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PACKING DIMENSIONS, TRANSVERSAL MAPPINGS AND GEODESIC FLOWS

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Abstract. In this work we first generalize the projection result by K. Falconer and J. Howroyd concerning packing dimensions of projected measures on $\mathbf{R}^n$ to parametrized families of transversal mappings between smooth manifolds and measures on them. After this we compute the packing dimension of the natural projection of a probability measure which is invariant under the geodesic flow on the unit tangent bundle of a two-dimensional Riemannian manifold.

1. Introduction

The behavior of the Hausdorff dimension, $\dim_{\mathrm{H}}$, under projection-type mappings is well known. In the 1950's Marstrand [Mar] proved that the Hausdorff dimension of a planar set is preserved under typical orthogonal projections. In [K] Kaufman verified the same result using potential theoretic methods, and in [Mat1] Mattila generalized it to higher dimensions. For measures the analogous principle, discovered by Kaufman [K], Mattila [Mat2], Hu and Taylor [HT], and Falconer and Mattila [FM], can be formulated in the following way: If μ is a compactly supported Radon measure on $\mathbf{R}^n$, then for almost all $V \in G(n,m)$

$$\dim_{\mathrm{H}} P_V\mu = \dim_{\mathrm{H}} \mu \quad \text{provided that } \dim_{\mathrm{H}} \mu \leq m.$$

On the other hand, if $\dim_{\mathrm{H}} \mu > m$, then

$$P_V\mu \ll \mathscr{H}^m|_V$$

for almost all $V \in G(n,m)$. In addition, if the m-energy of μ is finite, then

$$P_V\mu \ll \mathscr{H}^m|_V \quad \text{with the Radon–Nikodym derivative in } L^2(V, \mathscr{H}^m|_V)$$

for almost all $V \in G(n,m)$. Here $G(n,m)$ is the Grassmann manifold of all m-dimensional linear subspaces of $\mathbf{R}^n$, $P_V\colon \mathbf{R}^n \to V$ is the orthogonal projection

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onto $V \in G(n,m)$, and $P_V\mu$ is the image of μ under P_V defined by the formula $P_V\mu(A) = \mu(P_V^{-1}A)$ for $A \subset V$. Moreover, by $\mu \ll \nu$ we denote the absolute continuity of a measure μ with respect to a measure ν, and $\mu|_A$ is the restriction of a measure μ to a set A, that is $\mu|_A(B) = \mu(A \cap B)$.

In [FH], Falconer and Howroyd proved an analogous result for the packing dimension, $\dim_{\mathrm{p}}$, of projected measures. They showed that if μ is a finite Borel measure on $\mathbf{R}^n$, then

$$\dim_{\mathrm{p}} P_V\mu = \dim_m \mu \tag{1.1}$$

for almost all $V \in G(n,m)$, where $\dim_m \mu$ is a packing-type dimension defined by using a certain m-dimensional kernel. This result tells that the packing dimension is the same for almost all projections, but it may happen that $\dim_m \mu < \dim_{\mathrm{p}} \mu$.

The above results are "almost all"-results giving no information about any specific projection. However, as discovered by Ledrappier and Lindenstrauss [LL], similar methods work for one particular projection. In [LL] they studied measures on the unit tangent bundle SM of a compact two-dimensional Riemannian manifold M. They showed that, if μ is a probability measure on SM and μ is invariant under the geodesic flow, then

(1) $\dim_{\mathrm{H}} \Pi\mu = \dim_{\mathrm{H}} \mu$, if $\dim_{\mathrm{H}} \mu \le 2$, and
(2) $\Pi\mu \ll \mathscr{H}^2|_M$, if $\dim_{\mathrm{H}} \mu > 2$.

Here $\Pi\colon SM \to M$ is the natural projection. Ledrappier and Lindenstrauss proved also that the Radon–Nikodym derivative of $\Pi\mu$ is an L^2-function, if μ has finite α-energy for some $\alpha > 2$.

Inspired by the results in [LL], in [JJLe] we reproved the above theorem and showed that if the α-energy of μ is finite for some $\alpha > 2$, then $\Pi\mu$ has fractional derivatives of order γ in L^2 for all $\gamma < \frac{1}{2}(\alpha - 2)$. To achieve this new proof we used the generalized projection formalism introduced by Peres and Schlag in [PS]. Our proof also explains why this kind of projection result fails when the dimension of the base manifold is greater than two.

