Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 407–418

BMO-INVARIANCE OF QUASIMINIMIZERS

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Abstract. We consider the BMO-invariance of quasiminimizers by means of quasihyperbolic metric. It is shown that

$$\|u \circ \varphi\|_{\mathrm{BMO}(\Omega)} \le C \|u\|_{\mathrm{BMO}(\Omega')}$$

for all quasiminimizers u in Ω' whenever $\varphi\colon \Omega \to \Omega'$ is uniformly continuous with respect to quasihyperbolic metrics in the domains $\Omega \subset \mathbf{R}^n$ and $\Omega' \subset \mathbf{R}^n$. It is also shown that the quasihyperbolic uniform continuity is a necessary condition for the BMO-invariance under the additional assumption that φ is a quasiregular mapping.

1. Introduction

Let $\varphi\colon \Omega \to \Omega'$ be a mapping between strict subdomains of $\mathbf{R}^n$. In this paper we consider the following question: Under what assumption on φ we have the condition

$$\|u \circ \varphi\|_{\mathrm{BMO}(\Omega)} \le C \|u\|_{\mathrm{BMO}(\Omega')} \tag{1.1}$$

for all K-quasiminimizers u in Ω'? The famous result due to Reimann [Re] says that (1.1) holds for all functions $u \in \mathrm{BMO}(\Omega)$ if and only if $\varphi\colon \Omega \to \Omega'$ is quasiconformal. Therefore, if we consider only those BMO-functions which are in some sense harmonic, it is natural to hope that a larger class of the mappings φ satisfy (1.1). This appears to be true, and the correct class consists of those mappings $\varphi\colon \Omega \to \Omega'$, which are uniformly continuous with respect to the quasihyperbolic metrics in the domains Ω and Ω'. The quasihyperbolic uniform continuity is sufficient for the condition (1.1) without any differentiability assumption on φ as far as we do not require $u \circ \varphi$ to be harmonic in some sense (Theorem 4.1).

The quasihyperbolic uniform continuity is also a necessary condition for (1.1) at least if φ is quasiregular and u is n-harmonic (Theorem 4.5). In particular, the quasihyperbolic uniform continuity characterizes those analytic functions $\varphi\colon \Omega \to \Omega'$ for which (1.1) holds for all classical harmonic BMO-functions u in Ω'. Even this result seems to be untouched in the literature. The special case in which Ω and Ω' both coincide with the unit disk in the plane is clear by the Schwarz lemma, see [RU, Section 1]. In the quasiregular case Vuorinen has characterized

2000 Mathematics Subject Classification: Primary 35J60.

in [V, Chapter 12] the class we consider by means of a Harnack condition. The proof of Theorem 4.5 relies on this characterization.

This paper is organized as follows. In Section 2, we briefly introduce those properties of the quasihyperbolic metric and BMO that are needed for understanding the arguments in later sections. In Section 3, we introduce the necessary properties of quasiminimizers. Among else we characterize quasiminimizers with bounded mean oscillation by means of a weak Bloch condition. This weak Bloch condition has appeared at least in the papers [N1], [N2], [N3], [K], [L1] and [L2]. The key results of Section 3 are previously known for solutions of quasilinear elliptic equations. Although the proofs for quasiminimizers are essentially the same, we have written many of the details in order make the presentation reasonably self-contained. The main results of the paper are included in Section 4, where we focus on the BMO-invariance and prove the results introduced above.

2. Quasihyperbolic metric

In this section, we introduce the quasihyperbolic metric and the BMO-norm. Throughout this paper, Ω and Ω' are strict subdomains of $\mathbf{R}^n$, $n \geq 2$. For a function $u\colon \Omega \to \mathbf{R}$ and $E \subset \Omega$, we denote

$$\mathrm{osc}_E u = \sup_E u - \inf_E u.$$

All the constants are denoted by C. This should not cause any confusion since we always make precise how the constant C depend on the given parameters.

Quasihyperbolic metric. The quasihyperbolic metric was introduced by Gehring and Palka in [GP]. For each $x, y \in \Omega$, the *quasihyperbolic distance* $k_\Omega(x,y)$ is defined by

$$k_\Omega(x,y) = \inf_\gamma \int_\gamma \frac{ds}{d(z,\partial\Omega)},$$

where the infimum is taken over all rectifiable curves γ joining x and y in Ω.

