Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 43–58

TRIANGULATIONS OF CLOSED SETS AND BASES IN FUNCTION SPACES

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Abstract. Triangulations of closed sets F in $\mathbf{R}^n$ with certain properties, here called regular sequences of triangulations, appear in a natural way when studing bases in function spaces on F. In this paper a characterization of sets permitting a regular sequence of triangulations is given in the one-dimensional case. Interpolating bases in spaces of functions on such sets are also discussed.

1. Introduction

In the paper [2] bases in different spaces of functions defined on compact subsets F of $\mathbf{R}^n$ were constructed. It turned out that such bases could be constructed if the set F permits what was called a regular sequence of triangulations. Examples of fractal sets permitting such triangulations were given, but in general it is not easy to see for which sets such triangulations exist. However, it was shown that if a regular sequence of triangulations exists, then the set preserves Markov's inequality.

In this paper we consider the one-dimensional case only, which of course simplifies things. We show in Section 3 in our main theorem that the existence of a regular sequence of triangulations is then in fact equivalent to the condition that the set preserves Markov's inequality. We also exhibit, in Section 5, for such sets an interpolating basis in the the Lipschitz spaces $\Lambda_\alpha(F)$, $\alpha > 0$, α noninteger, thereby extending a result from [2] given for $0 < \alpha < 1$, benefitting from working in one dimension only.

2. Regular triangulations and sets preserving Markov's inequality

Let F be a compact subset of R^n. A finite set $\mathscr{T}$ of n-dimensional closed, non-degenerated, simplices is called *a triangulation of* F if the following conditions hold.

A1. For each pair $\Delta_1, \Delta_2 \in \mathscr{T}$, the intersection $\Delta_1 \cap \Delta_2$ is empty or a common face of lower dimension.

A2. Every vertex of a simplex $\Delta \in \mathscr{T}$ is in F.

A3. $F \subset \bigcup_{\Delta \in \mathscr{T}} \Delta$.

2000 Mathematics Subject Classification: Primary 28A80, 42C15, 46E35.

For a triangulation $\mathscr{T}$, let $\delta = \max_{\Delta \in \mathscr{T}} \operatorname{diam}(\Delta)$ be the diameter of the triangulation. When considering a sequence $\{\mathscr{T}_i\}_{i=0}^{\infty} = \{\mathscr{T}_i\}$ of triangulations, we denote by δ_i the diameter of the triangulation $\mathscr{T}_i$. In the sequel, we deal with sequences $\{\mathscr{T}_i\}$ of triangulations satisfying the following conditions:

B1. For each $i \geq 0$, $\mathscr{T}_{i+1}$ is a refinement of $\mathscr{T}_i$, i.e., for each $\Delta \in \mathscr{T}_{i+1}$ there is $\tilde{\Delta} \in \mathscr{T}_i$ such that $\Delta \subset \tilde{\Delta}$.

B2. $\delta_i \to 0$, $i \to \infty$.

It is easy to see that if a triangulation satisfies B1 and B2, then the following condition holds. We denote by $\mathscr{U}_i$ the set of vertices of $\mathscr{T}_i$.

B3. For $i \geq 0$, $\mathscr{U}_i \subset \mathscr{U}_{i+1}$.

We now define the class of triangulations which we will use. It is taken from [2] (see also the references given there), but similar classes have been considered elsewhere, for example in connection with the finite element method. Since the definition of this class simplifies in one dimension, we will only give it for this case, but preserve the notation from [2]. In the n-dimensional case, there is also a condition T3 guaranteeing that simplices are not too flat, and the condition T4 is more involved.

Definition 1. Let $F \subset \mathbf{R}$, and let $\{\mathscr{T}_i\}$ be a sequence of triangulations satisfying B1. Then $\{\mathscr{T}_i\}$ is a *regular sequence of triangulations* if the following conditions hold.

T1. There is a constant $c_2 > 0$, independent of i, such that, for all $\Delta_1, \Delta_2 \in \mathscr{T}_i$,

$$c_2^{-1} \operatorname{diam}(\Delta_2) \leq \operatorname{diam}(\Delta_1) \leq c_2 \operatorname{diam}(\Delta_2).$$

T2. There are constants $0 < c_3 < c_4 < 1$ such that, for all $i \geq 0$,

$$c_3 \delta_i \leq \delta_{i+1} \leq c_4 \delta_i.$$

T4. There exist a constant $a > 0$, independent of i, such that if $\Delta \in \mathscr{T}_i$ and $\Delta' \in \mathscr{T}_i$ and the distance between these intervals is less than or equal to $a\delta_i$, then the intervals intersect.

The following concept was introduced in connection with the study of function spaces on fractals, see e.g. [3], where the concept is discussed in detail. It can be seen as a generalization of the classical Markov inequality, which states that if P is a polynomial in one variable of degree at most m, then $\max_{x \in [0,1]} |dP/dx| \leq 2n^2 \max_{x \in [0,1]} |P(x)|$.

Definition 2. Denote by P_m the set of all polynomials in n variables of total degree less than or equal to m. A closed set $F \subset R^n$ *preserves Markov's inequality* if for every fixed positive integer m there exists a constant c, such that for all polynomials $P \in P_m$ and all closed balls $B = B(x_0, r)$, $x_0 \in F$, $0 < r \le 1$, holds

$$\max_{F\cap B} |\nabla P| \le \frac{c}{r} \max_{F\cap B} |P|,$$

where ∇ denotes the gradient.

Examples of sets preserving Markov's inequality are e.g. selfsimilar fractals not contained in an $n-1$-dimensional subspace of $\mathbf{R}^n$, and d-sets, as defined in [3], with $d > n-1$. There are several different geometric characterizations of such sets. We will use the one given in [3, Section 2.2], stating it in one dimension, only. If $F \subset \mathbf{R}$, then a set preserves Markov's inequality if and only if the following condition holds.

(G) There is an $\varepsilon > 0$ such that for any r with $0 < r \le 1$ and any $x_0 \in F$, the set $F \cap \{x : \varepsilon r \le |x - x_0| \le r\}$ is nonempty.

The concept of sets preserving Markov's inequality has appeared also in different contexts, then in general in a geometrical form, see for example [4], where they are called n-thick sets and their invariance properties under different classes of maps are studied.

3. The main theorem

In [2] it is shown that if $F \subset \mathbf{R}^n$ admits a regular sequence of triangulations, then F preserves Markov's inequality. The converse does not hold, as is seen by the trivial example of the unit ball in $\mathbf{R}^n$, $n > 1$. Here we shall prove that in one dimension the converse holds, and thus we have the following theorem.

Theorem 1. *Let F be a compact subset of $\mathbf{R}$. Then there exists a regular triangulation of F if and only if F preserves Markov's inequality.*

As a preparation for the proof of the theorem we give one more charactarization of sets preserving Markov's inequality, valid in one dimension. For a given compact set $F \subset \mathbf{R}$, the complement of F is a numerable union of disjoint open intervals which we denote by O_ν, $\nu = 1, 2, 3, \dots$. Let $0 < c < 1$, and let, for a given constant c, and for $n \ge 1$,

$$\mathscr{A}_n = \{O_\nu; \operatorname{diam} O_\nu \ge c^{n-1}\} \quad \text{and} \quad A_n = \cup\{O_\nu \in \mathscr{A}_n\}.$$

Then $A_n^c = \mathbf{R}^n \setminus A_n$ consists of a finite union of finite closed intervals which we denote, ordered from left to right, by B_{nm}, $m = 1, 2, 3, \dots, m_n$. Note that we then have that

(i) the distance between two consecutive intervals B_{nm} is at least c^{n-1}.