In this paper we consider the natural question of how the packing dimension of an invariant measure behaves under the natural projection. To achieve this, in Section 3 we first generalize (1.1) to parametrized families of transversal mappings between manifolds (Theorem 3.6). The methods of Falconer and Howroyd do not directly work in our setting, but circumventing some technical problems eventually leads to a bit easier proof. After this, in Section 4 we compute the packing dimension of $\Pi\mu$, when the setting is similar to that in [LL] (Theorem 4.2). Finally, in Section 5 we show that, unlike the Hausdorff dimension, the packing dimension of a (locally) invariant measure may decrease under the projection even in the two-dimensional case.

2. Preliminaries and definitions

In the following definitions (X, d_X) is a metric space, and $B(x,r)$ is the open ball with center at $x \in X$ and radius $r > 0$.

Definition 2.1. Let μ be a finite Borel measure on X. The (lower) *packing dimension* of μ is

$$\begin{aligned}\dim_{\mathrm{p}} \mu &:= \sup\Big\{t \ge 0 : \liminf_{r\to 0} r^{-t}\mu\big(B(x,r)\big) = 0 \text{ for } \mu\text{-a.a. } x \in X\Big\}\\ &= \inf\big\{\dim_{\mathrm{p}} A : A \subset X \text{ is a Borel set with } \mu(A) > 0\big\}\\ &= \mu\text{- }\operatorname*{ess\,inf}_{x\in X}\left(\limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}\right),\end{aligned}$$

and the *upper packing dimension* of μ is

$$\begin{aligned}\dim_{\mathrm{p}}^* \mu &:= \inf\Big\{t > 0 : \liminf_{r\to 0} r^{-t}\mu\big(B(x,r)\big) > 0 \text{ for } \mu\text{-a.a. } x \in X\Big\}\\ &= \inf\big\{\dim_{\mathrm{p}} A : A \subset X \text{ is a Borel set with } \mu(X \setminus A) = 0\big\}\\ &= \mu\text{- }\operatorname*{ess\,sup}_{x\in X}\left(\limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log r}\right).\end{aligned}$$

As in [FH], for every $k \in \mathbf{N}$ we define a new dimension of a measure, which may be regarded as a packing-type dimension defined in terms of a certain k-dimensional kernel. For more properties of this k-dimension, see [FH].

Definition 2.2. Let μ be a finite Borel measure on X. For every $x \in X$, $r > 0$ and $k \in \mathbf{N}$, we define

$$F_k^\mu(x,r) := \int_X \min\big\{1, r^k d_X(x,y)^{-k}\big\}\, d\mu(y) = kr^k \int_r^\infty \frac{\mu\big(B(x,h)\big)}{h^{k+1}}\, dh,$$

where the last equality follows directly from Fubini's theorem. Furthermore, we define

$$\dim_k \mu := \sup\Big\{t \ge 0 : \liminf_{r\to 0} r^{-t} F_k^\mu(x,r) = 0 \;\text{ for } \mu\text{-a.a. } x \in X\Big\}$$

and

$$\dim_k^* \mu := \inf\Big\{t > 0 : \liminf_{r\to 0} r^{-t} F_k^\mu(x,r) > 0 \;\text{ for } \mu\text{-a.a. } x \in X\Big\}$$

for all $k \in \mathbf{N}$.

The following theorem is from [FH, Corollary 3].

Theorem 2.3. *Let ν be a finite Borel measure on $\mathbf{R}^n$. Then*

$$\dim_{\mathrm{p}} \nu = \dim_n \nu \qquad \textit{and} \qquad \dim_{\mathrm{p}}^* \nu = \dim_n^* \nu.$$

Falconer and Howroyd proved also the following [FH, Theorem 6]:

Theorem 2.4. *Let ν be a finite Borel measure on $\mathbf{R}^n$. Then*

$$\dim_{\mathrm{p}} P_V\nu = \dim_m \nu \qquad \text{and} \qquad \dim^*_{\mathrm{p}} P_V\nu = \dim^*_m \nu$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$, where $\gamma_{n,m}$ is the unique orthogonally invariant probability measure on $G(n,m)$.

