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ASYMPTOTIC GROWTH OF CAUCHY TRANSFORMS

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Abstract. Let μ be a complex measure on the real line. We denote by $P\mu$ and $Q\mu$ the Poisson and the conjugate Poisson integrals of μ in the upper half-plane. In this note we study the relative asymptotic growth of $P\mu$ and $Q\mu$ near the support of μ. In particular, we show that on μ almost every vertical line $Q\mu$ grows no slower than $P\mu$. We also discuss applications to the theory of Cauchy transform in the plane and related questions on Riesz transforms in $\mathbf{R}^n$.

1. Introduction

Let $M(\mathbf{R})$ be the space of all complex measures μ on the real line satisfying

$$\int_{\mathbf{R}} \frac{d|\mu|(t)}{1+|t|} < \infty.$$

We will denote by $P\mu$ and $Q\mu$ the Poisson and the conjugate Poisson integrals of $\mu \in M(\mathbf{R})$ in the upper half-plane $\mathbf{C}_+$ respectively:

$$P\mu(x+iy) = \int_{\mathbf{R}} \frac{y}{(x-t)^2+y^2}\, d\mu(t)$$

and

$$Q\mu(x+iy) = \int_{\mathbf{R}} \frac{x-t}{(x-t)^2+y^2}\, d\mu(t).$$

The Poisson kernel is an example of the so-called approximate unity, whereas the conjugate Poisson kernel is a typical singular kernel. Hence the boundary behavior of the Poisson and the conjugate Poisson integrals reflects different properties of the measure. The growth of $P\mu(z)$ as $z \to x \in \mathbf{R}$ depends on the concentration of the mass near x. The behavior of $Q\mu$ depends on the "symmetry" of μ around x.

Nonetheless, as we will find out, the growth of $P\mu$ and $Q\mu$ must be almost the same near "most" points on the boundary. Roughly speaking, we show that the fast growth of mass near a point "usually" implies the lack of symmetry around it.

More precisely, we prove the following result. If $\mu \in M(\mathbf{R})$ we denote by μ_s its singular part with respect to the Lebesgue measure.

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Theorem 1.1. *Let $\mu \in M(\mathbf{R})$ and let Σ be a Borel subset of $\mathbf{R}$. Then the following conditions are equivalent*:

(1) $Q\mu(x+iy) = o\big(P\mu(x+iy)\big)$ *as* $y \to 0+$ *for* μ_s*-a.e.* $x \in \Sigma$;
(2) *the restriction of* μ_s *on* Σ *is discrete.*

Note that if a singular measure μ is such that

$$P\mu(x+iy) = o\big(Q\mu(x+iy)\big)$$

as $y \to 0+$ for μ-almost all $x \in E$ then $\mu(E) = 0$, see [7]. Hence, if μ is singular continuous, neither $P\mu$ nor $Q\mu$ can dominate its counterpart except near a zero set of points.

In Section 3 we consider the Riesz transform in $\mathbf{R}^n$. Let $M(\mathbf{R}^{n-1})$ be the space of all measures in the hyperplane $\mathbf{R}^{n-1}$ satisfying

$$\int_{\mathbf{R}^{n-1}} \frac{1}{1+|x|^{n-1}}\, d\mu(x) < \infty.$$

For each $\mu \in M(\mathbf{R}^{n-1})$ the Riesz transform $R\mu(x)$ is defined on the half-space $\mathbf{R}^n_+ = \{(x_1,\dots,x_n) \mid x_n > 0\}$ as $R\mu(x) = \langle R_1\mu(x),\dots,R_n\mu(x)\rangle$,

$$R_i\mu(x) = \int_{\mathbf{R}^{n-1}} \frac{x_i - y_i}{|x-y|^n}\, d\mu(y), \quad i = 1,2,\dots,n.$$

It is well known, that Riesz transforms can be viewed as generalizations of the Poisson integral (the nth transform R_n) and the conjugate Poisson integral (all other transforms $R_1,\dots,R_{n-1}$) to higher dimensions.

