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QUASIREGULAR MAPPINGS OF MAXIMAL LOCAL MODULUS OF CONTINUITY

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Abstract. We study the behavior of a K-quasiregular mapping near points where its local modulus of continuity has order $1/K$. We prove that the mapping is spherically analytic at such points and is asymptotically a rotation on circles. This result is used to prove sharp distortion estimates, including a version of Schwarz's lemma.

1. Introduction

We consider a K-quasiregular mapping $f\colon \Omega \to \mathbf{C}$, where Ω is a domain in $\mathbf{C}$. Recall that $f \in W^{1,2}_{\mathrm{loc}}(\Omega; \mathbf{R}^2)$ is K-quasiregular if $|Df(z)|^2 \leq KJ_f(z)$ for a.e. $z \in \Omega$. Here $|Df(z)|$ is the operator norm of the 2×2 differential matrix $Df(z)$ and $J_f(z) = \det Df(z)$. It is well known that f is locally Hölder continuous with exponent $1/K$ [1]. Consequently, for $z_0 \in \Omega$ and small $\delta > 0$ the local modulus of continuity

$$\omega_f(z_0, \delta) = \max\{|f(z) - f(z_0)| : z \in \Omega,\ |z - z_0| \leq \delta\}$$

is majorized by $A\delta^{1/K}$ for some $A > 0$. A number of sharper upper bounds have been established for $\omega_f(z_0, \delta)$, see [2], [17] and references therein. The upper limit

$$\omega_f(z_0) = \limsup_{\delta \to 0} \frac{\omega_f(z_0, \delta)}{\delta^{1/K}}$$

is always finite, but can be strictly positive. An example is provided by the radial stretch map $f_K(z) = |z|^{1/K-1}z$, for which $\omega_{f_K}(0) = 1$.

The purpose of this paper is to study the behavior of f near points where $\omega_f > 0$, which are called *points of maximal stretch.* Unless $K = 1$, the mapping f is not differentiable at a point of maximal stretch; nevertheless, Theorem 2.1 shows that it behaves in a rather regular way near it. In particular, for every $z_0 \in \Omega$

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|^{1/K}} = \omega_f(z_0)$$

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(which is trivially true when $\omega_f(z_0)=0$). Recall that the linear distortion function of f

$$H_f(z_0)=\limsup_{r\to 0}\sup_{z_1,z_2}\left\{\frac{|f(z_1)-f(z_0)|}{|f(z_2)-f(z_0)|}:|z_1-z_0|=r=|z_2-z_0|\right\}$$

is bounded by a constant depending on K and the degree of f at z_0 [14, Theorem 4.5]. Theorem 2.1 implies that $H_f(z_0)=1$ whenever $\omega_f(z_0)>0$. In the present paper a common course of reasoning "small distortion $\Longrightarrow$ small modulus of continuity" is reversed in an unusual way: "large modulus of continuity $\Longrightarrow$ small distortion".

In Section 3 we give sharp estimates of the distance between images of two points of maximal stretch. For f a K-quasiregular mapping of the unit disk to itself, Theorem 3.3 gives a sharp upper bound for $|f(z)|$ in terms of $|z|$, $\omega_f(0)$, and $\omega_f(z)$. In contrast to the classical Hersch–Pfluger distortion theorem [11, p. 64], the bound in Theorem 3.3 has a simple closed form.

2. Local behavior at a point of maximal stretch

Our first result describes the behavior of a K-quasiregular mapping f near a point z_0 with $\omega_f(z_0)>0$. Recall that

$$\omega_f(z_0)=\limsup_{z\to z_0}|f(z)-f(z_0)|/|z-z_0|^{1/K}.$$

Theorem 2.1. *Let $f\colon\Omega\to\mathbf{C}$ be a K-quasiregular mapping, $z_0\in\Omega$. If $\omega_f(z_0)>0$, then f is injective in a neighborhood of z_0 and there exists a continuous function θ: $(0,1)\to\mathbf{R}$ such that*

$$\lim_{z\to z_0}\left\{\frac{f(z)-f(z_0)}{|z-z_0|^{1/K-1}(z-z_0)}-\omega_f(z_0)e^{i\theta(|z-z_0|)}\right\}=0. \tag{2.1}$$

According to Theorem 2.1, f is *spherically analytic* [8], [9] and is *asymptotically a rotation on circles* [3], [4] at any point of maximal stretch. The proof of Theorem 2.1 is a combination of estimates of conformal capacity and recent results of M. Brakalova and J. A. Jenkins [4]. First, we define the conformal capacity of a compact set E with respect to domain $\Omega\supset E$ as follows.

$$\operatorname{cap}(\Omega,E)=\inf\left\{\int_\Omega|\nabla u(z)|^2\,d\mathscr{L}^2(z):u\in C_0^\infty(\Omega)\text{ and }u\geq 1\text{ on }E\right\},$$

where $\mathscr{L}^2$ is the two-dimensional Lebesgue measure. When E is connected, $\operatorname{cap}(\Omega,E)$ is equal to the reciprocal of the module of ring domain $\Omega\setminus E$ (see, e.g. [16]).