We shall use the fact that if F preserves Markov's inequality, then the length of each B_{nm} is at least c^n, where c can be taken as any constant ε admissible in the geometric characterization of sets preserving Markov's inequality. This property can, in fact, be used to characterize such sets.

Proposition 1. *Let $F \subset \mathbf{R}$ be a compact set. Then F preserves Markov's inequality if and only if there is a constant c with $0 < c < 1$, such that constructing B_{nm} with this constant as the given constant c, one has that*
(ii) *the length of each interval B_{nm}, $n \geq 1$, $m = 1, 2, \ldots, m_n$, is at least c^n.*

Proof. Assume that F preserves Markov's inequality, and let c be equal to an admissible constant ε in the geometric characterization (G) of such sets. Let $[a, b]$ be one of intervals B_{nm} and denote by l its length. Assume $l < c^n$, and take l' with $l < l' < c^n$. Then the set $\{x; l' \leq |x - a| \leq l'/c\}$ contains no points from F because of the statement (i) above, since $l'/c < c^{n-1}$. On the other hand, since F preserves Markov's inequality, the set contains points from F (use the geometric characterization (G) with $r = l'/c$). Thus, $l \geq c^n$.

Conversely, let c be a constant such that (ii) holds. Take $x_0 \in F$ and $r \leq 1$, and assume that $\left\{x; \frac{1}{2}r < |x - x_0| < r\right\}$ contains no points from F; otherwise, the condition in (G) holds for these x_0 and r with $\varepsilon = \frac{1}{2}$. Take n_0 so that $c^{n_0-1} < \frac{1}{2}r \leq c^{n_0-2}$. Then each one of the two intervals forming the set $\left\{x; \frac{1}{2}r < |x - x_0| < r\right\}$ is a subset of an interval in A_{n_0}, which means that, with m such that $x_0 \in B_{n_0 m}$, the endpoints of $B_{n_0 m}$ are in the interval $\left[x_0 - \frac{1}{2}r, x_0 + \frac{1}{2}r\right]$. By (ii), since the endpoints are in F, this means that there is an $x \in F$ with $|x - x_0| \leq \frac{1}{2}r$ such that $|x - x_0| \geq \frac{1}{2}c^{n_0} \geq \frac{1}{4}c^2 r$, so (G) is fulfilled with $\varepsilon = \frac{1}{4}c^2$.

Proof of Theorem 1. Assume that F preserves Markov's inequality, and let $c < \frac{1}{2}$ be a constant as in Proposition 1. Taking $c < \frac{1}{2}$ is allowed, since c can be taken as the constant ε in (G), which, clearly, can be taken arbitrarily small. We prove the theorem inductively in steps, choosing in the n^{th} step a set $\mathscr{P}_n$ of points from F, such that the distance between two consecutive points in the same B_{nm} is comparable to c^n, and in such a way that $\mathscr{P}_1 \subset \mathscr{P}_2 \subset \cdots$. A triangulation $\mathscr{T}_n$ is then chosen with the points in $\mathscr{P}_n$ as vertices.

Assumptions on the set $\mathscr{P}_{n-1}$. Assume that we have already chosen the set $\mathscr{P}_{n-1}$ of points from F, with $\mathscr{P}_0$ meaning the empty set $\emptyset$.

Let $B_{nm} = [a_{nm}, b_{nm}]$ be as before. Every point $x \in F$ belongs to exactly one interval B_{sm} for each $s \geq 1$; we denote the corresponding m by $m(s, x)$ and so the interval is $B_{s,m(s,x)}$. If x is an endpoint of an interval B_{sm}, then it is, for $t > s$, an endpoint of some B_{tm}, and we denote by s_x the smallest s, if any, such that x is an endpoint of some B_{sm}. If x is not an endpoint of any interval B_{sm} with $s \geq 1$, then we put $s_x = \infty$.

For a given $n \geq 1$, we now assume the following about the set $\mathscr{P}_{n-1}$, denoting it *the assumption* $A(n-1)$. (If $n = 1$ the interpretation is that nothing is assumed.) It describes how points $z \in \mathscr{P}_{n-1}$ are positioned with respect to the endpoints of the sets B_{sm} with $s \geq n$ and to each other, in terms of points z' which will appear naturally in the construction. If $z \in \mathscr{P}_{n-1}$ and $s_z > n$ (so

z is not an endpoint of some B_{nm}), then we assume that to z there is associated a point $z' \in F$, such that the following conditions hold.

1) $z' - a_{s,m(s,z)} > \frac{1}{2}c^s$ and $b_{s,m(s,z)} - z' > \frac{1}{2}c^s$ for $n \le s < s_z$.

2) $|z - z'| \le \frac{1}{2}c^{s_z}$.

3) If z_* is a point in $\mathscr{P}_{n-1}$ closest to z satisfying $z < z_* < b_{n,m(n,z)}$, which in particular means that $s_{z_*} > n$, then $z'_* - z' > \frac{1}{2}c^n$.

4) All endpoints of the intervals $B_{n-1,m}$, $1 \le m \le m_{n-1}$, are in $\mathscr{P}_{n-1}$.

Note that 1) implies that for $z \in \operatorname{int} B_{nm}$ we also have $z' \in \operatorname{int} B_{nm}$.

Choice of points in $\mathscr{P}_n$. Let, for some B_{nm}, $\alpha \in B_{nm} \cap F$ and $\beta \in B_{nm} \cap F$ be points satisfying $\beta - \alpha > \frac{1}{2}c^n$. We claim that the set $\left(\alpha + \frac{1}{2}c^n, \alpha + c^{n-1} + \frac{1}{2}c^n\right] \cap (\alpha, \beta] \cap F$ is nonempty. Since $\beta \in F$ and $\beta > \alpha + \frac{1}{2}c^n$, this is of course the case if $\beta \le \alpha + c^{n-1} + \frac{1}{2}c^n$. Otherwise the condition is the same as saying that $\left(\alpha + \frac{1}{2}c^n, \alpha + c^{n-1} + \frac{1}{2}c^n\right] \cap F$ is nonempty. This follows from the fact that any interval I of length $> c^{n-1}$ and satisfying $I \subset B_{nm}$ must intersect F, as otherwise $I \subset F^c$ and $|I| > c^{n-1}$ implies $I \subset O_\nu$ for some $O_\nu \in \mathscr{A}_n$ and thus $I \subset A_n$ and $I \cap B_{nm} = \emptyset$.