The natural question that arises after Theorem 1.1 is whether an analogous statement holds for Riesz transforms in $\mathbf{R}^n$. However, if one lets μ be the $(n-2)$-dimensional Lebesgue measure on an $(n-2)$-dimensional cube in $\mathbf{R}^{n-1}$ and considers its Riesz transform in $\mathbf{R}^n_+$, one obtains an example of a singular continuous measure satisfying

$$\big|\big\langle R_1\mu\big((x_1,\dots,x_n)\big),\dots,R_{n-1}\mu\big((x_1,\dots,x_n)\big)\big\rangle\big| = o\big(R_n\mu\big((x_1,\dots,x_n)\big)\big)$$

as $x_n \to 0+$ for μ-a.e. $(x_1,\dots,x_{n-1},0) \in \mathbf{R}^{n-1}$.

Nonetheless an analogue of Theorem 1.1 for Riesz transforms exists. One just has to replace the "radial" convergence with non-tangential. If $x \in \mathbf{R}^{n-1}$ and $0 < \phi < \frac{1}{2}\pi$ we denote by Γ^ϕ_x the truncated cone:

$$\Gamma^\phi_x = \big\{y = \langle y_1,\dots,y_n >\in \mathbf{R}^n_+ \mid\ y_n/|y-x|\rangle \sin\phi,\ y_n < 1\big\}.$$

As usual we write $y \underset{\sphericalangle}{\to} x$ if $y \to x$ and there exists $\phi = \phi(x)$ such that $y \in \Gamma^\phi_x$.

Theorem 1.2. *Let $\mu \in M(\mathbf{R}^{n-1})$ and let $\Sigma \subset \mathbf{R}^{n-1}$ be such that*

$$|\langle R_1\mu(y), \dots, R_{n-1}\mu(y)\rangle| = o\big(R_n\mu(y)\big) \tag{1}$$

as $y \underset{\triangleleft}{\to} x$ for μ-a.e. $x \in \Sigma$. Then the restriction of μ on Σ is absolutely continuous with respect to the Lebesgue measure in $\mathbf{R}^{n-1}$.

For $\mu \in M(\mathbf{R})$ we denote by $H\mu(x,\varepsilon)$ its Hilbert transform: The kernel $H_{x,\varepsilon}(t)$ is defined as 0 on $\big(x-\frac{1}{2}\varepsilon, x+\frac{1}{2}\varepsilon\big)$ and as $1/(x-t)$ elsewhere and

$$H\mu(x,\varepsilon) = \int_{\mathbf{R}} H_{x,\varepsilon}(t)\,dt.$$

The standard argument shows (see Lemma 2.1) that the relation (1) in Theorem 1.1 is equivalent to

$$H\mu(x,y) = o\big(P\mu(x+iy)\big) \tag{2}$$

as $y \to 0$ for μ-a.e. x.

An analog of Theorem 1.1 can be applied in the theory of the Cauchy transform in the plane. If $\mu \in M(\mathbf{R}^2)$ we denote by $C\mu$ the convolution of μ with $1/z$ in the sense of principal value: denote

$$C\mu(z,\varepsilon) = \int_{\{|z-\xi|>\varepsilon\}} \frac{1}{z-\xi}\,d\mu(\xi)$$

and put

$$C\mu(z) = \lim_{\varepsilon\to 0} C\mu(z,\varepsilon).$$

It is a well-known phenomenon that under various conditions on the Cauchy transform a large part of the measure lies on a collection of smooth curves, see for instance [3], [4], [5] or [9]. The Cauchy transform of the restriction of the measure to one of such curves is similar to the Hilbert transform on the line. This may allow one to apply an analog of Theorem 1.1 and conclude that the measure does not have a singular continuous part on any of those curves.