3. Packing dimensions and transversal mappings between manifolds

In this section we generalize Theorem 2.4 to families of transversal mappings. First we notice that Theorem 2.3 holds also on manifolds.

Theorem 3.1. *Let (N,d_N) be a smooth n-dimensional Riemannian manifold equipped with the distance function d_N induced by the Riemannian metric, and let μ be a finite, compactly supported Borel measure on N. Then*

$$\dim_{\mathrm{p}} \mu = \dim_n \mu \qquad \text{and} \qquad \dim^*_{\mathrm{p}} \mu = \dim^*_n \mu.$$

Proof. The proof of this theorem is the same as the proof of Theorem 2.3 [FH, Corollary 3]. The lemmas they need from [FM] are true also in our setting. □

For the generalization of Theorem 2.4 we first prove a lemma which gives the natural upper bound for the packing dimension of the image measure whenever the mapping is Lipschitz continuous.

Lemma 3.2. *Let (N,d_N) and (M,d_M) be smooth Riemannian manifolds with dimensions n and m, respectively. Let $P\colon N \to M$ be a Lipschitz continuous mapping, and let μ be a finite Borel measure on N. Then*

$$\dim_m P\mu \le \dim_m \mu \qquad \text{and} \qquad \dim^*_m P\mu \le \dim^*_m \mu.$$

Proof. We denote $C := \mathrm{Lip}(P) + 1 \ge 1$. Let $0 \le t < \dim_m P\mu$, and define

$$M(t) := \Big\{ u \in M : \liminf_{r\to 0} r^{-t} F_m^{P\mu}(u,r) = 0 \Big\}.$$

First we notice that by standard arguments $u \mapsto \liminf_{r\to 0} r^{-t} F_m^{P\mu}(u,r)$ is a Borel function and thus $M(t)$ is a Borel set. From the definition of $\dim_m P\mu$ we get that $\mu\big(N \setminus P^{-1}\big(M(t)\big)\big) = P\mu\big(M\setminus M(t)\big) = 0$. Now take $x \in P^{-1}\big(M(t)\big) \subset N$ and denote $u := P(x) \in M(t)$. Then

$$\begin{aligned}
\liminf_{r\to 0} r^{-t} F_m^{\mu}(x,r) &= \liminf_{r\to 0}\, r^{-t} \int_N \min\big\{1, r^m d_N(x,y)^{-m}\big\}\, d\mu(y) \\
&\overset{(*)}{\le} C^m \liminf_{r\to 0}\, r^{-t} \int_N \min\big\{1, r^m d_M\big(P(x),P(y)\big)^{-m}\big\}\, d\mu(y) \\
&= C^m \liminf_{r\to 0}\, r^{-t} \int_{P(N)} \min\big\{1, r^m d_M\big(P(x),v\big)^{-m}\big\}\, dP\mu(v) \\
&\overset{(**)}{=} C^m \liminf_{r\to 0}\, r^{-t} \int_M \min\big\{1, r^m d_M(u,v)^{-m}\big\}\, dP\mu(v) \\
&= C^m \liminf_{r\to 0}\, r^{-t} F_m^{P\mu}(u,r) = 0,
\end{aligned}$$

where $(*)$ follows from the fact that $d_M\big(P(x),P(y)\big) \le C d_N(x,y)$ and $(**)$ holds because $P\mu\big(M \setminus P(N)\big) = 0$. Hence $t \le \dim_m \mu$, and so $\dim_m P\mu \le \dim_m \mu$ proving the first inequality. Next take $0 \le t < \dim_m^* P\mu$. Again by definition $\mu\big(P^{-1}\big(M(t)\big)\big) = P\mu\big(M(t)\big) > 0$. The same calculation as above shows that

$$\liminf_{r\to 0} r^{-t} F_m^{\mu}(x,r) = 0$$

for all $x \in P^{-1}\big(M(t)\big)$ and thus $t \le \dim_m^* \mu$. This proves the claim. $\square$