Now we define a point $w'' \in B_{nm} \cap F$ (depending on α, β, and n) by

$$w'' = \max\left\{x; x \in \left(\alpha + \tfrac{1}{2}c^n, \alpha + c^{n-1} + \tfrac{1}{2}c^n\right] \cap (\alpha, \beta] \cap F\right\}. \tag{1}$$

If for some $\nu \ge n$ the distance from w'' to the boundary of $B_{\nu,m(\nu,w'')}$ is less than or equal to $\frac{1}{2}c^\nu$, let $\nu(w'')$ denote the first ν for which this occurs, otherwise put $\nu(w'') = \infty$. In other words, $\nu(w'')$ is the smallest $\nu \ge n$, if any, such that the interval $\left[w'' - \frac{1}{2}c^\nu, w'' + \frac{1}{2}c^\nu\right]$ is not "strictly inside" but contains an endpoint of $B_{\nu,m(\nu,w'')}$.

We next select a point w associated to w'', and consequently it will depend on α, β, and n, too. When doing this we assume more about β. The point β is either the point y' associated to a point $y \in \mathscr{P}_{n-1} \cap B_{nm}$ with $s_y > n$ by the assumption $A(n-1)$, or the right endpoint b_{nm} of B_{nm}. We make the following choices.

(a) If $\beta = b_{nm}$ and the interval $\left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$ contains b_{nm}, then we put $w = b_{nm}$.

(b) If $\beta = y'$, and the interval $\left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$ contains y', then we put $w = y$.

(c) If the interval $\left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$ does not contain b_{nm} or y', respectively, i.e. β, which implies, as we shall see below, that $\nu(w'') > n$, then, if $\nu(w'') = +\infty$ we let $w = w''$, and if $n < \nu(w'') < \infty$ we let w be an endpoint of $B_{\nu(w''),m(\nu(w''),w'')}$ which is closest to w''.

Before continuing the construction of $\mathscr{P}_n$, we make some observations related to this choice of w. In the cases (a) and (b) we obviously have, respectively,

$$b_{nm} - w'' \leq \tfrac{1}{2}c^n \quad \text{and} \quad y' - w'' \leq \tfrac{1}{2}c^n, \tag{2}$$

while in case (c) we have

$$|w - w''| \leq \tfrac{1}{2}c^{\nu(w'')} \leq \tfrac{1}{2}c^{n+1}. \tag{3}$$

The inequality $\nu(w'') > n$ in case (c) follows by the following remarks. The point a_{nm} does not belong to $\left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$, since, by (1), $w'' > \alpha + \frac{1}{2}c^n$, and $\alpha \geq a_{nm}$. This means that $b_{nm} \in \left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$ if and only if $\nu(w'') = n$. If $\beta = y'$ as above, then $b_{nm} \notin \left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$, since $b_{nm} - y' > \frac{1}{2}c^n$ by 1) in $A(n-1)$, and $w'' \leq y'$. Thus, in case (c), we have that $b_{nm} \notin \left[w'' - \frac{1}{2}c^n, w'' + \frac{1}{2}c^n\right]$, so $\nu(w'') > n$. We also remark that in case (c), if $\beta = y'$, then $w < y$, since, from 2) in $A(n-1)$ and (3), $y - w = y - y' + y' - w'' + w'' - w \geq -\frac{1}{2}c^{n+1} + \frac{1}{2}c^n - \frac{1}{2}c^{n+1} = c^n\left(\frac{1}{2} - c\right) > 0$, since $c < \frac{1}{2}$.

Now we use the above construction to divide an interval $B_{nm} = [a_{nm}, b_{nm}]$ into subintervals whose endpoints will be the points in $B_{nm} \cap \mathscr{P}_n$. This shall be done in such a way that the points in $B_{nm} \cap \mathscr{P}_{n-1}$ are included. We will temporarily denote the points in the new division of B_{nm} by $x_0, x_1, \ldots$.

The interval B_{nm} is in a natural way divided by $\mathscr{P}_{n-1} \cap B_{nm}$ into intervals which are of the form $[a_{nm}, y]$, $[y, y_*]$, $[y, b_{nm}]$, or $[a_{nm}, b_{nm}]$, where y and y_* are consecutive points in $\mathscr{P}_{n-1}$ which are not endpoints of B_{nm}. The last case appears if there are no points from $\mathscr{P}_{n-1}$ strictly inside B_{nm}, which is always the case if $n = 1$. We start by subdividing an interval of the form $[a_{nm}, b_{nm}]$ or $[a_{nm}, y]$, and take a_{nm} as the first division point, denoting it by x_0. Let $\alpha = a_{nm}$ and $\beta = b_{nm}$ or $\beta = y'$, respectively, where y' is the point associated to y in the assumption $A(n-1)$. If $\beta = b_{nm}$, then by (ii) in Proposition 1, we have $\beta > a_{nm} + \frac{1}{2}c^n$, and if $\beta = y'$, then $\beta > a_{nm} + \frac{1}{2}c^n$ holds by the assumption 1) used with $s = n$ and $z' = y'$. Thus we can use the construction above, and associate to α and β points w'' and w, which we denote by x_1'' and x_1, respectively. If x_1 equals b_{nm} or y, i.e., in case (a) or (b), then the subdivision of the interval $[a_{nm}, b_{nm}]$ or $[a_{nm}, y]$, respectively, is completed (without giving new points, except possibly a_{nm} or b_{nm}), otherwise we put $\alpha = x_1''$ but let β be as before. By our construction we have $\beta > x_1'' + \frac{1}{2}c^n$, for if the interval $\left[x_1'' - \frac{1}{2}c^n, x_1'' + \frac{1}{2}c^n\right]$ had contained β, then x_1 would have been equal to b_{nm} or y. Thus, we can again use the construction to give us new points w'' and w, which we denote by x_2'' and x_2. We continue in this way until we reach b_{nm} or y, say in the k^{th} step, so $x_k = b_{nm}$ or $x_k = y$.

Note that in the j^{th} step, if $x_{j-1} \neq b_{nm}$ or y, we always have $x_j'' > \alpha + \frac{1}{2}c^n = x_{j-1}'' + \frac{1}{2}c^n$ by (1) and the choice of $x_j'' := w''(x_{j-1}'', \beta, n)$. Thus after a finite number of steps the construction ends up with a selection of b_{nm} or y, respectively. We also remark that we have $x_{j-1} < x_j$, $j = 1, 2, \ldots, k$. We omit the verification of this for the moment; the calculations leading to it are given when estimating $z_2 - z_1$ at the end of this section.

If $x_k = b_{nm}$ the division of B_{nm} is completed, otherwise we continue to divide the next subinterval in B_{nm}, which is of the form $[x_k, b_{nm}]$ or $[x_k, y_*]$. In these cases we put $\alpha = x'_k$ and let β be b_{nm} or the point y'_*, respectively. Here the points x'_k and y'_* are the points associated to $x_k \in \mathscr{P}_{n-1}$ and $y_* \in \mathscr{P}_{n-1}$, in the assumption $A(n-1)$. Then, by the assumptions 1) and 3), respectively, we again have $\beta > \alpha + \frac{1}{2}c^n$, and we can use our construction again to give us new points w'' and w which we denote by x''_{k+1} and x_{k+1}. Unless we already reached b_{nm} or y_*, we take $\alpha = x''_{k+1}$ and let β be as before, and continue as above until the present interval is divided and then until all subintervals of B_{nm} have been divided, the last chosen point in B_{nm} being b_{nm}.