Quasihyperbolic metric extends the classical hyperbolic metric to arbitrary domains. We only need two basic facts on quasihyperbolic metric. Firstly,

$$k_\Omega(x,y) \geq \log\left(1 + \frac{|x-y|}{d(x,\partial\Omega)}\right) \tag{2.1}$$

for all $x, y \in \Omega$ by [GP, Lemma 2.1]. Secondly, if $0 < \varrho < 1$, $x \in \Omega$, and $y \in B\bigl(x, \varrho d(x,\partial\Omega)\bigr)$, then

$$C^{-1}\frac{|x-y|}{d(x,\partial\Omega)} \leq k_\Omega(x,y) \leq C\frac{|x-y|}{d(x,\partial\Omega)} \tag{2.2}$$

for some constant $C > 0$ only depending on ϱ, see e.g. [V, p. 34].

Lemma 2.1. *Let $\varphi\colon \Omega \to \Omega'$ be uniformly continuous between the metric spaces (Ω, k_Ω) and $(\Omega', k_{\Omega'})$. Then for each $0 < \kappa < 1$ there is $0 < \varrho < 1$ such that $\varphi(y) \in B\big(\varphi(x), \kappa d\big(\varphi(x), \partial\Omega'\big)\big)$ whenever $y \in B\big(x, \varrho d(x, \partial\Omega)\big)$.*

Proof. Let $0 < \kappa < 1$. A mapping $\varphi\colon \Omega \to \Omega'$ is uniformly continuous between the metric spaces (Ω, k_Ω) and $(\Omega', k_{\Omega'})$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $k_{\Omega'}\big(\varphi(x), \varphi(y)\big) < \varepsilon$ whenever $k_\Omega(x, y) < \delta$. We are free to assume that $0 < \varepsilon < \log\frac{3}{2}$ and $0 < \delta < \log\frac{3}{2}$. Now (2.1) implies that

$$|x - y| \le \tfrac{1}{2} d(x, \partial\Omega) \quad \text{and} \quad |\varphi(x) - \varphi(y)| \le \tfrac{1}{2} d\big(\varphi(x), \partial\Omega'\big).$$

Hence by (2.2),

$$\frac{|\varphi(x) - \varphi(y)|}{d(\varphi(x), \partial\Omega')} \le C\varepsilon \quad \text{if } \frac{|x - y|}{d(x, \partial\Omega)} \le C\delta.$$

It is enough to choose $C\varepsilon < \kappa$. □

Functions of bounded mean oscillation. For any $u \in L^1_{\text{loc}}(\Omega)$, we say that u is of *bounded mean oscillation* in Ω, write $u \in \mathrm{BMO}(\Omega)$, if

$$\|u\|_{\mathrm{BMO}(\Omega)} = \sup_{B \Subset \Omega} \frac{1}{|B|} \int_B |u - u_B|\, dx < \infty.$$

Here and elsewhere $|E|$ is the n-dimensional Lebesgue measure of the set $E \subset \mathbf{R}^n$, dx refers to the integration with respect to measure $|\cdot|$, and u_B is the integral average

$$u_B = \frac{1}{|B|} \int_B u\, dx.$$

The following result due to Staples [S, Corollary 2.26] plays the key role in our arguments:

Lemma 2.2. *Let $\alpha > 1$. If*

$$\|u\|_{\mathrm{BMO}_\alpha(\Omega)} := \sup_{\alpha B \subset \Omega} \frac{1}{|B|} \int_B |u - u_B|\, dx < \infty,$$

then

$$\|u\|_{\mathrm{BMO}(\Omega)} \le C \|u\|_{\mathrm{BMO}_\alpha(\Omega)}.$$

The constant C depends only on n and α.

We also need the following well-known fact, see [St, p. 144]:

Lemma 2.3. *Assume that $u \in \mathrm{BMO}(\Omega)$ and let $q > 1$. Then*

$$\frac{1}{|B|} \int_B |u - u_B|^q\, dx \le C \|u\|^q_{\mathrm{BMO}(\Omega)}$$

for all balls $B \Subset \Omega$. The constant C depends only on n and q.