To show an example of such an application, let us consider the following result by P. Mattila. We denote by $B(a,r)$ the ball of radius r centered at a. We say that $\underline{D}\mu(a) > 0$ if

$$\liminf_{r\to 0+} \frac{\mu(B(a,r))}{r} > 0.$$

Theorem 1.3 ([3]). *Let $\mu \in M(\mathbf{C})$ be a non-negative measure. If $\underline{D}\mu(z) > 0$ and $C\mu(z)$ exists for μ-a.e. $z \in \mathbf{C}$ then μ is concentrated on a countable set of C^1-curves, i.e. there exist C^1-curves $\gamma_1, \gamma_2, \dots$ such that*

$$\mu(\mathbf{C} \setminus \cup\gamma_i) = 0.$$

For this particular situation one can prove the following version of Theorem 1.1, see Section 4. We denote by H^1 the one-dimensional Hausdorff measure.

Theorem 1.4. *Let $\mu \in M(\mathbf{C})$, $\mu \geq 0$ and let γ be a C^1-curve in $\mathbf{C}$. Suppose that $C\mu$ exists μ-a.e. on γ. Then the restriction of μ on γ is the sum of a discrete measure and a measure absolutely continuous with respect to H^1.*

(Instead of the existence of $C\mu$ on γ one can actually require a slightly weaker condition, like $C_\varepsilon\mu(\xi) = o\big(\mu(B(\xi,\varepsilon)/\varepsilon\big)$ as $\varepsilon \to 0+$ for a.e. ξ with respect to the singular part of μ on γ, see Section 4.)

Together Theorems 1.4 and 1.3 give the following result:

Corollary 1.5. *If $\mu \in M(\mathbf{C})$, $\mu \geq 0$ is such that $\underline{D}\mu(z) > 0$ and $C\mu(z)$ exists for μ-a.e. z, then μ is the sum of a discrete measure and a measure absolutely continuous with respect to H^1, concentrated on a countable union of C^1 curves.*

In particular, note that the continuous part of μ will automatically satisfy the so-called linear growth condition:

$$\limsup_{\varepsilon\to 0} \frac{\mu_c\big(B(x,\varepsilon)\big)}{\varepsilon} < \infty$$

for μ_c-a.e. x.

Another application of Theorem 1.1 concerns inner functions in the unit disk. It is well known that certain geometric properties of a conformal mapping may imply the existence of its derivative on the boundary of the domain. Our next theorem shows a similar property in a non-conformal situation.

Let Σ be a subset of $\mathbf{T}$. We say that an inner function θ in the unit disk $\mathbf{D}$ is *radial* near Σ if it maps almost every radius that ends at Σ into a curve that is tangent to a radius. More precisely, θ is radial near Σ if for a.e. $\xi \in \Sigma$

$$\operatorname{Im} \theta(r\xi)\bar{\theta}(\xi) = o\big(1 - |\theta(r\xi)|\big)$$

as $r \to 1-$. Here $\theta(\xi)$ stands for the non-tangential limit of θ at ξ.

Theorem 1.6. *An inner function in the unit disk is radial near Σ if and only if it has non-tangential* (*angular*) *derivatives almost everywhere on Σ.*

I.e. an inner function is radial near Σ if and only if its zeros a_n and the singular measure σ corresponding to its singular factor satisfy

$$\sum_n \frac{1-|a_n|^2}{|a_n-\xi|^2} < \infty \qquad \textit{and} \qquad \int_{\mathbf{T}} \frac{d\sigma(z)}{|z-\xi|^2} < \infty \qquad \textit{for a.e. } \xi \in \Sigma.$$

We say that an inner function θ has an angular derivative at $\xi \in \mathbf{T}$ if $|\theta(\xi)| = 1$ and there exists a finite limit $\lim_{z \underset{\sphericalangle}{\to} \xi} (\theta(z)-\theta(\xi))/(z-\xi)$. (Note that the condition $|\theta(\xi)| = 1$ is redundant in the situation of Theorem 1.6.) There are several other equivalent definitions of the angular derivative related to each other by the Carathéodory theorem, see [8] for a detailed discussion.

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2. The growth of Cauchy integrals in the upper half-plane

We begin this section with the lemma, which illustrates the well-known fact that the conjugate Poisson integral is "close" to the Hilbert transform. For convenience we will often write $H\mu(x+iy)$ instead of $H\mu(x,y)$ (thus stressing the fact that $H\mu(x,y)$ is close to $Q\mu(x+iy)$). The notation H_{x+iy} will be used for the corresponding Hilbert kernel $H_{x,y}$. We will also use notations P_z and Q_z for the Poisson and conjugate Poisson kernels.