Dividing all intervals B_{nm} in this way we get the set $\mathscr{P}_n$, letting it consist of all the chosen points x_i. To complete the construction, we state how points z' are associated to points $z \in \mathscr{P}_n$. We do this for points z which are not endpoints of some B_{nm}. If such a z belongs to $\mathscr{P}_{n-1}$, there is already a point z' associated to it by our assumption, and we take it as the point associated to z also in $\mathscr{P}_n$. Otherwise, if $z \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, we take z' as the point z'' (in the notation above w'') used in the construction of z.

We remark that we have, with z_1 and z_2 being consecutive points with $z_1 < z_2$ from $\mathscr{P}_n$ in B_{nm}, with $z_2 \in \mathscr{P}_{n-1}$ or $z_2 = b_{nm}$, replacing z'_1 by a_{nm} if $z_1 = a_{nm}$, that

$$(4) \qquad z'_2 - z'_1 > \tfrac{1}{2}c^n, \quad z_2 \neq b_{nm} \qquad \text{and} \qquad z_2 - z'_1 > \tfrac{1}{2}c^n, \quad z_2 = b_{nm}.$$

These estimates can be found above in the description of the subdivision of B_{nm}, taking into account that $z'_1 = z''_1$ if $z_1 \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, $z_1 \neq a_{nm}$.

The set $\mathscr{P}_n$ satisfies $A(n)$. To complete the induction step we must check that this subdivision fulfills the condition $A(n)$. Let us make some observations first. Assume that $w \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, $w \neq a_{nm}, b_{nm}$. Then w is chosen in step n, and emanates, by means of case (c) in the construction, from a point w'' with $\nu(w'') > n$, in such a way that w is an endpoint of the interval B_{sm} with $s = \nu(w'')$ which contains w'' if $n < \nu(w'') < \infty$, and $w = w''$ if $\nu(w'') = \infty$. Because of 4) in $A(n-1)$, w is not an endpoint of some B_{sm} with $s < n$, and since by our construction the intervals $\left[w'' - \frac{1}{2}c^s, w'' + \frac{1}{2}c^s\right]$ do not contain an endpoint of any B_{sm} for all s with $n \leq s < \nu(w'')$ but contain w, it follows that $s = \nu(w'')$ is the first s such that w is an endpoint of some B_{sm}, i.e. $s_w = \nu(w'')$. Note also that, if $\nu(w'') < \infty$, w and w'' are both in the same $B_{\nu(w''),m}$, so, since the intervals are nesting, they are also in the same interval B_{sm} if $s < \nu(w'')$, which means that for $s \leq \nu(w'')$ we have e.g. $a_{s,m(s,w'')} = a_{s,m(s,w)}$.

To check $A(n)$, let $z \in \mathscr{P}_n$ with $s_z > n+1$. If $z \in \mathscr{P}_{n-1}$, then the conditions 1), and 2), in $A(n)$ hold by assumption and the choice of z'. If instead $w \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, $s_w > n+1$, then as we saw above the interval $\left[w'' - \frac{1}{2}c^s, w'' + \frac{1}{2}c^s\right]$ does not contain $a_{s,m(s,w'')} = a_{s,m(s,w)}$ or $b_{s,m(s,w'')} = b_{s,m(s,w)}$ for $n \le s < \nu(w'')$, but does for $s = \nu(w'') = s_w$. Thus, recalling that $w' = w''$ if $w \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, $s_w > n$, we see that we have have 1) even for $n \le s < s_w$, and 2) (with z in 1) and 2) equal to w). Let z and z_* be as in 3) in $A(n)$. Then the point α in the construction of z_* is taken as z', and if $z_* \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, so $z'_* = z''_*$, then (1) gives $z'_* - z' > \frac{1}{2}c^n > \frac{1}{2}c^{n+1}$, and if $z_* \in \mathscr{P}_{n-1}$, then we have the same estimate by (4). Condition 4) is immediate from the construction.

A regular triangulation from the constructed points. Having defined the point sets $\mathscr{P}_1$, $\mathscr{P}_2$, $\mathscr{P}_3, \dots$, we define the triangulation $\mathscr{T}_{nm}$ as consisting of the intervals connecting consecutive points of $\mathscr{P}_n \cap B_{nm}$, and then the triangulation $\mathscr{T}_n$ as the union of $\mathscr{T}_{nm}$ over $1 \le m \le m_n$. We must check that the triangulation obtained in this way is regular; it will turn out that to assure that the condition T2 is fulfilled, one has to choose a subsequence of $\mathscr{T}_n$.

Recall that if $z \in \mathscr{P}_{n-1} \cap B_{nm}, z \ne a_{nm}$, or $z = b_{nm}$, then z'' is the point which in the construction step above (there denoted w'') leads to the incorporation of z in $\mathscr{P}_n$, and if $z \in (\mathscr{P}_n \setminus \mathscr{P}_{n-1}) \cap B_{nm}$, $s_z > n$, then z'' is the point w'' which leads to a new point z and then $z'' = z'$. Note also that if $z \in \mathscr{P}_n \cap B_{nm}$, $s_z > n$, then

$$|z - z'| \le c^{s_z} \le \tfrac{1}{2}c^{n+1} \tag{5}$$

holds by 2) in $A(n-1)$ if $z \in \mathscr{P}_{n-1}$ and by (3) otherwise.

Suppose z_1 and z_2 are consecutive points with $z_1 < z_2$ from $\mathscr{P}_n$ in the same $B_{nm} = [a_{nm}, b_{nm}]$. Then z_2 was constructed with the aid of the point z''_2 chosen as the point w'' obtained from (1), with $\alpha = z'_1$ in (1) if $z_1 \ne a_{nm}$ and $\alpha = a_{nm}$ if $z_1 = a_{nm}$. Thus we have, respectively,

$$\tfrac{1}{2}c^n < z''_2 - z'_1 \le \tfrac{1}{2}c^n + c^{n-1} \quad \text{and} \quad \tfrac{1}{2}c^n < z''_2 - a_{nm} \le \tfrac{1}{2}c^n + c^{n-1}. \tag{6}$$

Assume now, furthermore, that z_1 and z_2 are not endpoints of B_{nm}. Then we have the following estimates.

