(h(z))/(n+3)} = e^{-L(z)}$. Thus 0 is an asymptotic value of G and Γ is the corresponding asymptotic curve, $\gamma = G(\Gamma)$ is a simple curve connecting $G(0)$ and 0. Let B be the length of γ, which is a finite number. And $dw = e^{-L(z)} L'(z) dz$. By this, (4) and (7), if $z \in \Gamma$, we have

$$\begin{aligned}
\big|g\big(h(z)\big)\big| &= \left| \int_{z_0 \text{ along } \Gamma}^{z} g'\big(h(z)\big) h'(z)\, dz + g\big(h(z_0)\big) \right| \\
&\leq \int_{z_0 \text{ along } \Gamma}^{z} \big|g'\big(h(z)\big) h'(z)\big|\, |dz| + \big|g\big(h(z_0)\big)\big| \\
&\leq \int_{w_0 \text{ along } \gamma}^{w} \frac{1}{|L'(z)|}\, |dw| + \big|g\big(h(z_0)\big)\big| \\
&\leq \int_{w_0 \text{ along } \gamma}^{w} |dw| + \big|g\big(h(z_0)\big)\big| \\
&\leq B + \big|g\big(h(z_0)\big)\big|.
\end{aligned}$$

Thus we can find a sequence of $\{z_k\}_{k=1}^{\infty}$ such that $z_k \to \infty$ as $k \to \infty$, and

$$f(z_k) \sim -P(z_k), \qquad f'(z_k) \sim -P'(z_k).$$

A contradiction follows from this and Lemma 1.

If g' has just one pole w_1, so does g, then $h(z)$ does not assume w_1, i.e., $h(z) = e^{\beta(z)} + w_1$. Moreover, if g' has a zero c, then $h(z) = c$ has infinitely many roots. One can derive a contradiction by arguing similarly as in Case 1. Hence g' has no zeros, i.e.,

$$g'(w) = \frac{1}{(w - w_1)^n} e^{\alpha(w)},$$

and

$$g'\big(h(z)\big)h'(z) = \beta'(z)\exp\big(\alpha\big(e^{\beta(z)+w_1}\big) + (1-n)\beta(z)\big).$$

By the same argument as that in Case 2 above, we can get a contradiction. Thus $F(z) = f(z) + P(z)$ is pseudo-prime. Now we assume that $F(z)$ has the following factorization:

$$F(z) = f(z) + P(z) = Q\big(g(z)\big),$$

where Q is rational, g is a transcendental meromorphic function. If Q is a polynomial, then g is entire. If Q has a pole w_1, then $g(z)$ does not assume w_1. Thus $h(z) = 1/(g(z) - w_1)$ is an entire function and $F(z) = Q_1\big(h(z)\big)$, where Q_1 is a rational function. Without loss of generality, we may assume that $g(z)$ is entire, and $Q(w)$ has at most one pole. Now we discuss the following two sub-cases.

Subcase 1: Q has one pole, say w_0, i.e., $Q(w) = Q_1(w)/(w - w_0)^n$, where $Q_1(w)$ is a polynomial with degree m and $Q_1(w_0) \neq 0$. Then $g(z) = w_0 + e^{h(z)}$, where $h(z)$ is a nonconstant entire function. Thus we have

$$\begin{aligned} f(z) &= Q_1(w_0 + e^{h(z)})e^{-nh(z)} - P(z) \\ &= a_0 e^{-nh(z)} + a_1 e^{-(n-1)h(z)} + \cdots + a_m e^{(m-n)h(z)} - P(z), \end{aligned}$$

where $a_0, a_1, \ldots, a_m$ are constants and $a_m \neq 0$, $a_0 = Q_1(w_0) \neq 0$. Thus

$$\begin{aligned} f'(z) &= (-na_0 e^{-nh(z)} - (n-1)a_1 e^{-(n-1)h(z)} + \cdots \\ &\qquad + (m-n)a_m e^{(m-n)h(z)})h'(z) - P'(z) \\ &= \big[-na_0 - (n-1)a_1 e^{h(z)} + \cdots \\ &\qquad + (m-n)a_m e^{mh(z)}\big]e^{-nh(z)}h'(z) - P'(z) \\ &= P_1(e^{h(z)})e^{-nh(z)}h'(z) - P'(z), \end{aligned}$$

where $P_1(w)$ is a polynomial and $P_1(0) = -na_0 \neq 0$. If $P_1(w)$ is a nonconstant polynomial, then $P_1(w)$ has a zero $c \neq 0$ and $e^{h(z)} = c$ has infinitely many roots. Let $\{z_k\}_{k=1}^{+\infty}$ be zeros of $e^{h(z)} - c$, then $f'(z_k) = -P'(z_k)$ and

$$f(z_k) = \frac{Q_1(w_0 + c)}{c^n} - P(z_k).$$

Again, by Lemma 1, we have a contradiction. If $P_1(w)$ is a constant polynomial, then

$$f(z) = a_0 e^{-nh(z)} + a_m - P(z), \qquad f'(z) = -na_0 e^{-nh(z)} h'(z) - P'(z).$$