Lemma 2.1. *Let $\mu\in M(\mathbf{R})$ be a positive measure.*

(1) *Suppose that $|Q\mu(iy)|<\alpha(y)P\mu(iy)$ for some positive function α: $\mathbf{R}_+\to\mathbf{R}_+$, $\alpha(y)\to 0$ as $y\to 0$. Then $|H\mu(iy)|<\beta(y)P\mu(iy)$ for some positive function β: $\mathbf{R}_+\to\mathbf{R}_+$, $\beta(y)\to 0$ as $y\to 0$ depending only on α;*

(2) *Conversely, if $|H\mu(iy)|<\alpha(y)P\mu(iy)$ for some positive function α: $\mathbf{R}_+\to\mathbf{R}_+$, $\alpha(y)\to 0$ as $y\to 0$ then $|Q\mu(iy)|<\beta(y)P\mu(iy)$ for some positive function β: $\mathbf{R}_+\to\mathbf{R}_+$, $\beta(y)\to 0$ as $y\to 0$ depending only on α.*

Proof. (1) Denote $H^\varepsilon\mu(x+iy)=\int_{(1+\varepsilon)^{-1}y}^{(1+\varepsilon)y}H\mu(x+is)\,ds$. Then $H^\varepsilon\mu$ is an integral transform of μ with the continuous kernel $H^\varepsilon_{x+iy}=\int_{(1+\varepsilon)^{-1}y}^{(1+\varepsilon)y}H_{x+is}\,ds$.

Linear combinations of functions

$$\frac{Q_{iy}-Q_i}{P_i}(t)=(y^2-1)Q_{iy}(t)$$

are dense in the space of odd continuous functions on $\widehat{\mathbf{R}}$. Therefore for any $\varepsilon>0$ one can choose constants c_k and points iy_k so that

$$\left|\frac{H^\varepsilon_i-Q_i}{P_i}-\sum_1^n c_k\frac{Q_{iy}-Q_i}{P_i}\right|<\varepsilon$$

on $\mathbf{R}$. Then

$$\left|H^\varepsilon_i-\sum_1^{n+1}c_kQ_{iy_k}\right|<\varepsilon P_i$$

on $\mathbf{R}$ where $c_{n+1}=1-\sum_{k=1}^n c_k$, $y_{n+1}=1$. From the properties of Poisson, conjugate Poisson and Hilbert kernels this implies that for any $0<s<1$

$$\left|H^\varepsilon_{is}-\sum_1^n c_kQ_{isy_k}\right|<\varepsilon P_{is}\qquad\text{on } \mathbf{R}.$$

Therefore, $|H^\varepsilon\mu(iy)|<\beta^\varepsilon(y)P\mu(iy)$ for some positive function β^ε: $\mathbf{R}_+\to\mathbf{R}_+$, $\beta^\varepsilon(y)\to 0$ as $y\to 0$, depending only on α.

To pass from H^ε to H notice that

$$\left|\frac{\partial H\mu(iy)}{\partial y}\right|<C_1\frac{P\mu(iy)}{y}.$$

Hence, $|H\mu(iy)|\le C_2\big(H^\varepsilon\mu(iy)+\varepsilon P\mu(iy)\big)$.

(2) In a similar way, one can notice that Q_i can be approximated by linear combinations of H_{iy} (such an approximation can actually be constructed much easier than in part (1)). □

Note: the lemma shows that the conjugate Poisson transform can be replaced with the Hilbert transform in the statement of Theorem 1.1 in the case when the measure is positive. To pass to the general case one can represent the measure as a linear combination of mutually singular positive measures and use Theorem 2.4 below.

Now we can proceed with the proof of Theorem 1.1. We start with the definition of a porous set. After that we prove a lemma, showing that any set E, such that the condition $|H\mu(x,y)| = o\big(P\mu(x+iy)\big)$ as $y \to 0+$ is satisfied uniformly for $x \in E$, has to be porous. The proof of Theorem 1.1 will then be completed by showing that a singular continuous measure μ, satisfying $Q\mu = o(P\mu)$ on μ-a.e. vertical line, cannot see a porous set, thus obtaining a contradiction.