Proof of Theorem 2.1. Without loss of generality we may assume that $z_0 = f(z_0) = 0$. The mapping f admits the Stoilow factorization $f = \psi \circ h$, where h is K-quasiconformal and ψ is a holomorphic function [11, p. 247]. Now if ψ' vanishes at $h(0)$, then as $z \to 0$ we have $\left|\psi\big(h(z)\big)\right| = O\big(|h(z)|^2\big) = O(|z|^{2/K})$, hence $\omega_f(0) = 0$, a contradiction. Thus $\psi'\big(h(0)\big) \neq 0$, which implies that f is homeomorphic in a neighborhood of 0. We may assume that such a neighborhood contains $\overline{\mathbf{D}}$. Here and in the sequel $\mathbf{D}(a,r) = \{z : |z-a| < r\}$, $\mathbf{D}(r) = \mathbf{D}(0,r)$, and $\mathbf{D} = \mathbf{D}(0,1)$.

Let Ω' denote the image of $\mathbf{D}$ under f. Then $\Omega' \subset \mathbf{D}(R)$ for some R, because f is continuous in $\overline{\mathbf{D}}$. Choose a sequence $r_j \to 0$ such that $\omega_f(0, r_j) \geq \rho_j = \frac{1}{2}\omega_f(0) r_j^{1/K}$ for every j. The set $F_j = f\big(\overline{\mathbf{D}}(r_j)\big)$ is connected and meets both 0 and $\partial\mathbf{D}(0,\rho_j)$. By results in [1, Chapter III] we have

$$\begin{aligned}\operatorname{cap}(\Omega', F_j) &\geq \operatorname{cap}\big(\mathbf{D}(R), F_j\big) \geq \frac{2\pi}{\log(4R/\rho_j)} \\ &= \frac{2\pi K}{\log(1/r_j)} - \frac{2\pi K^2 \log\big(8R/\omega_f(0)\big)}{\big(\log(1/r_j)\big)^2} + o\bigg(\frac{1}{\big(\log(1/r)\big)^2}\bigg).\end{aligned}$$

Therefore,

$$\operatorname{cap}(\Omega', F_j) \geq \frac{2\pi K}{\log(1/r_j)} - \frac{C}{\big(\log(1/r_j)\big)^2}, \tag{2.2}$$

where C does not depend on j.

Now we fix j and obtain an upper bound for $\operatorname{cap}(\Omega', F_j)$ as follows. Let $g\colon \Omega' \to \mathbf{D}$ be the inverse of f and define $u(w) = \big(\log|g(w)|\big)/\log r_j$ for $w \in \Omega'$. Note that u is Hölder continuous in $\Omega' \setminus F_j$, $\min\{u, 1\} \in W_0^{1,2}(\Omega')$, and $u_{|F_j} \geq 1$. Therefore,

$$\begin{aligned}\operatorname{cap}(\Omega', F_j) &\leq \int_{\Omega'\setminus F_j} |\nabla u(w)|^2 \, d\mathscr{L}^2(w) \\ &= \big(\log(1/r_j)\big)^{-2} \int_{\Omega'\setminus F_j} \big|\nabla \log|g(w)|\big|^2 \, d\mathscr{L}^2(w).\end{aligned} \tag{2.3}$$

The integrand can be expressed in terms of differential operators $\partial = \partial/\partial z$ and $\bar\partial = \partial/\partial\bar z$ (at almost every point of $\Omega' \setminus F_j$).

$$\big|\nabla \log|g|\big|^2 = 4\big|\partial \log|g|\big|^2 = |\partial \log g + \partial \log \overline{g}|^2 = \left|\frac{\partial g}{g} + \overline{\left(\frac{\bar\partial g}{g}\right)}\right|^2.$$

Now if $w = f(z)$, an easy calculation yields $\partial g(w) = \overline{\partial f(z)} J_f(z)^{-1}$ and $\bar\partial g(w) = -\bar\partial f(z) J_f(z)^{-1}$. Changing the variable of integration and introducing the notation