Let $K(z) = e^{nh(z)}$ and $|z'| = r$, $|K(z')| = M(r, K)$. Then by Lemma 2, we have, for $r \notin F$,

$$|-na_0 e^{-nh(z')} h'(z')| = \left| a_0 \frac{1}{K(z')} \frac{K'(z')}{K(z')} \right| = |a_0| \frac{1}{M(r,K)} \frac{\nu(r,K)}{r} (1 + o(1)),$$
$$|a_0 e^{-nh(z')}| = \frac{|a_0|}{M(r,K)}.$$

Noting $\lim_{r\to\infty} \big(\nu(r,K)/M(r,K)\big) = 0$ for a transcendental entire function K, we can find a sequence of $\{z_k\}_{k=1}^{+\infty}$ such that $|f(z_k)| \sim |P(z_k)|$, $|f'(z_k)| \sim |P'(z_k)|$. A contradiction follows from this and Lemma 1.

Subcase 2: $Q(w)$ has no pole, i.e., $Q(w)$ is a polynomial with degree ≥ 2. If $Q'(w)$ has at least two distinct zeros, then there exists a zero w_1 of $Q'(w)$ such that $g(z) = w_1$ has infinitely many zeros $\{z_n\}_{n=1}^{+\infty}$. Then

$$f'(z_n) = Q'\big(g(z_n)\big) - P'(z_n) = -P'(z_n), \qquad f(z_n) = Q(w_1) + P(z_n).$$

However, by Lemma 1,

$$|f'(z_n)| \geq \frac{|f(z_n)| \log |f(z_n)|}{16\pi |z_n|},$$

which will lead to a contradiction. Therefore, we only need to treat the case that $Q'(w)$ has only one zero w_0. If $g(z) - w_0$ has infinitely many zeros, again a contradiction follows from Lemma 1. Hence, we have

$$g(z) = w_0 + p_1(z) e^{h(z)} \qquad \text{and} \qquad Q'(z) = A(w - w_0)^{n-1},$$

where $p_1(z)$ is a polynomial, $h(z)$ a nonconstant entire function. Thus

$$Q(w) = \frac{A}{n}(w - w_0)^n + B,$$
$$f(z) = \frac{A}{n} p_1(z)^n e^{nh(z)} + B - P(z),$$
$$f'(z) = \frac{A}{n}\big(p_1'(z) + p_1(z) n h'(z)\big) e^{nh(z)} - P'(z).$$

Set $K(z) = e^{-nh(z)}$ and let $|z'| = r$, $K(z') = M(r,K)$. Then it follows from Lemma 2, for $r \notin F$, that

$$\left|\frac{A}{n}\big(p_1'(z') + p_1(z')nh'(z')\big)e^{nh(z')}\right| = \left|\frac{A}{n}\left(\frac{p_1'(z')}{K(z')} - \frac{p_1(z')}{K(z')}\frac{K'(z')}{K(z')}\right)\right|$$
$$\leq \frac{cr^t}{M(r,K)} + \frac{dr^t\nu(r,K)}{M(r,K)},$$

where c, d are positive constants, $t = \deg p_1 - 1$. Noting

$$\lim_{r\to\infty}\frac{r^t\nu(r,K)}{M(r,K)} = 0$$

for a transcendental entire function K, there exists a sequence of $\{z_n\}_{n=1}^{+\infty}$ such that

$$f(z_n) \sim -P(z_n), \qquad f'(z_n) \sim -P(z_n).$$

Again by Lemma 1, we get a contradiction. Thus we have proved that $F(z) = f(z) + P(z)$ is left-prime. Next we show that F is right-prime. Let

$$F(z) = g\big(q(z)\big),$$

where g is a transcendental entire function and $q(z)$ a polynomial with degree ≥ 2. Thus