3. Quasiminimizers

Quasiminimizers are the functions which minimize up to the constant the p-energy integral amongst all functions with the same (Sobolev) boundary values. For the precise definition, we first recall the notion of Sobolev function. Let $\Omega \subset \mathbf{R}^n$ be an open set and let $p \geq 1$. The *Sobolev space* $W^{1,p}(\Omega)$ consists of functions $u \in L^p(\Omega)$ for which the $L^p(\Omega)$-norm $\|\nabla u\|_p$ is finite. Here and elsewhere ∇u is the distributional gradient of u. The local Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega)$ consists of functions u satisfying $u \in W^{1,p}(G)$ for all open sets G compactly contained in Ω. Finally, the Sobolev space with zero boundary values is denoted by $W^{1,p}_0(\Omega)$.

Definition 3.1. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a K-*quasiminimizer in* Ω if there is a constant $K > 0$ such that

$$\int_\Omega |\nabla u|^p \, dx \leq K \int_\Omega |\nabla \varphi|^p \, dx$$

for all $\varphi \in W^{1,p}(\Omega')$ with $u - \varphi \in W^{1,p}_0(\Omega')$ and for all open sets $\Omega' \subset \Omega$ with compact closure in Ω.

Definition 3.2. A function $u \in W^{1,p}(\Omega)$ belongs to the *De Giorgi class in* Ω, denoted $u \in \mathrm{DG}(\Omega)$, if there is a constant $C > 0$ such that for all $k \in \mathbf{R}$ and $0 < r < R$

$$\int_{B(x,r)} |\nabla (u-k)^+|^p \, dx \leq \frac{C}{(R-r)^p} \int_{B(x,R)} ((u-k)^+)^p \, dx$$

whenever $B(x,R) \subset \Omega$.

It is known and vital to our consideration that $u \in \mathrm{DG}(\Omega)$ and $-u \in \mathrm{DG}(\Omega)$ with a constant C only depending on n, p and K whenever u is a K-quasiminimizer in Ω, see [KS, Section 3]. This easily implies the following Caccioppoli type estimate:

Lemma 3.3. *Let u be a K-quasiminimizer in Ω and let $0 < r < R$. Then*

$$\int_{B(x,r)} |\nabla u|^p \, dx \leq \frac{C}{(R-r)^p} \int_{B(x,R)} |u-k|^p \, dx$$

for all $k \in \mathbf{R}$ whenever $B(x,R) \subset \Omega$. The constant C depends only on n, p and K.

Proof. Let $k \in \mathbf{R}$. Since $u \in \mathrm{DG}(\Omega)$, we obtain

$$\int_{B(x,r)\cap\{u>k\}} |\nabla u|^p \, dx = \int_{B(x,r)} |\nabla (u-k)^+|^p \, dx \leq \frac{C}{(R-r)^p} \int_{B(x,R)} \big((u-k)^+\big)^p \, dx.$$

Similarly, since $-u \in \mathrm{DG}(\Omega)$ and $|\nabla u| = 0$ a.e. in $\{x \in \Omega : u(x) = k\}$, we have

$$\begin{aligned}\int_{B(x,r)\cap\{u<k\}} |\nabla u|^p \, dx &= \int_{B(x,r)\cap\{-u>-k\}} |\nabla u|^p \, dx \\ &\le \frac{C}{(R-r)^p} \int_{B(x,R)} \big((k-u)^+\big)^p \, dx.\end{aligned}$$

By combining the estimates,

$$\begin{aligned}\int_{B(x,r)} |\nabla u|^p \, dx &\le \frac{C}{(R-r)^p} \int_{B(x,R)} \big((u-k)^+\big)^p + \big((k-u)^+\big)^p \, dx \\ &= \frac{C}{(R-r)^p} \int_{B(x,R)} |u-k|^p \, dx. \ \square\end{aligned}$$

The celebrated De Giorgi method yields the following *weak Harnack inequality*, see [KS, p. 413]:

Lemma 3.4. *Let u be a K-quasiminimizer in Ω and let $\alpha > 1$. Then there is a constant $C > 0$ depending only on n, p, K and α such that*

$$\sup_{B(y,r)} |u| \le C \left(\frac{1}{|B(y,\alpha r)|} \int_{B(y,\alpha r)} |u|^p \, dx \right)^{1/p}$$

for all balls $B(y,r)$ with $B(y,\alpha r) \subset \Omega$.

As an application of Lemma 3.4 we obtain a useful oscillation estimate.