$\mu = \bar{\partial} f / \partial f$ and $\varphi = \arg z$, we obtain

$$\begin{aligned}\int_{\Omega' \setminus F_j} \big|\nabla \log|g(w)|\big|^2 \, d\mathscr{L}^2(w) &= \int_{\mathbf{D}\setminus\overline{\mathbf{D}}(r_j)} \left| \frac{\overline{\partial f(z)}}{zJ_f(z)} - \overline{\left(\frac{\bar{\partial} f(z)}{zJ_f(z)}\right)} \right|^2 J_f(z) \, d\mathscr{L}^2(z) \\ &= \int_{\mathbf{D}\setminus\overline{\mathbf{D}}(r_j)} \frac{|\partial f(z) - e^{-2i\varphi} \bar{\partial} f(z)|^2}{|\partial f(z)|^2 - |\bar{\partial} f(z)|^2} |z|^{-2} \, d\mathscr{L}^2(z) \\ &= \int_{\mathbf{D}\setminus\overline{\mathbf{D}}(r_j)} \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} |z|^{-2} \, d\mathscr{L}^2(z).\end{aligned}$$

Combining this with (2.2) and (2.3) yields

$$\int_{\mathbf{D}\setminus\overline{\mathbf{D}}(r_j)} \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} |z|^{-2} \, d\mathscr{L}^2(z) \geq 2\pi K \log(1/r_j) - C,$$

which is equivalent to

$$\int_{\mathbf{D}\setminus\overline{\mathbf{D}}(r_j)} \left(K - \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} \right) |z|^{-2} \, d\mathscr{L}^2(z) \leq C.$$

Since the integrand is non-negative, we conclude that

$$\int_{\mathbf{D}} \left| K - \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} \right| |z|^{-2} \, d\mathscr{L}^2(z) < \infty. \tag{2.4}$$

Furthermore, we have an estimate

$$\begin{aligned} 0 &\leq \frac{|1 + e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} - \frac{1}{K} \\ &= 2\frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} - \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} - \frac{1}{K} \\ &= \left(2\frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} - \frac{K^2 + 1}{K} \right) + \left(K - \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} \right) \\ &\leq K - \frac{|1 - e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2}, \end{aligned}$$

where the last inequality follows from $|\mu(z)| \leq (K-1)/(K+1)$. Therefore, (2.4) implies

$$\int_{\mathbf{D}} \left| K^{-1} - \frac{|1 + e^{-2i\varphi}\mu(z)|^2}{1 - |\mu(z)|^2} \right| |z|^{-2} \, d\mathscr{L}^2(z) < \infty \tag{2.5}$$

By (2.4) and (2.5) the mapping f satisfies the conditions of [4, Theorem 1], which says that there exists $A > 0$ such that $|f(z)|/|z|^{1/K} \to A$ as $z \to 0$. Thus f is spherically analytic at 0. It is easy to see that the constant A must be equal to $\omega_f(0)$. Moreover, by [4, Lemma 3] the mapping f is asymptotically a rotation on circles, that is,

$$\lim_{r\to 0}\bigl(\arg f(re^{\theta_2}) - \arg f(re^{\theta_1}) - (\theta_2 - \theta_1)\bigr) = 0$$

uniformly in $\theta_1, \theta_2 \in \mathbf{R}$. For $r \in (0,1)$ define $\theta(r) = \arg f(r)$, where the argument is chosen so that θ is continuous. It remains to observe that (2.1) holds with this choice of θ. □

Corollary 2.1. *Let $f\colon \Omega \to \mathbf{C}$ be a K-quasiconformal mapping. Suppose that $\liminf_{z\to z_0} |f(z) - f(z_0)|/|z - z_0|^K < \infty$ for some $z_0 \in \Omega$. Then there exist a constant A and a continuous function $\theta\colon (0,1) \to \mathbf{R}$ such that*

$$\lim_{z\to z_0}\left\{\frac{f(z) - f(z_0)}{|z - z_0|^{K-1}(z - z_0)} - Ae^{i\theta(|z-z_0|)}\right\} = 0.$$

Proof. Apply Theorem 2.1 to f^{-1}. □

Even though the statement of Corollary 2.1 almost mirrors that of Theorem 2.1, the similarity is not complete. Corollary 2.1 does not apply to general K-quasiregular mappings, as follows from considering the mapping $f(z) = z^2$ at $z_0 = 0$.

Since 1-quasiregular mappings are holomorphic, for $K = 1$ the conclusion of Theorem 2.1 can be strengthened as follows:

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

exists. One might ask if the same is true for $K > 1$. The answer is given by the following theorem.