(i) If $z_2 \in \mathscr{P}_{n-1}$, then, from (5) and (4),

$$\begin{aligned} z_2 - z_1 = z_2 - z'_2 + z'_2 - z'_1 + z'_1 - z_1 &\ge -\tfrac{1}{2}c^{n+1} + \tfrac{1}{2}c^n - \tfrac{1}{2}c^{n+1} \\ &= \tfrac{1}{2}c^n - c^{n+1}. \end{aligned}$$

(ii) If $z_2 \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, so $z'_2 = z''_2$, then, from (5) and (6), we have,

$$\begin{aligned} z_2 - z_1 &= z_2 - z'_2 + z'_2 - z'_1 + z'_1 - z_1 \\ &\ge -\tfrac{1}{2}c^{n+1} + \tfrac{1}{2}c^n - \tfrac{1}{2}c^{n+1} = \tfrac{1}{2}c^n - c^{n+1}. \end{aligned}$$

(iii) If $z_2 \in \mathscr{P}_{n-1}$, then, from (5), (2), and (6),

$$\begin{aligned} z_2 - z_1 &= z_2 - z_2' + z_2' - z_2'' + z_2'' - z_1' + z_1' - z_1 \\ &\le \tfrac{1}{2}c^{n+1} + \tfrac{1}{2}c^n + \tfrac{1}{2}c^n + c^{n-1} + \tfrac{1}{2}c^{n+1} \\ &= c^{n-1} + c^n + c^{n+1}. \end{aligned}$$

(iv) If $z_2 \in \mathscr{P}_n \setminus \mathscr{P}_{n-1}$, then, from (5) and (6),

$$\begin{aligned} z_2 - z_1 &= z_2 - z_2'' + z_2'' - z_1' + z_1' - z_1 \\ &\le \tfrac{1}{2}c^{n+1} + \tfrac{1}{2}c^n + c^{n-1} + \tfrac{1}{2}c^{n+1} \\ &= c^{n-1} + \tfrac{1}{2}c^n + c^{n+1}. \end{aligned}$$

If instead $z_2 = b_{nm}$, then we make the same estimates as in (i) and (iii), except that we now do not add and subtract z_2', and if $z_1 = a_{nm}$ we do not add and subtract z_1' in the various cases. In any case this clearly leads to slightly better estimates. Thus, in general we have

$$c^n\left(\tfrac{1}{2} - c\right) = \tfrac{1}{2}c^n - c^{n+1} \le z_2 - z_1 \le c^{n-1} + c^n + c^{n+1} = c^n\left(\frac{1}{c} + 1 + c\right),$$

from which T1, and also the left inequality in T2, clearly follows since $c < \frac{1}{2}$. Condition T4 is easily checked. Concerning the right inequality in T2, the construction gives $\delta_{i+1} \le \delta_i$, only. The inequality $\delta_{i+1} \le c_4\delta_i$ is obtaind by considering a subsequence of the triangulations $\{\mathscr{T}_n\}$, considering the sequence $\{\mathscr{T}_n'\}$ where $\mathscr{T}_n' = \mathscr{T}_{nk}$, $n = 1, 2, \ldots$, for a positive integer k big enough. This concludes the proof of the theorem.

4. The spaces $C^k(F)$ and $\Lambda_\alpha(F)$

The spaces $C^k(F)$ of k times differentiable functions on F and the Lipschitz spaces $\Lambda_\alpha(F)$ can be defined on arbitrary closed subsets of $\mathbf{R}^n$, but the definitions simplify if F is a set in one dimension preserving Markov's inequality, and we define them here on such sets only, referring to e.g. [3] for the general case.

Let F be a compact subset of $\mathbf{R}$ preserving Markov's inequality and f a function defined on F. Then, since F is perfect, derivatives can be defined in the usual way by $Df(x_0) = \lim_{x\to x_0, x\in F}\big(f(x) - f(x_0)\big)/(x - x_0)$, $x_0 \in F$. For a given $k \ge 0$, assuming that derivatives of orders $\le k$ exist, denote by R_j the Taylor remainders given by

$$D^j f(x) = \sum_{l=0}^{k-j}(x-y)^l D^{j+l}f(y)/l! + R_j(x, y), \qquad 0 \le j \le k.$$

A function f belongs to *the space* $C^k(F)$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|R_j(x,y)| < \varepsilon|x-y|^{k-j}$ for $0 \le j \le k$ and $|x-y| < \delta$. The Whitney extension theorem given in [5] gives that there is a linear extension operator E from $C^k(F)$ to $C^k(\mathbf{R})$. It should be noted that in the present setting derivatives are uniquely determined by f, which means that elements in $C^k(F)$ are functions rather than families of functions as in the general Whitney extension theorem.

For $k < \alpha < k+1$, a function f is in *the Lipschitz space* $\Lambda_\alpha(F)$ if $|D^j f(x)| \le M$ and $|R_j(x,y)| \le M|x-y|^{\alpha-j}$ for $x, y \in F$ and $0 \le j \le k$. The norm of f in $\Lambda_\alpha(F)$ is the infimum of the possible constants M. If $F = \mathbf{R}$, we will in general work with the equivalent norm given by the infimum of the constants M such that $|D^j f(x)| \le M$, $0 \le j \le k$ and $|D^k f(x) - D^k f(y)| \le M|x-y|^{\alpha-k}$, $|x-y| \le 1$. For this case the Whitney extension theorem gives that there is a bounded linear operator E from $\Lambda_\alpha(F)$ to $\Lambda_\alpha(\mathbf{R})$, see e.g. [3].

If instead $\alpha = k > 0$, we say, following [3, p. 62], that $f \in \Lambda_\alpha(F)$ if there is a sequence $\{f_n\}_{n=1}^\infty$ of functions in $\Lambda_\gamma(F)$ where $k < \gamma < k+1$, such that, for $n \ge 1$, $\|D^j(f-f_n)\|_{\infty,F} \le M2^{-n(k-j)}$ for $0 \le j < k$, $\|D^j(f_{n+1}-f_n)\|_{\infty,F} \le M$ for $j = k$, and $\|f_n\|_{\Lambda_\gamma(F)} \le M2^{-n(k-\gamma)}$ (here $\|\cdot\|_{\infty,F}$ denotes the maximum norm on F). The norm in $\Lambda_\alpha(F)$ is again the infimum of the possible constants M. The version of the Whitney extension theorem given by Theorem 2 in Chapter 3 in [3] gives the same conclusion as in the noninteger case.

We will need the following lemma on Hermite interpolation. We give it for functions defined on $\mathbf{R}$, but because of the Whitney extension theorem we will be able to apply it for functions defined on sets preserving Markov's inequality as well. For $f \in C(\mathbf{R})$, let $\omega(f,t) = \sup\{|f(x)-f(y)|; |x-y| \le t\}$.

Lemma 1. *Let $k \ge 0$, $f \in C^k(\mathbf{R})$, $a \in \mathbf{R}$, $h > 0$, and let P be the polynomial of degree at most $2k+1$ with $D^\nu P(a) = D^\nu f(a)$ and $D^\nu P(a+h) = D^\nu f(a+h)$ for $\nu = 0, 1, \dots, k$. Then*

$$|D^\nu P(x) - D^\nu f(x)| \le ch^{k-\nu}\omega(D^k f, h), \qquad x \in [a, a+h],\ \nu = 0, 1, \dots, k,$$

where c is a constant depending on k, only.

Proof. Assume first that $a = 0$ and $h = 1$, let $0 \le x \le 1$, and let T denote the Taylor polynomial of f around 0 of degree k. Then, for $\nu < k$,

$$\begin{aligned} D^\nu f(x) - D^\nu T(x) &= D^\nu f(x) - \sum_{l=0}^{k-\nu-1} x^l D^{\nu+l} f(0)/l! - x^{k-\nu} D^k f(0)/(k-\nu)! \\ &= x^{k-\nu}\big(D^k f(\xi) - D^k f(0)\big)/(k-\nu)! \end{aligned}$$

for some ξ between 0 and x, so we get $|D^\nu f(x) - D^\nu T(x)| \le 1/(k-\nu)!\omega(D^k f, 1)$. For $\nu = k$, the same estimate is immediate since $D^k T(x) = D^k f(0)$.