Lemma 3.5. *Let u be a K-quasiminimizer in Ω and let $\alpha > 1$. Then there is a constant $C > 0$ depending only on n, p, K and α such that*

$$\operatorname{osc}_B u \le Cr \left(\frac{1}{|\alpha B|} \int_{\alpha B} |\nabla u|^p \, dx \right)^{1/p}$$

for all balls $B = B(y,r)$ such that $\alpha B = B(y,\alpha r)$ is contained in Ω.

Proof. By Lemma 3.4 and the Poincaré inequality [MZ, Theorem 1.51]

$$\begin{aligned}\operatorname{osc}_B u &\le 2 \sup_B |u - u_{\alpha B}| \le C \left(\frac{1}{|\alpha B|} \int_{\alpha B} |u - u_{\alpha B}|^p \, dx \right)^{1/p} \\ &\le Cr \left(\frac{1}{|\alpha B|} \int_{\alpha B} |\nabla u|^p \, dx \right)^{1/p}. \ \square\end{aligned}$$

Remark. Manfredi has obtained in [M] the estimate of Lemma 3.5 even for all weakly monotone functions in the case $p > n - 1$.

We are now prepared to give a useful characterization for quasiminimizers with bounded mean oscillation. In order to do this, we first define the weak Bloch norm by means of certain averages of p-energy integrals. In what follows, we denote $B_x = B\big(x, \frac{1}{2}d(x,\partial\Omega)\big)$, where $d(x,\partial\Omega)$ is the distance between $x \in \Omega$ and the boundary $\partial\Omega$.

Definition 3.6. Let u be a K-quasiminimizer in Ω. We call u a *Bloch function*, denoted $u \in \mathbf{B}(\Omega)$, if

$$\|u\|_{\mathbf{B}(\Omega)} := \sup_{y\in\Omega} d(y,\partial\Omega)\left(\frac{1}{|B_y|}\int_{B_y} |\nabla u|^p\,dx\right)^{1/p} < \infty.$$

The following lemma is essentially known, see [N1], [N3], [L1], [L2], but we present the proof in detail since it is needed later in Section 4.

Lemma 3.7. *Let u be a K-quasiminimizer in Ω. Then there is a constant $C > 0$ depending only on n, p and K such that*

$$C^{-1}\|u\|_{\mathbf{B}(\Omega)} \le \|u\|_{\mathrm{BMO}(\Omega)} \le C\|u\|_{\mathbf{B}(\Omega)}.$$

Proof. By Lemma 3.5,

$$\operatorname{osc}_{2/3B_y} u \le C d(y,\partial\Omega)\left(\frac{1}{|B_y|}\int_{B_y} |\nabla u|^p\,dx\right)^{1/p} \le C\|u\|_{\mathbf{B}(\Omega)}$$

for all $y \in \Omega$. Hence it follows from Lemma 2.2 that

$$\|u\|_{\mathrm{BMO}(\Omega)} \le C\|u\|_{\mathbf{B}(\Omega)}$$

with the constant C depending only on n, p and K. On the other hand, Lemma 3.3 yields

$$\int_{B_y} |\nabla u|^p\,dx \le \frac{C}{d(y,\partial\Omega)^p}\int_{3/2B_y} |u - u_{3/2B_y}|^p\,dx.$$

By multiplying with $d(y,\partial\Omega)^{p-n}$,

$$d(y,\partial\Omega)^p \frac{1}{|B_y|}\int_{B_y} |\nabla u|^p\,dx \le \frac{C}{|\frac{3}{2}B_y|}\int_{3/2B_y} |u - u_{3/2B_y}|^p.$$

Since Lemma 2.3 implies that

$$\left(\frac{1}{|\frac{3}{2}B_x|}\int_{3/2B_x} |u - u_{3/2B_x}|^p\,dx\right)^{1/p} \le C\|u\|_{\mathrm{BMO}(\Omega)},$$

we conclude

$$\|u\|_{\mathbf{B}(\Omega)} \le C\|u\|_{\mathrm{BMO}(\Omega)}$$

with a constant C only depending on n, p and K. □

Remark 3.8. Let u be a K-quasiminimizer in Ω and let $1 < \alpha < 2$. We define the weak Bloch norm more generally by setting

$$\|u\|_{\alpha;\mathbf{B}(\Omega)} := \sup_{y\in\Omega} d(y,\partial\Omega)\left(\frac{1}{|\alpha B_y|}\int_{\alpha B_y} |\nabla u|^p\,dx\right)^{1/p} < \infty.$$