Theorem 2.2. *For any $K > 1$ there exists a K-quasiconformal mapping $f\colon \mathbf{C} \to \mathbf{C}$ such that the assumptions of Theorem 2.1 are satisfied, but the limit*

$$\lim_{z\to z_0}\frac{f(z) - f(z_0)}{|z - z_0|^{1/K-1}(z - z_0)}$$

does not exist.

Proof. Let $\gamma\colon (0,1] \to (0,1]$ be an increasing continuous function such that

$$\int_0^1 \gamma(r)\,\frac{dr}{r} = \infty \quad \text{and} \quad \int_0^1 \gamma(r)^2\,\frac{dr}{r} < \infty. \tag{2.6}$$

For instance, $\gamma(r) = \bigl(\log(e+1/r)\bigr)^{-1}$ would work. Fix $K > 1$ and let $k = (K-1)/(K+1)$. Let $f\colon \mathbf{C} \to \mathbf{C}$ be the normal solution [1, p. 91] of the Beltrami equation $\bar\partial f = \mu \partial f$, where

$$\mu(re^{i\varphi}) = \begin{cases} -ke^{i(2\varphi+\gamma(r))}, & r \le 1; \\ 0, & r > 1. \end{cases}$$

Then

$$\begin{aligned} \int_{\mathbf{D}} \left| K - \frac{|1-e^{-2i\varphi}\mu(z)|^2}{1-|\mu(z)|^2} \right| \frac{d\mathscr{L}^2(z)}{|z|^2} &= 2\pi \int_0^1 \left| \frac{2k\bigl(1-\cos\gamma(r)\bigr)}{1-k^2} \right| \frac{dr}{r} \\ &= \frac{8\pi k}{1-k^2} \int_0^1 \sin^2\bigl(\tfrac12\gamma(r)\bigr)\,\frac{dr}{r} < \infty, \end{aligned}$$

so (2.4) holds. It follows that f satisfies the assumptions of Theorem 2.1 with $z_0 = 0$.

Since $e^{-2i\varphi}\mu(re^{i\varphi})$ does not depend on φ, it follows from the uniqueness of normal solutions that f is a rotation on circles. That is, there exist continuous functions $g\colon [0,\infty) \to [0,\infty)$ and $\delta\colon (0,\infty) \to \mathbf{R}$ such that $f(re^{i\varphi}) = g(r)e^{i(\varphi+\delta(r))}$. Actually, g and δ are absolutely continuous on every compact subinterval of $(0,\infty)$, which can be seen as follows. Fix $r_0 > 0$ and choose ε so that $0 < \varepsilon < r_0$ and $|\delta(r) - \delta(r_0)| < \frac14\pi$ whenever $|r - r_0| < \varepsilon$. Now consider the mapping

$$h(x+iy) = \log f(e^{x+iy}) = \log g(e^x) + i\bigl(y + \delta(e^x)\bigr)$$

defined on the rectangle

$$R = \bigl\{(x,y) : \log(r_0-\varepsilon) < x < \log(r_0+\varepsilon),\ |y| < \tfrac14\pi\bigr\}.$$

Since h is a quasiconformal mapping, it is absolutely continuous on almost every horizontal segment lying in R [11, p. 162]. Thus $\log g(e^x)$ and $\delta(e^x)$ are absolutely continuous in a neighborhood of $\log r_0$, which implies the absolute continuity of g and δ in a neighborhood of r_0.

It remains to prove that $\delta(r)$ does not have a finite limit as $r \downarrow 0$. For almost every $r \in (0,1)$ we can compute ∂f and $\bar\partial f$ to find that

$$\mu(re^{i\varphi}) = e^{2i\varphi}\frac{g'(r) - g(r)/r + ig(r)\delta'(r)}{g'(r) + g(r)/r + ig(r)\delta'(r)}.$$

Rearranging the terms, obtain

$$ke^{i\gamma(r)} = \frac{1 - \bigl(rg'(r)/g(r) + ir\delta'(r)\bigr)}{1 + \bigl(rg'(r)/g(r) + ir\delta'(r)\bigr)}.$$

Therefore,

$$\frac{rg'(r)}{g(r)} + ir\delta'(r) = \frac{1 - ke^{i\gamma(r)}}{1 + ke^{i\gamma(r)}} = \frac{1 - k^2 - 2ik\sin\gamma(r)}{|1 + ke^{i\gamma(r)}|^2}.$$