Next we introduce the setting we are going to work with. Let (L,d_L) be a smooth, bounded l-dimensional Riemannian manifold equipped with the distance function d_L induced by the Riemannian metric, let (N,d_N) be a smooth n-dimensional Riemannian manifold, and let (M,d_M) be a smooth m-dimensional Riemannian manifold. We suppose that $l,n \ge m$ so that in our setting L corresponds to $G(n,m)$, and furthermore, N and M correspond to $\mathbf{R}^n$ and $\mathbf{R}^m \cong V \in G(n,m)$, respectively. Let $P\colon L \times N \to M$ be a continuous function such that for all $j \in \{0,1,\dots\}$ there exists a constant C_j such that whenever $k_1 + \dots + k_l = j$,

$$\|\partial_{\lambda_1}^{k_1} \cdots \partial_{\lambda_l}^{k_l} P(\lambda,x)\| \le C_j$$

for all $(\lambda,x) \in L \times N$. Later on we will use the notation $P_\lambda(x) := P(\lambda,x)$. The basic assumptions we need are the following:

(1) There are finite collections $\{\phi,V\}$ and $\{\varphi,U\}$ of charts on L and M, respectively, with the following property: there exists $R>0$ such that for all $\lambda \in L$ and $u \in M$

$$B(\lambda,R) \subset V \qquad \text{and} \qquad B(u,R) \subset U$$

for some V and U.

(2) The Lipschitz constants of the mappings φ, φ^{-1}, ϕ, and ϕ^{-1} are uniformly bounded from above by a positive constant K.

(3) Mapping $T\colon \big\{(x,y,\lambda) \in N^2 \times L : x \ne y,\ d_M\big(P_\lambda(x),P_\lambda(y)\big) \le R\big\} \to \mathbf{R}^m$,

$$T_{x,y}(\lambda) = \frac{\varphi \circ P_\lambda(x) - \varphi \circ P_\lambda(y)}{d_N(x,y)}$$

is transversal, i.e., there exists a constant $C_T > 0$ such that

$$\det\big(DT_{x,y}(\lambda)\big(DT_{x,y}(\lambda)\big)^T\big) \ge C_T^2 \qquad \text{whenever } |T_{x,y}(\lambda)| \le C_T.$$

We may refer to this property also by saying that P is transversal. Here A^T stands for the transpose of a matrix A. Moreover, we assume that

$$|\partial_{\lambda_j}\partial_{\lambda_k}(T_{x,y})_i(\lambda)| \le L < \infty$$

for all $j,k \in \{1,\dots,l\}$, $i \in \{1,\dots,m\}$, $x,y \in N$, and $\lambda \in L$.

Throughout the rest of this section the manifolds (L,d_L), (N,d_N) and (M,d_M), and the mapping $P\colon L\times N\to M$ will satisfy the above assumptions.

Before stating the main result of this chapter we prove three short lemmas which will help us in the proof of Theorem 3.6. Analogous lemmas can be found in [FH], but since our setting is a bit more general, we have to modify their methods. In particular, in our case we do not have the lower bound for the $\mathscr{H}^l$-measure of the set of exceptional parameters λ (see inequality (3.1)). In fact, circumventing this problem leads to a little bit simpler proof than the one in [FH].

Lemma 3.3. *Let (L,d_L), (N,d_N), (M,d_M), and $P\colon L\times N\to M$ be as above, and let μ be a finite, compactly supported Borel measure on N. In addition, assume that P is transversal. Then there exist constants $C'>0$ and $r_0>0$ such that*

$$C'F_m^\mu(x,r)\ge\int_L\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big)\,d\mathscr{H}^l(\lambda)$$

whenever $0<r<r_0$. Here $\mu_\lambda:=P_\lambda\mu$.

Proof. We know that under our assumptions there exist constants $C'>0$ and $r_0>0$ such that

$$\mathscr{H}^l\big\{\lambda\in L: d_M\big(P_\lambda(x),P_\lambda(y)\big)\le r\big\}\le C'\min\big\{1,r^m d_N(x,y)^{-m}\big\} \tag{3.1}$$

for all $0<r<r_0$ and distinct $x,y\in N$ (see Lemma 2.1 in [JJN] for the proof). Hence we see by Fubini's theorem that