It is trivial that

$$\|u\|_{\mathbf{B}(\Omega)} \le \alpha^{n/p}\|u\|_{\alpha;\mathbf{B}(\Omega)}.$$

It is also true that

$$\|u\|_{\alpha;\mathbf{B}(\Omega)} \le C\|u\|_{\mathbf{B}(\Omega)} \tag{3.1}$$

with a constant C only depending on n, p, K and α. In fact, by Lemma 3.3

$$\int_{\alpha B_y} |\nabla u|^p\,dx \le \frac{C}{d(y,\partial\Omega)^p}\int_{(\alpha+2)/2B_y} |u - u_{(\alpha+2)/2B_y}|^p\,dx,$$

and Lemma 2.3 implies that

$$\left(\frac{1}{\left|\frac{1}{2}(\alpha+2)B_y\right|}\int_{(\alpha+2)/2B_y} |u - u_{(\alpha+2)/2B_y}|^p\,dx\right)^{1/p} \le C\|u\|_{\mathrm{BMO}(\Omega)}.$$

The assertion (3.1) follows by similar reasoning as in the proof of Lemma 3.7.

Remark 3.9. For $\mathbf{C}^1(\Omega)$-functions, the Bloch seminorm is most naturally defined by

$$\|u\|^*_{\mathbf{B}(\Omega)} := \sup_{y\in\Omega} d(y,\partial\Omega)|\nabla u(y)|.$$

The class of $\mathbf{C}^1(\Omega)$-functions u with $\|u\|^*_{\mathbf{B}(\Omega)} < \infty$ has been studied e.g. in [J] and [KX]. It is true that the condition $\|u\|^*_{\mathbf{B}(\Omega)} < \infty$ implies $u \in \mathrm{BMO}(\Omega)$ (this is proved first in [J]), but the converse seems to require a kind of harmonicity assumption. For p-harmonic functions, the Bloch seminorms $\|\cdot\|^*_{\mathbf{B}(\Omega)}$ and $\|\cdot\|_{\mathbf{B}(\Omega)}$ are equivalent, see [L1, Lemma 4.2].

4. BMO-invariance

This section contains our results on BMO-invariance. We first prove that the quasihyperbolic uniform continuity is a sufficient condition for the BMO-invariance in the class of quasiminimizers.

Theorem 4.1. *Let $\varphi\colon \Omega \to \Omega'$ be uniformly continuous as a mapping between the metric spaces (Ω, k_Ω) and $(\Omega', k_{\Omega'})$. Then*

$$\|u\circ\varphi\|_{\mathrm{BMO}(\Omega)} \le C\|u\|_{\mathrm{BMO}(\Omega')}$$

for all K-quasiminimizers u in Ω'. The constant C depends only on n, p and K.

Proof. Let u be a K-quasiminimizer in Ω' and let $y \in \Omega'$. Applying Lemma 3.4 and Lemma 2.3 to a K-quasiminimizer $u - u_{B_y}$ yields

$$\sup_{1/2B_y} |u - u_{B_y}| \le C\left(\frac{1}{|B_y|}\int_{B_y} |u - u_{B_y}|^p\, dx\right)^p \le C\|u\|_{\mathrm{BMO}(\Omega')}.$$

Therefore

$$\operatorname{osc}_{1/2B_y} u \le C\|u\|_{\mathrm{BMO}(\Omega')} \tag{4.1}$$

for all $y \in \Omega'$. By Lemma 2.1, there is $0 < \varrho < 1$ such that $\varphi(\varrho B_z) \subset \frac{1}{2}B_{\varphi(z)}$ for all $z \in \Omega$. Hence

$$\sup_{z\in\Omega} \operatorname{osc}_{\varrho B_z}(u\circ\varphi) \le \sup_{y\in\Omega'} \operatorname{osc}_{1/2B_y} u \le C\|u\|_{\mathrm{BMO}(\Omega)},$$

so that

$$\sup_{z\in\Omega} \frac{1}{|\varrho B_z|}\int_{\varrho B_z} |(u\circ\varphi) - (u\circ\varphi)_{\varrho B_z}|\, dx \le C\|u\|_{\mathrm{BMO}(\Omega)}.$$