Comparing the imaginary parts, we conclude that

$$-\delta'(r) = \frac{2k\sin\gamma(r)}{r|1 + ke^{i\gamma(r)}|^2} \ge \frac{2k}{(1+k)^2}\frac{\sin\gamma(r)}{r}.$$

By (2.6) the right-hand side is not integrable, so $\lim_{r\downarrow 0}\delta(r) = +\infty$. □

3. Distortion of distance between points of maximal stretch

The main tool used in the present section is the generalized reduced module of a domain $\Omega \subseteq \mathbf{C}$ which was introduced in a series of papers by V. N. Dubinin [5], [6]. Besides Ω itself, the module depends on parameters $Z = (z_1, \dots, z_n)$, $\Delta = (\delta_1, \dots, \delta_n)$, and $\Psi = (\psi_1, \dots, \psi_n)$. Here Z is an n-tuple of distinct points in Ω, $\delta_j \in \mathbf{R} \setminus \{0\}$, and $\psi_j \colon [0,\infty) \to [0,\infty)$ is a function of the form $\psi_j(r) = \mu_j r^{\nu_j}$ with $\mu_j, \nu_j > 0$ for each j.

The conformal capacity of Ω with respect to Z, Δ, Ψ, and $r > 0$ is defined as follows.

$$\operatorname{cap} C(r; \Omega, Z, \Delta, \Psi) = \inf\Biggl\{\int_\Omega |\nabla u(z)|^2 \, d\mathscr{L}^2(z) : u \in C_0^\infty(\Omega),$$
$$u = \delta_j \text{ on } \overline{\mathbf{D}}(z_j, \psi_j(r))\Biggr\}.$$

Let $\nu = \bigl(\sum_{j=1}^n \delta_j^2/\nu_j\bigr)^{-1}$. The generalized reduced module of $(\Omega, Z, \Delta, \Psi)$ is equal to

$$M(\Omega, Z, \Delta, \Psi) = \lim_{r\downarrow 0}\Bigl\{\bigl(\operatorname{cap} C(r; \Omega, Z, \Delta, \Psi)\bigr)^{-1} + \frac{\nu}{2\pi}\log r\Bigr\},$$

provided that the limit exists. It is evident from the definition that $M(\Omega, Z, \Delta, \Psi)$ is monotone with respect to Ω.

Before stating the basic result concerning the generalized reduced module we need a few definitions. Given a domain $\Omega \subset \mathbf{C}$, its Green's function $g_\Omega \colon \Omega \times \Omega \to (0, +\infty]$ is determined by the conditions that $g_\Omega(\,\cdot\,, \zeta)$ is harmonic in $\Omega \setminus \{\zeta\}$, vanishes at the regular boundary points of $\partial\Omega$, and

$$g_\Omega(z, \zeta) = -\log|z - \zeta| + \gamma(\Omega, \zeta) + o(1) \qquad \text{as } z \to \zeta.$$

The number $\gamma(\Omega, \zeta)$ is called the Robin constant of Ω at ζ. The domain Ω is called Greenian if g_Ω exists, or, equivalently, if its complement contains a compact set of positive logarithmic capacity [15]. We will use the following theorem, which is a combination of results found in [6], [7].

Theorem 3.1 ([6], [7]). *If Ω is a Greenian domain, then*

$$M(\Omega, Z, \Delta, \Psi) = \frac{\nu^2}{2\pi}\left\{\sum_{j=1}^{n} \frac{\delta_j^2}{\nu_j^2}\big(\gamma(\Omega, z_j) - \log \mu_j\big) + \sum_{j\neq k} \frac{\delta_j\delta_k}{\nu_j\nu_k} g_\Omega(z_j, z_k)\right\}. \tag{3.1}$$

If Ω is not Greenian and $\sum_{j=1}^{n} \delta_j/\nu_j = 0$, then

$$M(\Omega, Z, \Delta, \Psi) = -\frac{\nu^2}{2\pi}\left\{\sum_{j=1}^{n} \frac{\delta_j^2}{\nu_j^2}\log \mu_j + \sum_{j\neq k} \frac{\delta_j\delta_k}{\nu_j\nu_k} \log|z_j - z_k|\right\}. \tag{3.2}$$

Since $M(\Omega, Z, \Delta, \Psi)$ is defined in terms of conformal capacity, it is not surprising that it exhibits a similar quasiinvariance property. The special case $n = 1$ of the following proposition appeared in [13].