Consider $R = P - T$. Then R is a polynomial of degree at most $2k+1$ such that $D^\nu R(0) = 0$ and $|D^\nu R(1)| \le 1/(k-\nu)!\omega(D^k f, 1)$ for $\nu \le k$. Let P^s be the polynomial of degree $\le 2k+1$ defined in the beginning of Section 5. Then we have $R(x) = \sum_{\nu=0}^{k} D^\nu R(0)(-1)^\nu P^\nu(1-x) + D^\nu R(1) P^\nu(x)$. With $c_1 = \max_{x\in[0,1],0\le s\le 2k+1} |P^s(x)|$ this gives $|R(x)| \le c_1 \sum_{\nu=0}^{k} \big(|D^\nu R(0)| + |D^\nu R(1)|\big) \le c_1(k+1)\omega(D^k f, 1)$ for $0 \le x \le 1$. By using the classical Markov inequality (see Section 2) repeatedly one obtains $|D^\nu R(x)| \le 2^\nu (2k+1)^{2\nu} \max_{x\in[0,1]} |R(x)|$ and so

$$\begin{aligned}|D^\nu P(x) - D^\nu f(x)| &\le |D^\nu P(x) - D^\nu T(x)| + |D^\nu T(x) - D^\nu f(x)|\\ &\le c_1 2^\nu (2k+1)^{2\nu}(k+1)\omega(D^k f, 1) + 1/(k-\nu)!\omega(D^k f, 1)\\ &= c\omega(D^k f, 1).\end{aligned}$$

If instead f is defined on the interval $[a, a+h]$, then a linear change of variables gives, with $g(x) = hf(hx+a)$, that $\omega(D^\nu g, 1) = h^\nu \omega(D^\nu f, h)$, from which the lemma follows.

5. Interpolation bases in $C^k(F)$ and the characterization of Lipschitz spaces

Now we define the functions which will give a basis in $C^k(F)$. Let a sequence $\{\mathscr{T}_i\}_{i=0}^{\infty}$ of regular triangulations of F be given, and let as before $\mathscr{U}_i$ be the set of vertices of $\mathscr{T}_i$. Let $\mathscr{V}_i = \mathscr{U}_i \setminus \mathscr{U}_{i-1}$, $i > 0$, $\mathscr{V}_0 = \mathscr{U}_0$, and $\mathscr{V} = \bigcup_{i=0}^{\infty} \mathscr{V}_i$, and let $\preceq$ be a linear order on $\mathscr{V}$ satisfying the following condition: if $\xi \in \mathscr{V}_i$ and $\eta \in \mathscr{V}_j$ with $i < j$, then $\xi \preceq \eta$.

Let $k \ge 0$, and let, for $s = 0, 1, \dots, k$, P^s be the polynomial of degree $2k+1$ which satisfies $D^\nu P^s(0) = 0$ for $\nu = 0, 1, \dots, k$, $D^\nu P^s(1) = 0$ for $\nu = 0, 1, \dots, k$, $\nu \ne s$, and $D^\nu P^s(1) = 1$ if $\nu = s$. Let $\xi \in \mathscr{V}_i$. Then ξ is the endpoint of one or two intervals in $\mathscr{T}_i$. If ξ is the right endpoint of $\Delta \in \mathscr{T}_i$ of length l, define $\phi^s_{i,\xi}$ on Δ by $\phi^s_{i,\xi}(x) = l^s P^s\big((x-\xi+l)/l\big)$, and if ξ is the left endpoint of $\Delta' \in \mathscr{T}_i$ of length l', define $\phi^s_{i,\xi}$ on Δ' by $\phi^s_{i,\xi}(x) = l'^s(-1)^s P^s\big((\xi-x+l')/l'\big)$. If ξ is an endpoint of both Δ and Δ', this means that $\phi^s_{i,\xi}$ is a spline function defined on $\Delta \cup \Delta'$ which is k times differentiable, equals a polynomial of degree at most $2k+1$ on each of the intervals Δ and Δ', and whose derivatives of orders less than or equal to k are equal to zero at the endpoints of these intervals, except that $D^s\phi^s_{i,\xi}(\xi) = 1$. Note that for $x \in \Delta$ we have $D^\nu \phi^s_{i,\xi}(x) = l^{s-\nu}(D^\nu P^s)\big((x-\xi+l)/l\big)$ and consequently for $x \in \Delta$ we have, with a constant c depending on ν,

$$|D^\nu \phi^s_{i,\xi}(x)| \le c l^{s-\nu}, \qquad \nu \ge 0, \tag{7}$$

an estimate which clearly also holds if $x \in \Delta'$, with l replaced by l'.

For $\xi \in \mathscr{V}_i$, let $\psi^s_\xi = \phi^s_{i,\xi}$ on the one or two intervals in $\mathscr{T}_i$ which have ξ as an endpoint, and put $\psi^s_\xi = 0$ elsewhere on $\{\cup\Delta ; \Delta \in \mathscr{T}_i\}$. Note that ψ^s_ξ is defined with respect to the triangulation $\mathscr{T}_i$ in which ξ first appears as a vertex. We let $C^k(F)$ be normed by $\max_{0\le\nu\le k}\|D^\nu f\|_{\infty,F}$, although $C^k(F)$ is, in general, not complete in this norm, see e.g. [1]. Then the functions $\psi_\xi^s|_F$ form a basis in $C^k(F)$ which interpolates to f and its derivatives of degree at most k as explained by Proposition 2 below. Here $f|_F$ denotes the restriction of f to F.

Given $f\in C^k(F)$, let $S_i(f)$ denote the spline function defined on $\{\cup\Delta;\Delta\in\mathscr{T}_i\}$ which coincides with a polynomial of degree at most $2k+1$ on each interval in $\mathscr{T}_i$, and interpolates to f and all its derivatives of orders less than or equal to k at each point in $\mathscr{U}_i$.

Proposition 2. *Let $F\subset\mathbf{R}$ be a compact set with a regular sequence $\{\mathscr{T}_i\}_{i=0}^\infty$ of triangulations of F, and let k be a nonnegative integer. Then the system of functions $\{\psi_\xi^s|_F,\ \xi\in\mathscr{V},\ s=0,1,\dots,k\}$, ordered by $\preceq$, is an interpolating Schauder basis in $C^k(F)$.*

More precisely, every $f\in C^k(F)$ has a unique representation

$$f=\sum_{i=0}^\infty\sum_{\xi\in\mathscr{V}_i}\sum_{s=0}^k c_\xi^s\psi_\xi^s|_F \tag{8}$$

in $C^k(F)$, where $c_\xi^s=D^sf(\xi)-D^sS_{i-1}(f)(\xi)$ for $\xi\in\mathscr{V}_i$, $i>0$, and $c_\xi^s=D^sf(\xi)$ for $\xi\in\mathscr{V}_0=\mathscr{U}_0$. In addition, for $N\ge0$,

$$D^\nu f(\eta)=\sum_{i=0}^N\sum_{\xi\in\mathscr{V}_i}\sum_{s=0}^k c_\xi^s D^\nu\psi_\xi^s(\eta)$$

for $0\le\nu\le k$ and $\eta\in\mathscr{U}_N$.