The claim follows from Lemma 2.2. □

Remark 4.2. (a) Using the argument of Theorem 4.1 we easily obtain a variant of Theorem 4.1 for all functions satisfying Harnack's inequality. Let u be a non-negative function in Ω' satisfying Harnack's inequality

$$\sup_B u \le C \inf_B u$$

for all balls B such that $2B \subset \Omega'$. Then in particular

$$\sup_{B_y} u \le C \inf_{B_y} u$$

for all $y \in \Omega'$. Taking logarithms on both sides yields

$$\operatorname{osc}_{B_y}(\log u) \le \log C$$

for all $y \in \Omega'$. Hence $\log u$ satisfies (4.1) and we conclude from the proof of Theorem 4.1 that $\log u \circ \varphi \in \mathrm{BMO}(\Omega)$ whenever φ satisfies the quasihyperbolic uniform continuity assumption of Theorem 4.1.

(b) The argument of Theorem 4.1 works also in the metric spaces under the assumptions described in [KS] if the assumption concerning quasihyperbolic uniform continuity is replaced by the condition in Lemma 2.1. Notice that the fixed exponent p in Lemma 3.4 can be replaced by any exponent $q > 0$, see [KS, p. 414].

Quasiregular mappings. In this subsection we prove the converse for Theorem 4.1 under the additional assumptions that $\varphi\colon \Omega \to \Omega'$ is quasiregular and u is n-harmonic. Hence we consider the special case $p = n$ in what follows. For the reader's convenience we first define quasiregular and n-harmonic functions.

Definition 4.3. Let $\Omega \subset \mathbf{R}^n$ be open and $K \geq 1$. A continuous mapping $f\colon \Omega \to \mathbf{R}^n$ is called K-*quasiregular* if the coordinate functions of f belong to the Sobolev space $W^{1,n}_{\mathrm{loc}}(\Omega)$ and

$$\max_{|h|=1} |f'(x)h|^n \leq K J_f(x)$$

for almost every $x \in \Omega$. Here $f'(x)$ is the almost everywhere defined Jacobi matrix at x and $J_f(x)$ is the determinant of $f'(x)$.

For more on quasiregular mappings, see e.g. [HKM], [Ri] or [V].

Definition 4.4. A function $u\colon \Omega' \to \mathbf{R}$ is n-harmonic if $u \in W^{1,n}_{\mathrm{loc}}(\Omega')$ is continuous and

$$\int_\Omega |\nabla u|^{n-2} \nabla u \cdot \nabla \psi \, dx = 0$$

for all functions $\psi \in \mathbf{C}_0^\infty(\Omega')$.

The proof of our final theorem is based on the well-known fact that the function $x \mapsto \log|x-a|$ is n-harmonic in $\mathbf{R}^n \setminus \{a\}$ for all $a \in \mathbf{R}^n$, see [HKM, Theorem 14.19].

Theorem 4.5. *Let Ω and Ω' be proper subdomains of $\mathbf{R}^n$ and let $\varphi\colon \Omega \to \Omega'$ be a K-quasiregular mapping such that φ is not uniformly continuous between the metric spaces (Ω, k_Ω) and $(\Omega', k_{\Omega'})$. Then*

$$\sup_u \frac{\|u \circ \varphi\|_{\mathrm{BMO}(\Omega)}}{\|u\|_{\mathrm{BMO}(\Omega')}} = +\infty,$$

where the supremum is taken over all non-constant n-harmonic BMO*-functions u in Ω'.*

Proof. By [V] (see Theorem 12.5 and Corollary 12.16), there are sequences $x_i, y_i \in \Omega$ such that $|x_i - y_i| \leq \frac{1}{2} d(x_i, \partial\Omega)$ and

$$d\big(\varphi(x_i), \partial\Omega'\big) < \frac{1}{i} d\big(\varphi(y_i), \partial\Omega'\big) \tag{4.2}$$

for all $i = 1, 2, \ldots$. Pick $a_i' \in \partial\Omega'$ such that $d\big(\varphi(x_i), \partial\Omega'\big) = |\varphi(x_i) - a_i'|$ and consider the functions $u_i\colon \Omega' \to \mathbf{R}$,