Proposition 3.1. *Let $f\colon \Omega \to \mathbf{C}$ be a K-quasiregular mapping and suppose that $z_1, \dots, z_n \in \Omega$ are points of maximal stretch. Let Δ, Ψ, and ν be as above. For $1 \le j \le n$ let $w_j = f(z_j)$ and $\tilde\psi_j(r) = \omega_f(z_j)\psi_j(r)^{1/K}$. Suppose that either* (i) *f is quasiconformal, or* (ii) *$w_1, \dots, w_n$ are distinct and $\delta_1, \dots, \delta_n$ are of the same sign. Then*

$$M\big(f(\Omega), W, \Delta, \widetilde\Psi\big) \ge K^{-1} M(\Omega, Z, \Delta, \Psi). \tag{3.3}$$

Proof. It follows from Theorem 2.1 that for small $r > 0$

$$\overline{\mathbf{D}}\big(w_j, (1 + o(1))\tilde\psi_j(r)\big) \subseteq f\big(\overline{\mathbf{D}}(z_j, \psi_j(r))\big) \subseteq \overline{\mathbf{D}}\big(w_j, \big(1 + o(1)\big)\tilde\psi_j(r)\big).$$

If f is quasiconformal, these inclusions together with [6, Lemma 1] and quasiinvariance of capacity imply

$$\big(\operatorname{cap} C\big(r; f(\Omega), W, \Delta, \widetilde\Psi\big)\big)^{-1} \ge \big(K \operatorname{cap} C(r; \Omega, Z, \Delta, \Psi)\big)^{-1} + o(1). \tag{3.4}$$

Adding $(\nu/2\pi K)\log r$ to both sides and passing to the limit $r \downarrow 0$, we obtain (3.3).

Now suppose that the assumption (ii) holds. Without loss of generality we may assume that $\delta_1, \dots, \delta_n$ are positive. If $\delta_j = 1$ for every j, then the inequality (3.4) follows from [14, Theorem 7.1] and [6, Lemma 1]; the rest of the proof is as above. In the general case let $\delta_j^* = 1$, $\psi_j^*(r) = \psi_j(r^{1/\delta_j})$, and $\nu^* = \big(\sum_{j=1}^{n} \delta_j/\nu_j\big)^{-1}$ (recall that $\psi_j(r) = \mu_j r^{\nu_j}$). Then

$$\begin{aligned} M(\Omega, Z, \Delta, \Psi) &= \frac{\nu^2}{(\nu^*)^2} M(\Omega, Z, \Delta^*, \Psi^*) \\ &\le \frac{K\nu^2}{(\nu^*)^2} M\big(f(\Omega), W, \Delta^*, \widetilde\Psi^*\big) \\ &= K M(f(\Omega), W, \Delta, \widetilde\Psi). \end{aligned}$$

Here the first step is based on Theorem 3.1; the inequality for reduced modules is true because $\delta_j^* = 1$ for every j; and in the last step we have used the fact that transformations $\psi \mapsto \psi^*$ and $\psi \mapsto \tilde\psi$ commute. □

With an appropriate choice of Z, Δ and Ψ, Theorem 3.1 and Proposition 3.1 yield sharp distortion estimates for quasiregular mappings. Our first result is that the points of maximal stretch are mapped as far from each other as the Hölder continuity allows.

Theorem 3.2. *Let $f\colon \Omega \to \mathbf{C}$ be a K-quasiconformal mapping. For any domain $\Omega_1 \Subset \Omega$ there exists a constant $C = C(\Omega, \Omega_1)$ such that*

$$|f(z_1) - f(z_2)| \geq C\sqrt{\omega_f(z_1)\omega_f(z_2)}\,|z_1 - z_2|^{1/K} \tag{3.5}$$

for any $z_1, z_2 \in \Omega_1$. When $\Omega = \mathbf{C}$ one can take $C = 1$, and this value is best possible for every $K \geq 1$.

Proof. Assume that $\omega_f(z_1), \omega_f(z_2) > 0$ and $z_1 \neq z_2$; otherwise the statement is trivial. Set parameters $Z = (z_1, z_2)$, $\Delta = (1, -1)$, $\psi_1(r) = r = \psi_2(r)$, $W = \big(f(z_1), f(z_2)\big)$, $\tilde\psi_1(r) = \omega_f(z_1)r^{1/K}$, and $\tilde\psi_2(r) = \omega_f(z_2)r^{1/K}$. By Proposition 3.1 and monotonicity of the generalized reduced modulus we have

$$M(\mathbf{C}, W, \Delta, \widetilde\Psi) \geq M\big(f(\Omega), W, \Delta, \widetilde\Psi\big) \geq K^{-1}M(\Omega, Z, \Delta, \Psi). \tag{3.6}$$

If $\mathbf{C}\setminus\Omega$ is not Greenian, one can plug (3.2) into the inequality (3.6) to obtain (3.5) with $C = 1$.