Definition 2.2. We say that a set $K \subset \mathbf{R}$ is porous if for any $x \in K$ and any $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that $(x-\delta, x-\delta/100) \cup (x+\delta/100, x+\delta)$ does not intersect K.

Note that by the density theorem a porous set has to have zero Lebesgue measure.

Lemma 2.3. *Let $\mu \in M(\mathbf{R})$ be a positive measure and let E be a closed set such that*

(1) $\mu(E) > 0$,
(2) $d\mu/dm(x) = \infty$ *for any* $x \in E$ *and*
(3) $|H\mu(x+iy)| < \alpha(y)P\mu(x+iy)$ *for some positive function* $\alpha\colon \mathbf{R}_+ \to \mathbf{R}_+$, $\alpha(y) \to 0$ *as* $y \to 0$, *at any* $x \in E$.

Then E is porous.

Here $d\mu/dm$ denotes the Radon derivative of μ with respect to the Lebesgue measure m.

Proof. By the last lemma, we can assume that $Q\mu$ grows uniformly slower than $P\mu$ with some function β replacing α in the corresponding estimate. By Theorem 1.2 (see next section for the proof), for μ-a.e. $x \in E$ for any sector Γ_x^ϕ, $\phi > 0$,

$$\limsup_{z\to x,\ z\in\Gamma_x^\phi} \frac{|Q\mu(z)|}{P\mu(z)} > 0.$$

Hence, for μ-a.e. $x \in E$ one can consider the balls $B_n = B\big(x+i\varepsilon_n, \tfrac{1}{2}\varepsilon_n\big)$ such that

$$\liminf_{n\to\infty} \frac{\max_{B_n}|Q\mu|}{\max_{B_n} P\mu} > 0 \qquad \text{where } \varepsilon_n \to 0+. \tag{3}$$

Let ζ_n be the "rescaling" maps: $\zeta_n(z) = (x+\varepsilon_n i) + \frac{1}{2}\varepsilon_n z$. Then

$$u_n = \frac{1}{\max_{B_n} P\mu} K\mu \circ \zeta_n$$

is a sequence of functions in the unit disk whose real part is bounded by 1. The derivative of $Q\mu$ in B_n is bounded by $C\varepsilon_n^{-1}P\mu$. Thus the derivatives of the imaginary parts of u_n are uniformly bounded in the unit disc. Also, by the definition of the set E

$$\operatorname{Im} u_n(0) \leq C\frac{Q\mu(x+i\varepsilon_n)}{P\mu(x+i\varepsilon_n)} \to 0.$$

Thus the imaginary parts of u_n are also uniformly bounded. By the normal families argument, one can choose a subsequence u_{n_k} converging to an analytic function u pointvise in the disk. Let $u = p + iq$. By Harnak's lemma, the real parts of u_n are bounded away from zero, and so p is non-zero. By (3) $|\operatorname{Im} u_n| > c > 0$ on a subdisk of a fixed hyperbolic radius and hence q is non zero. By the definition of the set E, $q = 0$ on the vertical diameter $d = (-i, i)$. Hence the partial derivative q_y is 0 on d. If $q_x = 0$ on d then by the Cauchy–Riemann equations $q \equiv 0$ and we have a contradiction. Therefore, for k large enough in each ball B_{n_k} there is at least one point $z_k = x + iy_k$ on the vertical diameter where $y_k|(Q\mu)_x| > c\max_{B_{n_k}} P\mu$ for some positive c. Then the inequality $y_k|(Q\mu)_x| > \frac{1}{2}c\max_{B_{n_k}} P\mu$ must hold in the ball $D_k = B(z_k, \varepsilon y_k)$ for some $\varepsilon > 0$ and for $k = 1, 2, \ldots$. Since $Q\mu$ is continuous, WLOG we can assume that