$$f(z) = g\big(q(z)\big) - P(z)$$

and hence

$$f'(z) = g'\big(q(z)\big)q'(z) - P'(z).$$

First, we prove that $g'(w)$ has infinitely many zeros. In fact, if $g'(w)$ has only finitely many zeros, then $g'(w) = s(w)e^{h(w)}$, where $s(w)$ is a polynomial and $h(w)$ is a nonconstant entire function. Let $K(z) = e^{-h(z)/3}$. There exists a curve Γ tending to infinity such that if $z \in \Gamma$, then $|K(z)| = M(|z|, K)$. Noting that K is a transcendental entire function, we have that $M(r,K) \geq r^{2m+2}$ for $r \geq r_0$, where $m = \deg s$. Let $w = G(z) = e^{h(z)/3}$ and $\lambda = G(\Gamma)$. Then $dw = \frac{1}{3}h'(z)e^{h(z)/3}$. If $h(z)$ is nonconstant polynomial, then there exists a positive constant c such that $|h'(z)| \geq c$ for sufficiently large $|z| = r$. If $h(z)$ is transcendental, then $\left|\frac{1}{3}h'(z)\right| > 1$ for $z \in \Gamma$ and sufficiently large $|z| = r$, by (7). Hence, we have, for $z \in \Gamma$ and $|z| \geq r_0$,

$$|g'(z)| \leq \frac{1}{M(r,K)^2},$$
$$|g(z)| = \left|\int_{z_0 \text{ along } \Gamma}^{z} g'(z)\,dz + g(z_0)\right| \leq \left|\int_{w_0 \text{ along } \lambda}^{w} |dw|\right| \leq A,$$

where $w_0 = G(z_0)$, $w = G(z)$ and A is a positive constant. Let γ be a component of $q^{-1}(\Gamma)$, and denote $R = |q(z)|$ for $z \in \gamma$. Then for $z \in \gamma$, we have

$$\big|g\big(q(z)\big)\big| \le A, \qquad |g'(z)q'(z)| \le \frac{BR^{m+1}}{M(R,K)^2} \to 0, \qquad \text{as } z \to \infty,$$

where A and B are constants. Hence, for $z \in \gamma$, we have

$$|f(z)| \sim |P(z)|, \qquad |f'(z)| \sim |P'(z)|.$$

Again, by Lemma 1, the above estimates will lead to a contradiction as before. Thus g' has infinitely many zeros. Now let $n = \deg q$ and $m = \deg P$. Next we will prove that $n \mid m$, i.e., there is a positive integer r such that $m = nr$. Let $\{w_k\}_{k=1}^{\infty}$ denote the zeros of $g'(w)$ and set

$$q(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0.$$

We consider the roots of the equation

$$q(z) = w_k,$$

which implies

$$a_n z^n\big(1 + o(1)\big) = w_k. \tag{8}$$

On the other hand, the roots of the above equation can be expressed as

$$z_k^{(j)} = \left|\frac{w_k}{a_n}\right|^{1/n} e^{i(2j\pi+\phi_k)/n}\big(1+o(1)\big),$$

where

$$\phi_k = \arg\frac{w_k}{a_n}, \qquad j = 0,1,2,\dots,n-1.$$

Thus

$$\begin{aligned}
P(z_k^{(0)}) &\sim A|w_k|^{m/n},\\
P(z_k^{(1)}) &\sim e^{2m\pi i/n}A|w_k|^{m/n},\\
P'(z_k^{(0)}) &\sim B|w_k|^{(m-1)/n},\\
P'(z_k^{(1)}) &\sim e^{2(m-1)\pi i/n}B|w_k|^{(m-1)/n},
\end{aligned}$$

where A, B are constants depending on $q(z)$ and $P(z)$ only. Thus we have sequences $\{w_k\}_{k=1}^{\infty}$, with $w_k \to \infty$ as $k \to \infty$, $\{z_k^{(0)}\}_{k=1}^{\infty}$ and $\{z_k^{(1)}\}_{k=1}^{\infty}$ such that

$$q(z_k^{(0)}) = q(z_k^{(1)}) = w_k, \tag{9}$$

$$P(z_k^{(0)}) - P(z_k^{(1)}) \sim (1 - e^{2m\pi i/n}) A |w_k|^{m/n}, \tag{10}$$

$$f'(z_k^{(0)}) = -P'(z_k^{(0)}) \sim -B|w_k|^{(m-1)/n}, \tag{11}$$

$$f'(z_k^{(1)}) = -P'(z_k^{(1)}) \sim -e^{2(m-1)\pi i/n} B |w_k|^{(m-1)/n}, \tag{12}$$

$$f(z_k^{(0)}) = g(w_k) - P(z_k^{(0)}), \tag{13}$$

$$f(z_k^{(1)}) = g(w_k) - P(z_k^{(1)}), \tag{14}$$

$$f(z_k^{(1)}) - f(z_k^{(0)}) = P(z_k^{(0)}) - P(z_k^{(1)}). \tag{15}$$

If $n \nmid m$, then $1 - e^{2m\pi i/n} \neq 0$. Now we discuss two subcases.