$$\begin{aligned}C'F_m^\mu(x,r)&\ge\int_N\mathscr{H}^l\big\{\lambda\in L: d_M\big(P_\lambda(x),P_\lambda(y)\big)\le r\big\}\,d\mu(y)\\&=\int_L\mu\big\{y\in N: d_M\big(P_\lambda(x),P_\lambda(y)\big)\le r\big\}\,d\mathscr{H}^l(\lambda)\\&=\int_L\mu\big(P_\lambda^{-1}\big(B\big(P_\lambda(x),r\big)\big)\big)\,d\mathscr{H}^l(\lambda)\\&=\int_L\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big)\,d\mathscr{H}^l(\lambda)\end{aligned}$$

whenever $0<r<r_0$. □

Lemma 3.4. *Let (L,d_L), (N,d_N), (M,d_M), and $P\colon L\times N\to M$ be as above, and let μ be a finite, compactly supported Borel measure on N. In addition, assume that P is transversal. If $\liminf_{r\to0}r^{-t}F_m^\mu(x,r)=0$, then*

$$\liminf_{r\to0}r^{-t}\mu_\lambda\big(B(P_\lambda(x),r)\big)=0$$

for $\mathscr{H}^l$-almost all $\lambda\in L$, where $\mu_\lambda=P_\lambda\mu$.

Proof. Firstly, because

$$C' F_m^\mu(x,r) \geq \int_L \mu_\lambda\big(B\big(P_\lambda(x),r\big)\big)\, d\mathscr{H}^l(\lambda)$$

by Lemma 3.3, we have that

$$\mathscr{H}^l\big\{\lambda \in L : \mu_\lambda\big(B\big(P_\lambda(x),r\big)\big) > kr^t\big\} \leq k^{-1} r^{-t} C' F_m^\mu(x,r)$$

for all $r,k > 0$. Now fix $k > 0$ and choose a strictly decreasing sequence $r_j \to 0$ such that

$$\lim_{j\to\infty} r_j^{-t} F_m^\mu(x,r_j) = 0.$$

Since

$$\begin{aligned}
\mathscr{H}^l&\Big\{\lambda \in L : \liminf_{r\to 0} r^{-t}\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big) > k\Big\}\\
&= \mathscr{H}^l\bigg(\bigcup_{j=1}^{\infty}\big\{\lambda \in L : r^{-t}\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big) > k \text{ for all } r \leq r_j\big\}\bigg)\\
&= \lim_{j\to\infty} \mathscr{H}^l\big\{\lambda \in L : r^{-t}\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big) > k \text{ for all } r \leq r_j\big\}\\
&\leq \limsup_{j\to\infty} \mathscr{H}^l\big\{\lambda \in L : r_j^{-t}\mu_\lambda\big(B\big(P_\lambda(x),r_j\big)\big) > k\big\}\\
&\leq \limsup_{j\to\infty} k^{-1} r_j^{-t} C' F_m^\mu(x,r_j) = 0,
\end{aligned}$$

we get that for all $k > 0$

$$\liminf_{r\to 0} r^{-t}\mu_\lambda\big(B\big(P_\lambda(x),r\big)\big) \leq k$$

for $\mathscr{H}^l$-almost all $\lambda \in L$. Thus the claim follows when we let k tend to 0 through a countable sequence. □

Lemma 3.5. *Let (L,d_L), (N,d_N), (M,d_M), and $P: L \times N \to M$ be as above, and let μ be a finite, compactly supported Borel measure on N. In addition, assume that P is transversal. For all $t \geq 0$ we define*

$$E(t) = \Big\{x \in N : \liminf_{r\to 0} r^{-t} F_m^\mu(x,r) = 0\Big\}.$$

(a) *If $\mu(N \setminus E(t)) = 0$, then $\dim_{\mathrm{p}} \mu_\lambda \geq t$ for $\mathscr{H}^l$-almost all $\lambda \in L$.*
(b) *If $\mu(E(t)) > 0$, then $\dim_{\mathrm{p}}^* \mu_\lambda \geq t$ for $\mathscr{H}^l$-almost all $\lambda \in L$.*