Now suppose that G is Greenian. Consider the regular part of Green's function $h_\Omega\colon \Omega\times\Omega \to \mathbf{R}$, defined by $h_\Omega(\zeta_1, \zeta_2) = g_\Omega(\zeta_1, \zeta_2) + \log|\zeta_1 - \zeta_2|$ if $\zeta_1 \neq \zeta_2$, and $h_\Omega(\zeta_1, \zeta_1) = \gamma(\Omega, \zeta_1)$ otherwise. Since h_Ω is continuous, the function $|h_\Omega|$ attains a finite maximum H on the compact set $\overline{\Omega_1}\times\overline{\Omega_1}$. By (3.1) we have for any $z_1, z_2 \in \Omega_1$

$$\begin{aligned} M(\Omega, Z, \Delta, \Psi_1) &= \frac{1}{8\pi}\big\{\gamma(\Omega, z_1) + \gamma(\Omega, z_2) - 2g_\Omega(z_1, z_2)\big\} \\ &\geq \frac{1}{8\pi}\big\{2\log|z_1 - z_2| - 4H\big\}. \end{aligned}$$

This inequality together with (3.6) and (3.2) yields (3.5) with $C = e^{-2H}$.

It remains to show that the constant C in (3.5) cannot be greater than 1. Fix $K \geq 1$ and introduce functions $h_1(z) = z/(1-z)$, $h_2(z) = |z|^{1/K-1}z$, and $h_3(z) = z/(1+z)$, which are understood as bijective self-maps of the Riemann sphere $\overline{\mathbf{C}}$. The composition $h = h_3\circ h_2\circ h_1$ is a K-quasiconformal automorphism of $\mathbf{C}$. Straightforward computations show that $h(0) = 0$, $h(1) = 1$, and $\omega_h(0) = 1 = \omega_h(1)$. Thus the value $C = 1$ is best possible in the case $\Omega = \mathbf{C}$. □

Theorem 3.2 is not valid for K-quasiregular mappings, even when $K = 1$. Indeed, for a holomorphic function f every point z with $f'(z) \neq 0$ is a point of maximal stretch. Obviously, two non-critical points of f may be mapped by f into the same point.

Next we consider a quasiregular mapping $f\colon \mathbf{D} \to \mathbf{D}$ that has at least two points of maximal stretch. By composing f with appropriate Möbius transformations, we may assume that 0 is both a fixed point and a point of maximal stretch for f. Note that under these assumptions $\omega_f(0) \le 4^{1-1/K}$, as follows from the Hersch–Pfluger distortion theorem and calculations in [11, p. 65].

Theorem 3.3. *Let $f\colon \mathbf{D} \to \mathbf{D}$ be a K-quasiregular mapping such that $f(0) = 0$. Then for any $z \in \mathbf{D}$*

$$|f(z)| \le \bigl(1 + \omega_f(0)\omega_f(z)|z|^{-2/K}(1-|z|^2)^{1/K}\bigr)^{-1/2}. \tag{3.7}$$

For any $K \ge 1$ and any $z \in \mathbf{D}$ there exists a K-quasiconformal mapping f for which equality is attained in (3.7).

Proof. Assume that $f(z) \ne 0$, $\omega_f(0) > 0$, and $\omega_f(z) > 0$; otherwise there is nothing to prove. Let $Z = (0, z)$, $\Delta = (1, 1)$, $\psi_1(r) = r = \psi_2(r)$, $W = \bigl(0, f(z)\bigr)$, $\tilde\psi_1(r) = \omega_f(0) r^{1/K}$, and $\tilde\psi_2(r) = \omega_f(z) r^{1/K}$. By Proposition 3.1 and the monotonicity of M we have $M(\mathbf{D}, W, \Delta, \widetilde\Psi) \ge K^{-1} M(\mathbf{D}, Z, \Delta, \Psi)$. Using (3.1) and the identities $\gamma(\mathbf{D}, z) = \log(1-|z|^2)$ and $g_{\mathbf{D}}(z, 0) = -\log|z|$, obtain

$$\log\frac{1-|f(z)|^2}{\omega_f(0)\omega_f(z)} + 2\log\frac{1}{|f(z)|} \ge \frac{1}{K}\left\{\log(1-|z|^2) + 2\log\frac{1}{|z|}\right\}.$$

This inequality readily implies (3.7).