$$u_i(x) = \log|x - a_i'|.$$

The functions u_i are n-harmonic in Ω' and it is easy to compute that

$$|\nabla u_i(x)| = \frac{1}{|x - a_i'|}$$

for all $x \in \Omega'$. For each $y \in \Omega'$ and $x \in B_y$, we have

$$|y - a_i'| \le |x - y| + |x - a_i'| \le \tfrac{1}{2}|y - a_i'| + |x - a_i'|,$$

and therefore

$$\begin{aligned} \|u_i\|_{\mathbf{B}(\Omega')} &= \sup_{y\in\Omega'} d(y,\partial\Omega')\biggl(\frac{1}{|B_y|}\int_{B_y} |\nabla u_i(x)|^n\,dx\biggr)^{1/n} \\ &\le \sup_{y\in\Omega'} d(y,\partial\Omega')\biggl(\frac{1}{|B_y|}\int_{B_y} \frac{2^n}{|y-a_i'|^n}\,dx\biggr)^{1/n} \le 2. \end{aligned} \tag{4.3}$$

On the other hand,

$$|\varphi(y_i) - a_i'| \ge d\bigl(\varphi(y_i),\partial\Omega'\bigr) > i\cdot d\bigl(\varphi(x_i),\partial\Omega'\bigr) = i|\varphi(x_i) - a_i'|.$$

Hence

$$|(u_i\circ\varphi)(x_i) - (u_i\circ\varphi)(y_i)| = \left|\log\frac{|\varphi(y_i)-a_i'|}{|\varphi(x_i)-a_i'|}\right| \ge \log i. \tag{4.4}$$

Assume now that our claim does not hold. This in particular implies that

$$\|u_i\circ\varphi\|_{\mathrm{BMO}(\Omega)} \le C\|u_i\|_{\mathrm{BMO}(\Omega')} \tag{4.5}$$

for some constant C independent of i. Since the functions $u_i\circ\varphi$ are $\mathscr{A}$-harmonic in the sense of [HKM] with the structure constants only depending on K ([HKM, Lemma 14.38 and Theorem 14.39]) and $y_i \in B\bigl(x_i, \frac{1}{2}d(x_i,\partial\Omega)\bigr)$, we obtain from the oscillation estimate [HKM, Theorem 6.6] that

$$\operatorname{osc}_{B(x_i,|x_i-y_i|)}(u_i\circ\varphi) \le C\biggl(\frac{|x_i-y_i|}{d(x_i,\partial\Omega)}\biggr)^{\kappa}\operatorname{osc}_{3/2B_{x_i}}(u_i\circ\varphi).$$

By Lemma 3.5 and Remark 3.8,

$$\begin{aligned} \operatorname{osc}_{3/2B_{x_i}}(u_i\circ\varphi) &\le Cd(x_i,\partial\Omega)\biggl(\frac{1}{|\frac{5}{3}B_{x_i}|}\int_{5/3B_{x_i}} |\nabla(u_i\circ\varphi)|^n\,dx\biggr)^{1/n} \\ &\le C\|u_i\circ\varphi\|_{\mathbf{B}(\Omega)}. \end{aligned}$$

By combining the estimates and using (4.3) and (4.5) together with Lemma 3.7 yields

$$|(u_i\circ\varphi)(x_i) - (u_i\circ\varphi)(y_i)| \le C\biggl(\frac{|x_i-y_i|}{d(x_i,\partial\Omega)}\biggr)^{\kappa}\|u_i\|_{\mathbf{B}(\Omega')} \le 2C2^{-\kappa}.$$

This contradicts (4.4) and the claim follows. □

Remark 4.6. Let $\varphi\colon \Omega \to \varphi(\Omega)$ be a quasiconformal mapping such that Ω and $\varphi(\Omega)$ are strict subdomains of $\mathbf{R}^n$. Then φ is uniformly continuous as a mapping between the metric spaces (Ω, k_Ω) and $\big(\varphi(\Omega), k_{\varphi(\Omega)}\big)$, see [GO, Theorem 3] or [V, Corollary 12.19]. A similar assertion holds also for quasiregular mappings if the boundary $\partial\varphi(\Omega)$ satisfies a suitable connectedness assumption, see [V, Theorem 12.21]. There are analytic mappings φ defined in a strict subdomain Ω of $\mathbf{R}^2$, which are not uniformly continuous as a mapping between the metric spaces (Ω, k_Ω) and $\big(\varphi(\Omega), k_{\varphi(\Omega)}\big)$, see [V, Example 11.4].

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Received 22 January 2004