Proof. As in Lemma 3.2 we see by standard methods that $E(t)$ is a Borel set. The proof of (a) is similar to that of Proposition 5(a) in [FH]. For the reader's convenience we will prove claim (b). By Lemma 3.4 for all $x \in E(t)$

$$\liminf_{r\to 0} r^{-t}\mu_\lambda\big(B(P_\lambda(x),r)\big)=0$$

for $\mathscr{H}^l$-almost all $\lambda \in L$. Therefore Fubini's theorem implies that for $\mathscr{H}^l$-almost all $\lambda \in L$

$$\liminf_{r\to 0} r^{-t}\mu_\lambda\big(B(u,r)\big)=0$$

for μ_λ-almost all $u \in P_\lambda\big(E(t)\big) \subset M$. Because

$$\mu_\lambda\big(P_\lambda\big(E(t)\big)\big) \geq \mu\big(E(t)\big) > 0,$$

we have that for $\mathscr{H}^l$-almost all $\lambda \in L$

$$\mu_\lambda\Big(\Big\{u \in M : \liminf_{r\to 0} r^{-t}\mu_\lambda\big(B(u,r)\big)=0\Big\}\Big) > 0,$$

and so $\dim_{\mathrm{p}}^* \mu_\lambda \geq t$ for $\mathscr{H}^l$-almost all $\lambda \in L$ proving (b). □

After these lemmas we get the main theorem almost for free.

Theorem 3.6. *Let (L,d_L), (N,d_N), (M,d_M), and $P\colon L\times N \to M$ be as above, and let μ be a finite, compactly supported Borel measure on N. In addition, assume that P is transversal. Then*

$$\dim_{\mathrm{p}} \mu_\lambda = \dim_m \mu \qquad \text{and} \qquad \dim_{\mathrm{p}}^* \mu_\lambda = \dim_m^* \mu$$

for $\mathscr{H}^l$-almost all $\lambda \in L$, where $\mu_\lambda = P_\lambda\mu$ is the projected measure.

Proof. From Theorem 3.1 and Lemma 3.2 we immediately obtain that $\dim_{\mathrm{p}} \mu_\lambda \leq \dim_m \mu$ and $\dim_{\mathrm{p}}^* \mu_\lambda \leq \dim_m^* \mu$ for all $\lambda \in L$. Hence it is enough to show that the reverse inequalities hold for $\mathscr{H}^l$-almost all $\lambda \in L$. Assume that $0 < t < \dim_m \mu$. Then

$$\liminf_{r\to 0} r^{-t}F_m^\mu(x,r)=0$$

for μ-almost all $x \in N$. By Lemma 3.5(a) $\dim_{\mathrm{p}} \mu_\lambda \geq t$ for $\mathscr{H}^l$-almost all $\lambda \in L$. Taking a countable sequence of $t \to \dim_{\mathrm{p}} \mu$ gives the first claim. Assume next that $0 < t < \dim_m^* \mu$. Then

$$\liminf_{r\to 0} r^{-t}F_m^\mu(x,r)=0$$

in a set of points x of positive μ-measure. By Lemma 3.5(b) $\dim_{\mathrm{p}}^* \mu_\lambda \geq t$ for $\mathscr{H}^l$-almost all $\lambda \in L$. As above, this gives that $\mathscr{H}^l\{\lambda \in L : \dim \mu_\lambda < \dim_m \mu\} = 0$, proving the claim. □

Remark 3.7. In all the theorems and lemmas in Section 3 it is possible to replace (N,d_N) by a closure of an open and bounded subset of such a manifold, since essentially the only property we need is that the space is n-dimensional.

4. The packing dimension of the projection of a measure invariant under the geodesic flow

In this section we will move from "almost all"-results to the study of one specific projection. The behavior of the Hausdorff dimension of a locally invariant probability measure on the unit tangent bundle of a Riemannian surface under the natural projection was studied in [LL] and [JJLe]. Those results tell us that the Hausdorff dimension is preserved under the natural projection. Next we prove an analogous theorem for the packing dimension of the projected measure. But first we recall the definitions of the geodesic flow and the invariance of a measure under the flow.

Definition 4.1. Let SM be the unit tangent bundle of a smooth, compact Riemannian manifold M. For a given $t \in \mathbf{R}$, the geodesic flow $F_t\colon SM \to SM$ is a diffeomorphism defined by the condition

$$F_t(x,v) = \big(\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t)\big),$$

where $\gamma_{(x,v)}$ is the unique geodesic with initial conditions $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$ for every $(x,v) \in SM$.