To prove that (3.7) is sharp, fix $z_0 \in (0,1)$. There is $t \in (0,1)$ such that $z_0 = 2t/(1+t^2)$. The holomorphic function $h_1(z) = (z^2 - t^2)/(1 - t^2 z^2)$ maps the half-disk $\mathbf{D} \cap \{\operatorname{Re} z > 0\}$ conformally onto $\mathbf{D} \setminus (-1, -t^2)$. Composing h_1 with the radial stretch map $f_K(z) = |z|^{1/K-1} z$ and an appropriate branch of $h_2(z) = \sqrt{(z + t^{2/K})/(1 + t^{2/K} z)}$, we find that $h_2 \circ f_K \circ h_1$ is a K-quasiconformal automorphism of $\mathbf{D} \cap \{\operatorname{Re} z > 0\}$ which maps $\mathbf{D} \cap \{\operatorname{Re} z = 0\}$ onto itself. It can be extended by reflection to a K-quasiconformal map $F_1\colon \mathbf{D} \to \mathbf{D}$. Note that $F_1(\pm t) = \pm t^{1/K}$. Finally, let $F = h_3 \circ F_1 \circ h_0$, where $h_0(z) = (z-t)/(1-tz)$ and $h_3(z) = (z + t^{1/K})/(1 + t^{1/K} z)$.

We have $F(0) = 0$ and $F(z_0) = 2t^{1/K}/(1 + t^{2/K})$. Furthermore,

$$\omega_F(0) = |h_0'(0) h_1'(-t)|^{1/K} |h_2'(0) h_3'(-t^{1/K})| = \left(\frac{2}{1+t^2}\right)^{1/K} \frac{1 + t^{2/K}}{2};$$

$$\omega_F(z_0) = |h_0'(z_0) h_1'(t)|^{1/K} |h_2'(0) h_3'(t^{1/K})| = \left(\frac{2(1+t^2)}{(1-t^2)^2}\right)^{1/K} \frac{(1-t^{2/K})^2}{2(1+t^{2/K})}.$$

Hence

$$\begin{aligned} 1 + \omega_F(0)\omega_F(z_0)\left(\frac{1-|z_0|^2}{|z_0|^2}\right)^{1/K} &= 1 + \left(\frac{4}{(1-t^2)^2}\right)^{1/K} \frac{(1-t^{2/K})^2}{4} \left(\frac{(1-t^2)^2}{4t^2}\right)^{1/K} \\ &= 1 + \frac{(1-t^{2/K})^2}{4t^{2/K}} = \frac{(1+t^{2/K})^2}{4t^{2/K}} \\ &= |F(z_0)|^{-2} \end{aligned}$$

as required. □

4. Concluding remarks

One can generalize some of the above results by taking into account the local degree of a mapping. For example, let f be a K-quasiregular mapping of degree m at a point z. If $\limsup_{\delta\to 0} \delta^{-m/K}\omega_f(z,\delta) > 0$, then f is spherically analytic at z, as follows from Theorem 2.1 and the Stoilow factorization. The following problems indicate other possible directions for further research.

Problem 4.1. *Extend Theorem* 2.1 *to dimension* $n \geq 3$.

It is clear that in higher-dimensional setting the definition of $\omega_f(z)$ and the statement of Theorem 2.1 will have to be modified. In particular, the constant K should be replaced with $K_I^{1/(n-1)}$, where K_I is the inner distortion of f. See [10] for comparison of different distortion functions.

It is known [12] that the linear distortion $H_f(z)$ of a K-quasiconformal mapping in the plane is bounded by K whenever $Df(z)$ exists and $|Df(z)| > 0$. By Theorem 2.1 we have $H_f(z) = 1$ at the points where $\limsup_{\delta\to 0} \delta^{-1/K}\omega_f(z,\delta) > 0$. It may be possible to interpolate between the two estimates as follows.

Problem 4.2. *Prove that for any* $\alpha \in [1/K, 1]$ *and any planar* K*-quasiregular mapping* f *the condition* $\limsup_{\delta\to 0} \delta^{-\alpha}\omega_f(z,\delta) > 0$ *implies* $H_f(z) \leq \alpha K$.

As an example, consider $h_1(z) = |z|^{\alpha-1}z$ and $h_2(x+iy) = \alpha K x + iy$. The mapping $f = h_2 \circ h_1$ satisfies the above conditions at the origin, where its linear distortion is $H_f(0) = \alpha K$. There could be a higher-dimensional analogue of Problem 4.2, but the situation is complicated by the fact that despite recent progress (e.g. [9]), the local topological behavior of spatial quasiregular mappings is not yet fully understood.

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