$$y_k(Q\mu)_x > \tfrac{1}{2}cP\mu \text{ in } D_k. \tag{4}$$

Now if we choose n large, so that

$$\beta(y_n) \ll \varepsilon \tag{5}$$

then (4) implies that for any

$$x_1 \in (x - \varepsilon y_n, x - \varepsilon y_n/100) \cup (x + \varepsilon y_n/100, x + \varepsilon y_n)$$

we have

$$|Q\mu(x_1 + iy_n)| > \frac{\varepsilon c}{100} P\mu(x_1 + iy_n).$$

Together with (5) the last inequality implies that $x_1 \notin E$. □

We will also need the following

Theorem 2.4 ([6]). *Let $\mu, \nu \in M(\mathbf{R})$, $\mu = f\nu + \eta$ where $f \in L^1(|\nu|)$ and $\eta \perp \nu$. Then the limit*

$$\lim_{z \overset{\to}{\sphericalangle} x} \frac{K\mu(x)}{K\nu(x)}$$

exists ν-a.e. It is equal to $f(x)$ ν_s-a.e.

Lemma 2.5. *Let μ and E be the same as in Lemma 2.3. Denote by ν the restriction of μ on E. For ν-a.e. x and any $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$ such that*

(1) $(x-\delta, x-\delta/100) \cup (x+\delta/100, x+\delta)$ does not intersect E;

(2) for any $x_0 + iy_0$ such that $x_0 \in (x-\delta/100, x+\delta/100) \cap E$, $|x - x_0| < y_0 < 2|x-x_0|$,

$$|H\mu(x_0+iy_0)| + |H\nu(x_0+iy_0)| < \frac{1}{1000} P\mu(x_0+iy_0).$$

(Note: the condition $|x-x_0| < y_0 < 2|x-x_0|$ means that $x_0 + iy_0 \in \Gamma_x^{\pi/4} \setminus \Gamma_x^{\pi/6}$.)

Proof. The constant δ satisfying (1) exists by the last lemma.

Since $x_0 + iy_0 \in \Gamma_x^{\pi/4}$, by Theorem 2.4 (note that ν is a singular measure)

$$\lim_{\delta\to 0} \frac{K\mu(x_0+iy_0)}{K\nu(x_0+iy_0)} = \lim_{\delta\to 0} \frac{P\mu(x_0+iy_0) + iQ\mu(x_0+iy_0)}{P\nu(x_0+iy_0) + iQ\nu(x_0+iy_0)} = 1$$

for ν-a.e. x. Since $x_0 \in E$, for small enough δ,

$$|Q\mu(x_0+iy_0)| < \frac{1}{1000} P\mu(x_0+iy_0),$$

and therefore

$$|Q\nu(x_0+iy_0)| < \frac{1}{1000} P\nu(x_0+iy_0) \le \frac{1}{1000} P\mu(x_0+iy_0).$$

Now one can use Lemma 2.1. □

Proof of Theorem 1.1. (2) $\Rightarrow$ (1). The implication is easy to verify if μ_s has just one point mass. If μ_s has more than one point mass, Theorem 2.4 implies that near each of those point masses the contribution of other point masses is negligeably small.

(1) $\Rightarrow$ (2). By Theorem 2.4, one can reduce the statement to the case when μ is positive.

Let us denote by μ_{sc} the singular continuous part of μ. We need to show that $\mu_{sc}(\Sigma) = 0$. Suppose it is not so. Then one can choose a subset $E \subset \Sigma$, $\mu_{sc}(E) > 0$ such that

$$|H\mu(x,y)| < \alpha(y) P\mu(x+iy)$$

for some positive function $\alpha\colon \mathbf{R}_+ \to \mathbf{R}_+$, $\alpha(y) \to 0$ as $y \to 0$, at any $x \in E$. Let ε be such that $\alpha(\varepsilon) < 1/1000$. By Lemma 2.5 for any $x \in E$ there exists $\delta < \varepsilon$ satisfying conditions (1) and (2) from the statement of the lemma. Let us cover E with such δ-neighborhoods. Let I be one of the intervals of this covering: $I=(x-\delta,x+\delta)$ for some $x\in E$, $\delta=\delta(x)<\varepsilon$. Denote $a=\min E\cap I$, $b=\max E\cap I$ (note: $a,b\in(x-\delta/100,x+\delta/100)$). Let $\Delta=b-a$, $z_1=a+2\Delta i$, $z_2=b+2\Delta i$. Then

$$|H\nu(z_1)-H\nu(z_2)|\le|H\mu(z_1)-H\mu(z_2)|+|H\nu(z_1)-H\mu(z_1)|+|H\nu(z_2)-H\mu(z_2)|.$$