Proof. Assume that $f\in C^k(F)$. Consider the sum $\sum_{i=0}^N\sum_{\xi\in\mathscr{V}_i}\sum_{s=0}^k c_\xi^s\psi_\xi^s$. Using induction over N it is easy to see that this sum coincides with the function $S_N(f)$, which, in particular, gives the interpolation property. Denote by Ef the Whitney extension of f belonging to $C^k(\mathbf{R})$ satisfying $D^\nu f(x)=D^\nu(Ef)(x)$, $x\in F$, $\nu\le k$. We take, as we may, E so that Ef has compact support. Note that $S_N(f)=S_N(Ef)$ on $\{\cup\Delta;\Delta\in\mathscr{T}_N\}$. By Lemma 1 we then have the estimate $|D^\nu Ef(x)-D^\nu S_N(Ef)(x)|\le c(\operatorname{diam}\Delta)^{k-\nu}\omega\bigl(D^k(Ef),\operatorname{diam}\Delta\bigr)$ for $x\in\Delta\in\mathscr{T}_N$, so

$$|D^\nu f(x)-D^\nu S_N(f)(x)|\le c\delta_N^{k-\nu}\omega\bigl(D^k(Ef),\delta_N\bigr),\qquad x\in F, \tag{9}$$

which shows that the partial sums of the sum in (8) corresponding to summation over all terms with $i\le N$ converge to f in $C^k(F)$ as $N\to\infty$. The estimate (9) also gives $|c_\xi^s|\le c\delta_N^{k-s}\omega\bigl(D^k(Ef),\delta_N\bigr)$ for $\xi\in\mathscr{V}_{N+1}$, and from (7) it follows, using T1, that $|D^\nu\psi_\xi^s(x)|\le c\delta_{N+1}^{s-\nu}$, $\xi\in\mathscr{V}_{N+1}$. Thus we get with the aid of T2 that $|c_\xi^sD^\nu\psi_\xi^s|\le c\delta_N^{k-\nu}\omega\bigl(D^k(Ef),\delta_N\bigr)$, $\xi\in\mathscr{V}_{N+1}$, from which it follows that we have convergence to f in $C^k(F)$ for the sequence of all partial sums. Uniqueness follows from the fact that if we have a representation of f in $C^k(F)$ as in (8), then it is easy to realize that the coefficients must be the ones given in the proposition.

Remark. If $k = 0$, then the theorem holds under the much weaker condition that the triangulations $\{T_i\}$ satisfy the conditions B1 and B2, only. This is not true for $k > 0$, even if $F = [0, 1]$. To see this, let for $i > 0$ $\mathscr{T}_i$ denote a subdivision of [0,1] into equal intervals of length $\delta_i = 2^{-2^i}$. Define f_i on $I_i = [\delta_i, 2\delta_i]$ by $f_i(x) = \delta_i^{-2}(x - \delta_i)^2(x - 2\delta_i)^2$ and let $f_i = 0$ elsewere. Put $f = \sum_{i=1}^{\infty} f_i$. Then $f \in C^1[0, 1]$. Let $\xi = \frac{3}{2}\delta_i$. Then $\xi \in \mathscr{V}_{i+1}$, and it is easy to see that the derivative of $c_\xi^0 \psi_\xi^0$ is at least $\frac{1}{16}$ at some point, from which it follows that f can not have a representation in $C^1(F)$ as in the proposition.

Theorem 2. *Let $F \subset \mathbf{R}$ be a compact set with a regular sequence of triangulations $\{\mathscr{T}_i\}_{i=0}^{\infty}$ of F, and let $k < \alpha < k + 1$, $k \geq 0$. Assume that f has the representation in $C^k(F)$*

$$f = \sum_{i=0}^{\infty} \sum_{\xi \in \mathscr{V}_i} \sum_{s=0}^{k} a_\xi^s \psi_\xi^s|_F. \tag{10}$$

Then f belongs to $\Lambda_\alpha(F)$ if and only if the coefficients a_ξ^s satisfy

$$|a_\xi^s| \leq c\delta_i^{\alpha - s}, \qquad i \geq 0,\ \xi \in \mathscr{V}_i,\ 0 \leq s \leq k, \tag{11}$$

and the norm of f in $\Lambda_\alpha(F)$ is equivalent to the infimimum of the possible constants c in (11).

In addition, convergence of (10) in $C^k(F)$ follows from (11) assuming pointwise convergence of the sum in (10), only.

Proof. If $f \in \Lambda_\alpha(F)$ with the representation (10) in $C^k(F)$, then $a_\xi^s = c_\xi^s$ by the uniqueness of the representation in $C^k(F)$, and (11) follows immediately from the definition of the coefficients c_ξ^s and the estimate (9), with E in (9) being a bounded operator from $\Lambda_\alpha(F)$ to $\Lambda_\alpha(\mathbf{R})$ as in the Whitney extension theorem. In fact, for $\xi \in \mathscr{V}_i$, $i > 0$, (9) gives $|c_\xi^s| \leq c\delta_{i-1}^{k-s}\omega\big(D^k(Ef), \delta_{i-1}\big)$ which gives (11), using the boundedness of E, the definition of $\Lambda_\alpha(\mathbf{R})$, and T2.

To prove the converse, assume that (11) holds. We first extend the functions ψ_ξ^s to functions defined on $\mathbf{R}$. If ξ is the endpoint of just one interval $\Delta \in \mathscr{T}_i$, say the right endpoint, denote by Δ_1 the interval of length $\frac{1}{4}a\delta_i$ such that ξ is the left endpoint of the interval, where a is the constant in T4. Define ψ_ξ^s on Δ_1 in the same way as it was previously defined on Δ' when ξ was the left endpoint of Δ'. After this, whether ξ is the endpoint of one or two intervals in $\mathscr{T}_i$, put $\psi_\xi^s = 0$ everywhere where ψ_ξ^s is not already defined. We consider the sum in (10) with these extended functions ψ_ξ^s in the place of $\psi_\xi^s|_F$, denoting it again by f, and show that it is in $\Lambda_\alpha(\mathbf{R})$ using the classical definition of $\Lambda_\alpha(\mathbf{R})$ as in Section 4, and that the convergence of the sum is in $C^k(\mathbf{R})$. Since the extension does not change the the sum on F we clearly then have the desired result. We remark that this can be used to extend a function $f \in \Lambda_\alpha(F)$ to a function in $\Lambda_\alpha(\mathbf{R})$.

By (7) and T1 we have, for $\xi \in \mathscr{V}_i$, $|D^\nu \psi_\xi^s| \le c\delta_i^{s-\nu}$, and thus

$$|a_\xi^s D^\nu \psi_\xi^s| \le c\delta_i^{\alpha-\nu}. \tag{12}$$

For any point x, $\psi_\xi^s(x)$ is nonzero for at most two $\xi \in \mathscr{V}_i$, so for $0 \le \nu \le k$ we have $\sum_{i=0}^\infty \sum_{\xi\in\mathscr{V}_i} \sum_{s=0}^k |a_\xi^s D^\nu \psi_\xi^s| \le c\sum_{i=0}^\infty \delta_i^{\alpha-\nu} \le c\sum_{i=0}^\infty (c_4^i\delta_0)^{\alpha-\nu} \le c$, where c_4 is the constant in T2. This shows that $f \in C^k(\mathbf{R})$ and that convergence to f is in $C^k(\mathbf{R})$.