A measure μ on SM is invariant under the geodesic flow, if $F_t\mu = \mu$ for every $t \in \mathbf{R}$, that is $\mu\big(F_t^{-1}(A)\big) = \mu(A)$ for $A \subset SM$.

The main result of this section is the following:

Theorem 4.2. *Let M be a smooth, compact Riemannian surface, let μ be a Radon probability measure on the unit tangent bundle SM, and let $\Pi\colon SM \to M$ be the natural projection. If μ is invariant under the geodesic flow, then*

$$\dim_{\mathrm{p}} \Pi\mu = \dim_2 \mu.$$

Before going into the proof we fix our notation and prove some technical lemmas. For further details see Section 3 in [JJLe].

The invariance of the measure μ implies that

$$\mu = \sum_{j=1}^{K} \mu_j,$$

where $\mu_j = \psi(\nu_j \times \mathscr{L}^1)$, ψ is a bi-Lipschitz mapping from a compact set $\tilde{I} \subset \mathbf{R}^3$ (basically $\tilde{I} = [0,1]^3$) to its image $\psi(\tilde{I}) \subset SM$, and ν_j is a probability measure on $[0,1]^2$ for every $j \in \{1,\dots,K\}$. We call this kind of measures μ_j locally invariant. Since

$$\Pi\Big(\sum_{j=1}^{K} \mu_j\Big) = \sum_{j=1}^{K} \Pi\mu_j \qquad \text{and} \qquad \dim_{\mathrm{p}}\Big(\sum_{j=1}^{K} \mu_j\Big) = \min_{1\le j\le K} \dim_{\mathrm{p}} \mu_j,$$

it is enough to prove Theorem 4.2 for measures μ_j. So we fix some $j \in \{1,\dots,K\}$ and denote $\tilde\mu := \mu_j$ and $\nu := \nu_j$. In [JJLe] we proved that there exists a transversal mapping $P\colon [0,1]\times[0,1]^2 \to \mathbf{R}$, $(t,x)\mapsto P(t,x) =: P_t(x)$ such that the dimensional behavior of $(\Phi\circ\Pi)\tilde\mu$ is similar to that of the measure μ' defined by the condition

$$\int_{\mathbf{R}^2} f(x,t)\,d\mu'(x,t) = \int_0^1\int_{\mathbf{R}} f(x,t)\,dP_t\nu(x)\,d\mathscr{L}^1(t)$$

for all non-negative Borel functions $f\colon \mathbf{R}\times[0,1]\to[0,\infty]$. Here Φ is a chart defined on an open set $U\subset M$ such that $[0,1]^2\subset\Phi(U)$.

Next we prove a lemma which gives us the desired lower bound for the packing dimension of $\Pi\tilde\mu$.

Lemma 4.3. *Suppose that for all $t\in[0,1]$ we have a compactly supported Radon measure ν_t on $\mathbf{R}$. Suppose that μ is a Radon measure on $\mathbf{R}\times[0,1]$ such that*

$$\int_{\mathbf{R}^2} f(x,t)\,d\mu(x,t) = \int_0^1\int_{\mathbf{R}} f(x,t)\,d\nu_t(x)\,d\mathscr{L}^1(t)$$

for all non-negative Borel functions $f\colon\mathbf{R}\times[0,1]\to\mathbf{R}$. Then, if $\dim_{\mathrm{p}}\nu_t\ge\alpha$ for $\mathscr{L}^1$-almost all $t\in[0,1]$, we have that $\dim_{\mathrm{p}}\mu\ge\alpha+1$.

Proof. The proof of this theorem is similar to the proof of the corresponding result for the Hausdorff dimension, which can be found from [JJL, Lemma 3.4]. One essential part of the proof is the fact that since $P_{V^\perp}\mu \ll \mathscr{H}^1|_{V^\perp}$ (see Lemma 3.4 in [JJL]),

$$\mathscr{H}^1\text{-}\operatorname*{ess\,inf}_{a\in V^\perp}\bigl\{\dim_{\mathrm{p}}\mu_{V,a} : \mu_{V,a}\ne 0\bigr\} \le \dim_{\mathrm{p}}\mu - 1, \tag{4.1}$$