The first summand is small relative to $P\mu$ because $a,b\in E$ and the other two are small because of part (2) of the last lemma. Altogether we get

$$|H\nu(z_1)-H\nu(z_2)|\le\frac{5}{1000}P\nu(z_1).$$

On the other hand, the difference of kernels $H_{z_1}-H_{z_2}$ is 0 on (a,b) (both kernels are 0 there) and satisfies $H_{z_1}-H_{z_2}<-1/2P_{z_1}$ outside of $(x-100\delta,x+100\delta)$. Since ν is absent on $(x-100\delta,x+100\delta)\setminus(x-\delta,x+\delta)$,

$$|H\nu(z_1)-H\nu(z_2)|\ge\frac{1}{2}\int_{\mathbf{R}\setminus I}P_{z_1}\,d\nu.$$

Therefore

$$\frac{5}{1000}P\nu(z_1)\ge\frac{1}{2}\int_{\mathbf{R}\setminus I}P_{z_1}\,d\nu$$

and

$$\frac{\nu\big((a,b)\big)}{b-a}\ge P\nu(z_1)-\int_{\mathbf{R}\setminus I}P_{z_1}\,d\nu>\frac{1}{2}P\nu(z_1). \tag{6}$$

To obtain the final contradiction consider points $w_1=a+\Delta i$, $w_2=b+\Delta i$. Note that, in the same way as before, the choice of I (part (2) of Lemma 2.5) and the fact that $a,b\in E$ imply

$$|H\nu(w_1)-H\nu(w_2)|\le\frac{5}{1000}P\nu(w_1).$$

On the other hand, $(H_{w_1}-H_{w_2})>1/2(b-a)$ on (a,b) and $|H_{w_1}-H_{w_2}|<2P_{z_1}$ on $\mathbf{R}\setminus I$. Hence by (6),

$$\begin{aligned}|H\nu(w_1)-H\nu(w_2)|&\ge\left|\int_{(a,b)}(H_{w_1}-H_{w_2})\,d\nu\right|-\left|\int_{\mathbf{R}\setminus I}(H_{w_1}-H_{w_2})\,d\nu\right|\\&\ge\frac{1}{2}\frac{\nu((a,b))}{b-a}-\frac{10}{1000}P\nu(z_1)\ge\frac{1}{10}P\nu(z_1)\ge\frac{1}{100}P\nu(w_1).\ \square\end{aligned}$$

3. Non-tangential growth of Riesz transforms in the half-space

In this section we prove Teorem 1.2. We start with the following lemmas.

For $\mu \in M(\mathbf{R}^{n-1})$ we denote by μ_s the part of μ that is singular with respect to the $(n-1)$-dimensional Lebesgue measure on $\mathbf{R}^{n-1}$.

Lemma 3.1. *Let $\mu \in M(\mathbf{R}^{n-1})$, $0 < \phi < \frac{1}{2}\pi$ and let $E \subset \mathbf{R}^{n-1}$ be a set of zero $(n-1)$-dimensional Lebesgue measure satisfying $|\mu_s|(E) > 0$. Denote $\Gamma = \bigcup_{x\in E} \Gamma_x^{\phi}$. Then*

$$\int_{\partial\Gamma} |R\mu|\, ds = \infty. \tag{7}$$

(In the last integral and throughout this section ds corresponds to the standard integration with respect to the surface area.)

Proof. Here we only give the proof for the case of the upper half-plane ($n = 2$). For this case the proof seems especially natural. The same ideas could be modified to obtain the general proof.