Consider next

$$D^k f(x) - D^k f(y) = \left(\sum_{i=0}^{i_0-1} + \sum_{i=i_0}^{\infty}\right) \sum_{\xi\in\mathscr{V}_i} \sum_{s=0}^k a_\xi^s \big(D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y)\big) = \mathrm{I}+\mathrm{II},$$

and choose i_0 so that $\delta_{i_0+1} < |x-y| \le \delta_{i_0}$ (assume $|x-y| \le \delta_0$, and interpret I as vanishing if $i_0 = 0$). The sum taken over $i \ge i_0$ is estimated in an analoguous way as in the estimate above, using $|D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y)| \le |D^k\psi_\xi^s(x)| + |D^k\psi_\xi^s(y)|$, and gives the estimate

$$\begin{aligned}\mathrm{II} &\le \sum_{i=i_0}^\infty c\delta_i^{\alpha-k} \le \sum_{i=i_0}^\infty (c_4^{i-i_0}\delta_{i_0})^{\alpha-k} \\ &= c\delta_{i_0}^{\alpha-k} \le c\delta_{i_0+1}^{\alpha-k} \le c|x-y|^{\alpha-k}.\end{aligned}$$

Let next $i < i_0$ and let $x \in \Delta \in \mathscr{T}_i$ and $y \in \Delta' \in \mathscr{T}_i$; the case when x or y is in an interval of the type Δ_1 is treated in the same way. If $\Delta = \Delta'$ then, by the mean value theorem, (7), and T1, for $\xi \in \mathscr{V}_i$,

$$|D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y)| \le c\delta_i^{s-k-1}|x-y|. \tag{13}$$

If $\Delta \ne \Delta'$ and Δ and Δ' intersect at the point $\eta \in \mathscr{V}_i$, we get the same estimate by inserting $\pm D^k\psi_\xi^s(\eta)$. If $\Delta \ne \Delta'$ and Δ and Δ' do not intersect, then $|x-y| \ge a\delta_i$, so by (7) we have $|D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y)| \le c\delta_i^{s-k} = c\delta_i^{s-k}a \le c\delta_i^{s-k}|x-y|/\delta_i = c\delta_i^{s-k-1}|x-y|$, so we again have (13). In case one of the points x or y, say y, is not in some $\Delta \in \mathscr{T}_i$, or in some Δ_1, then we use, if $D^k\psi_\xi^s(x) \ne 0$, that $D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y) = D^k\psi_\xi^s(x) = D^k\psi_\xi^s(x) - D^k\psi_\xi^s(y*)$ where $y*$ is the endpoint of the interval Δ, or Δ_1, containing x not equal to ξ.

Note that $D^\nu\psi_\xi^s(x) - D^\nu\psi_\xi^s(y)$ is nonzero for at most four vertices $\xi \in \mathscr{V}_i$ and that, by T2, $\delta_i \ge c_4^{-(i_0-i)}\delta_{i_0}$ if $i < i_0$. Using this, (11), and (13), we get

$$\begin{aligned}\mathrm{I} &\le c\sum_{i=0}^{i_0-1}\sum_{s=0}^k \delta_i^{s-k-1}\delta_i^{\alpha-s}|x-y| \\ &\le c\sum_{i=0}^{i_0-1} c_4^{-(i_0-i)(\alpha-k-1)}\delta_{i_0}^{\alpha-k-1}|x-y| \\ &\le c|x-y|^{\alpha-k}.\end{aligned}$$

Thus we have $|D^k f(x) - D^k f(y)| \le c|x-y|^{\alpha-k}$, $|x-y| \le \delta_0$, and since the desired norm estimates are implicit in the proof, this concludes the proof of the theorem.

The construction above can be used to give a characterization of $\Lambda_\alpha(F)$ also if $\alpha = k$, where k is a positive integer, although the representation is not unique and somewhat less straightforward. We suggest a way to find a kind of atomic decomposition which in spirit is similar to the non-integer case.

Given a sequence $\delta_i \to 0$ satisfying T2, a definition of $\Lambda_k(F)$ equivalent to the one given in Section 4 is given by $f \in \Lambda_k(F)$ if and only if there is a sequence $\{f_i\}_{i=0}^{\infty}$ of functions in $\Lambda_\gamma(F)$ where $k < \gamma < k+1$, such that, for $i \ge 0$, $\|D^j(f - f_i)\|_{\infty,F} \le M\delta_i^{(k-j)}$ for $0 \le j < k$, $\|D^j(f_{i+1} - f_i)\|_{\infty,F} \le M$ for $j = k$, and $\|f_i\|_{\Lambda_\gamma(F)} \le M\delta_i^{(k-\gamma)}$, with the norm in $\Lambda_k(F)$ equal to the infimum of the possible constants M. We omit the straightforward verification.

Let $f \in \Lambda_k(F)$, let $\mathscr{T}_i$ be a regular triangulation of F with diameters δ_i, and let $\{f_i\}_{i=0}^{\infty}$ be a sequence associated to f and $\{\delta_i\}$ as in the above characterization of $\Lambda_k(F)$. Define, inductively for $i \ge 0$, g_i as the spline function which coincides with a polynomial of degree at most $2k+1$ on each interval in $\mathscr{T}_i$, and interpolates to $f_i - S_{i-1}(f_{i-1})$ (to f_i if $i = 0$) and all its derivatives of orders less than or equal to k at each point in $\mathscr{U}_i$. Then we will have $\sum_{i=0}^{n} g_i = S_n(f_n)$ on $\{\cup\Delta, \Delta \in \mathscr{T}_n\}$. The function g_i has a representation

$$g_i = \sum_{\xi \in \mathscr{U}_i} \sum_{s=0}^{k} c_{i,\xi}^s \phi_{i,\xi}^s,$$

where $c_{i,\xi}^s = D^s f_i(\xi) - D^s S_{i-1}(f_{i-1})(\xi)$ for $\xi \in \mathscr{U}_i$. Writing $f - S_n(f_n)$ as $f - f_n + f_n - S_n(f_n)$ and estimating with the aid of (9), one obtains that on F the function f has the representation, in $C^{k-1}(F)$,

$$f = \sum_{i=0}^{\infty} \sum_{\xi \in \mathscr{U}_i} \sum_{s=0}^{k} c_{i,\xi}^s \phi_{i,\xi}^s. \tag{14}$$

Proposition 3. *Let $F \subset \mathbf{R}$ be a compact set with a regular sequence of triangulations $\{\mathscr{T}_i\}_{i=0}^{\infty}$ of F, and let $k > 0$. Then, if $f \in \Lambda_k(F)$, in the representation* (14) *we have*

$$|c_{i,\xi}^s| \le c\delta_i^{k-s}. \tag{15}$$

Conversely, if f has a representation (14) *with some coefficients $c_{i,\xi}^s$ satisfying* (15), *then $f \in \Lambda_k(F)$.*

The proof is similar to, but more involved than, the proof of Theorem 2.

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Received 22 January 2002