where $V = x$-axis, $V^\perp$ is the orthogonal complement of V, and $\mu_{V,a}$ is the slice of measure μ by the affine subspace $V_a = V+a$, $a\in V^\perp$. For the definition of these slices see [Mat3, Chapter 10]. Inequality (4.1) in turn follows basically from the definition of the packing dimension and from Lemma 5.1 in [F], which says that for every $A\subset\mathbf{R}^n$ and subspace $V\in G(n,n-m)$

$$\dim_{\mathrm{p}}(A\cap V_a)\le\max\{\dim_{\mathrm{p}}A-m,0\}$$

for $\mathscr{H}^m$-almost all $a\in V^\perp$. □

Lemma 4.4. *We have*

$$\dim_{\mathrm{p}}(\Phi\circ\Pi)\tilde\mu = \dim_2\tilde\mu$$

Proof. First we notice that

$$\dim_2\tilde\mu = \dim_1\nu + 1. \tag{4.2}$$

Namely, since ψ is a bi-Lipschitz mapping, we have that

$$\begin{aligned} F_1^{\nu}(y,r) &= r\int_r^{\infty} \frac{\nu\big(B(y,h)\big)}{h^2}\,dh = r\int_r^{\infty} \frac{\nu\big(B(y,h)\big)\cdot h}{h^3}\,dh \\ &\asymp r\int_r^{\infty} \frac{(\nu\times\mathscr{L})\big(B\big((y,s),h\big)\big)}{h^3}\,dh \\ &\asymp r\int_r^{\infty} \frac{\psi(\nu\times\mathscr{L})\big(B\big(\psi(y,s),h\big)\big)}{h^3}\,dh \\ &\asymp \frac{F_2^{\psi(\nu\times\mathscr{L})}\big(\psi(y,s),r\big)}{r} = \frac{F_2^{\tilde\mu}\big(\psi(y,s),r\big)}{r} \end{aligned}$$

for all $(y,s)\in\tilde I$ and $r>0$. Equation (4.2) follows from this equality and the definition of the image measure. By the notation $A\asymp B$ we mean that there exists a constant $c>0$ such that $A/c\le B\le cA$. Above this constant does not depend on r.

Theorem 3.6 tells us that $\dim_{\mathrm{p}} P_t\nu = \dim_1\nu$ for $\mathscr{L}^1$-almost all $t\in[0,1]$, and so by Lemma 4.3 and equality (4.2), $\dim_{\mathrm{p}}(\Phi\circ\Pi)\tilde\mu \ge \dim_1\nu + 1 = \dim_2\tilde\mu$. On the other hand, by Theorem 3.1 and Lemma 3.2

$$\dim_{\mathrm{p}}(\Phi\circ\Pi)\tilde\mu = \dim_2(\Phi\circ\Pi)\tilde\mu \le \dim_2\tilde\mu$$

proving the claim. □

Proof of Theorem 4.2. Theorem 4.2 follows directly from the previous lemma, since Φ does not change the dimension as a bi-Lipschitz mapping. □

5. An example of a locally invariant measure whose packing dimension decreases under the projection

While the Hausdorff dimension of an invariant measure is preserved under the natural projection in the two-dimensional case, the packing dimension may change when the measure is projected to the base manifold. An example showing that the packing dimension of a locally invariant measure can really decrease under the projection can be obtained by using the measure of Example 5.1 in [FM]. In that example Falconer and Mattila constructed for every $0<d<s<2$, $d<1$ a finite, compactly supported measure $\nu_{s,d}$ on $[0,1]^2$, whose packing dimension is s, and

$$\dim_1\nu_{s,d} = \frac{s\big(1-\frac12 d\big)}{1+\frac12 s-d} < s.$$

Using the similar notation as in the previous section we define

$$\tilde\mu := \psi(\nu_{s,d}\times\mathscr{L}^1),$$

in which case $\tilde{\mu}$ is a locally invariant measure on SM and

$$\dim_{\mathrm{p}} \Pi\tilde{\mu} = \dim_2 \tilde{\mu} = \dim_1 \nu_{s,d} + 1 < s + 1 = \dim_{\mathrm{p}} \tilde{\mu}.$$

Remark 5.1. The example above suggests the existence of a globally invariant measure whose dimension decreases under the projection, but the construction of such a measure does not seem to be quite simple.

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