Subcase 1: $\{f(z_k^{(0)})\}_{k=1}^{\infty}$ is bounded. We have, by (10)–(15),

$$|f(z_k^{(1)})| \sim |(1 - e^{2m\pi i/n})A| \, |w_k|^{m/n}. \tag{16}$$

By this and Lemma 1, we obtain that

$$\begin{aligned} |B| \, |w_k|^{(m-1)/n} \sim |f'(z_k^{(1)}| &\geq \frac{|f(z_k^{(1)})| \log |f(z_k^{(1)})|}{16\pi |z_k^{(1)}|} \\ &\sim C|w_k|^{(m-1)/n} \log(|(1 - e^{2m\pi i/n})A| \, |w_k|^{m/n}), \end{aligned}$$

where

$$C = \frac{|(1 - e^{2m\pi i/n})A| \, |a_n|^{1/n}}{16\pi},$$

which is a contradiction.

Subcase 2: $\{f(z_k^{(0)})\}_{k=1}^{\infty}$ is unbounded. Then there exists a sub-sequence of $\{f(z_k^{(0)})\}_{k=1}^{\infty}$ tending to infinity, which we may, without confusing, denote by the original sequence: $\{f(z_k^{(0)})\}_{k=1}^{\infty}$. Thus by Lemma 1, we have

$$\begin{aligned} |B| \, |w_k|^{(m-1)/n} \sim |f'(z_k^{(0)}| &\geq \frac{|f(z_k^{(0)})| \log |f(z_k^{(0)})|}{16\pi |z_k^{(0)}|} \\ &\sim \frac{|a_n|^{1/n} |f(z_k^{(0)})| \log |f(z_k^{(0)})|}{16\pi |w_k|^{1/n}}. \end{aligned}$$

Hence,

$$|f(z_k^{(0)})| = o(|w_k^{(m/n)}|).$$

Thus

$$|f(z_k^{(1)})| \sim |(1 - e^{2m\pi i/n})A|\, |w_k^{m/n}|.$$

By arguing similarly as in Subcase 1, we will arrive at a contradiction. Hence $n \mid m$. Finally, we will prove that $q(z)$ is a common right factor of $f(z)$ and $P(z)$. If $q(z)$ is not a right factor of $P(z)$, then there exist polynomials Q and P_1 with $0 < \deg P_1 < n = \deg q$ such that

$$P(z) = Q\big(q(z)\big) + P_1(z).$$

Thus

$$G(z) = f(z) + P_1(z) = g\big(q(z)\big) - Q\big(q(z)\big) = g_1\big(q(z)\big),$$

where $g_1(w) = g(w) - Q(w)$ is a transcendental entire function. By arguing similarly as in the subcase above, it follows that $n \mid \deg P_1$, which is a contradiction. Thus, $P(z) = Q\big(q(z)\big)$ and $f(z) = g\big(q(z)\big) - Q\big(q(z)\big)$. The conclusion follows.

4. Concluding remarks

Corollary. *Let f be a transcendental entire function in B, then for any constant $a \neq 0$, $f(z) + az$ is prime.*

Remark 4. This corollary shows that if $f(z) - az \in B$ for some constant a, then $|NP(f)| \le 1$.

Remark 5. If h is a periodic entire function of order one and mean type, then $h \in B$. Thus if $G(z)$ is as stated in Theorem D, then $G^n \in B$.

Remark 6. The condition $f \in B$ in the above theorem and corollary is not removable. For example, $f(z) = e^z e^{e^z} + e^z$, then $f(z) = (we^w + w) \circ e^z$, and $f(z) + z = (e^w + w) \circ (e^z + z)$. This example shows the cardinality of $NP(f)$ may be greater than one if $f \notin B$.

Remark 7. If f is an entire function such that $\operatorname{sing}(f^{-1}) \subset \mathbf{R}$, then, by Lemma 3, $\sin\big(f(z)\big) \in B$ and $\cos\big(f(z)\big) \in B$. Thus, for any constant $a \neq 0$, $\sin\big(f(z)\big) + az$ and $\cos\big(f(z)\big) + az$ are prime. It was mentioned in [2] that the Pólya–Laguerre class LP consists of all entire functions f which have a representation

$$f(z) = \exp(-az^2 + bz + c) z^n \prod \left(1 - \frac{z}{z_k}\right) \exp\left(\frac{z}{z_k}\right),$$

where $a, b, c \in \mathbf{R}$, $a \ge 0$, $n \in \mathbf{N}_0$, $z_k \in \mathbf{R}\backslash\{0\}$ for all $k \in \mathbf{N}$, and $\sum_{k=1}^{\infty} |z_k|^{-2} < \infty$. Furthermore, if $f_1, f_2, \dots, f_n \in LP$, and $f = f_1 \circ f_2 \circ \cdots \circ f_n$, then $\operatorname{sing}(f^{-1}) \subset \mathbf{R}$. Thus, for example, $\sin\big(f(z)\big) + az$ is prime for $a \neq 0$, when $f \in LP$.

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