Let $\mu \in M(\mathbf{R})$ and $\Gamma = \bigcup_{t\in E} \Gamma_t^{\phi}$, where $|E| = 0$, $|\mu_s|(E) > 0$. Suppose that $K\mu$ is summable on $\partial\Gamma$ with respect to the arclength. Define the function f on $\mathbf{R}$ in the following way:

$$f(x) = \begin{cases} |K\mu(x+iy)| & \text{if } x+iy \in \partial\Gamma \text{ for some } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f \in L^1(\mathbf{R})$. Consider the Poisson integral Pf of f in the upper half-plane. One can show that then

$$|K\mu| < C_1 Pf \tag{8}$$

a.e. on $\partial\Gamma$ with respect to the arclength. Indeed, let $x+iy \in \partial\Gamma$ for some $y < 1$. Then at least on one half of the interval $\bigl(x-\frac{1}{2}y, x+\frac{1}{2}y\bigr)$ we have $f > C_3|K\mu(x+iy)|$ because of the Lipschitz properties of $K\mu$ in $B\bigl(x+iy, \frac{1}{2}y\bigr)$. Since the Poisson kernel P_{x+iy} is larger than C_4 on $\bigl(x-\frac{1}{2}y, x+\frac{1}{2}y\bigr)$, we get (8) at $x+iy$.

Since $K\mu$ is a function of the Smirnov class in Γ and Pf is harmonic, (8) holds inside Γ as well. Since $f\,dx \perp \mu_s$, we obtain a contradiction with Lemma 3.2 below (just put $f\,dx = \nu$). □

Lemma 3.2 ([6]). *Let $\mu, \nu \in M(\mathbf{R})$ and $\nu \perp \mu_s$. Then*

$$\lim_{z \underset{\sphericalangle}{\to} x} \frac{|P\mu|}{|P\nu|} = \infty \qquad \text{for } \mu_s\text{-a.e. } x.$$

Proof. The statement presents a version of the Lebesgue theorem saying that for a summable function almost every point is its Lebesgue point. The classical proof can be easily modified to work in our case. □

Lemma 3.3. *Let u be a harmonic function in a domain in $\mathbf{R}^n$. Denote by $u_1, \dots, u_n$ its partial derivatives. Then the function $u_n^2 - C_n \sum_{i=1}^{n-1} u_i^2$ is superharmonic in the same domain for any $C_n \geq n-1$.*

Proof. Note that since $\sum_{i=1}^n u_{ii} = 0$,

$$\triangle(u_j)^2 = \sum_{i=1}^n \partial_i \partial_i (u_j)^2 = \sum_{i=1}^n 2(\partial_i u_j)^2 + 2u_j \sum_{i=1}^n \partial_i^2 u_j = 2\sum_{i=1}^n (u_{ij})^2.$$

Therefore

$$\begin{aligned}\triangle\left(u_n^2 - C_n \sum_{i=1}^{n-1} u_i^2\right) &= 2\left(\sum_{i=1}^n (u_{ni})^2 - C_n \sum_{i=1}^n (u_{i(n-1)})^2 \cdots - C_n \sum_{i=1}^n (u_{i1})^2\right)\\ &= 2\left((u_{nn})^2 - C_n \sum_{i=1}^{n-1} (u_{i(n-1)})^2 \dots - C_n \sum_{i=1}^{n-1} (u_{i1})^2\right)\\ &\leq 2\big((u_{nn})^2 - C_n (u_{(n-1)(n-1)})^2 \cdots - C_n (u_{11})^2\big).\end{aligned}$$

Now using again the fact that $u_{nn} = -\sum_{i=1}^{n-1} u_{ii}$ one can conclude that the last expression is negative. □

Proof of Theorem 1.2. Let (1) be satisfied for μ-a.e. $x \in \Sigma$ but $|\mu_s|(\Sigma) > 0$. Denote by u the harmonic function in $\mathbf{R}_+^n$ such that $\nabla u = R\mu$, i.e. $R_k \mu = u_k$. Let $\frac{1}{4}\pi < \phi < \frac{1}{2}\pi$. Then there exists a closed set $E \subset \Sigma, |E| = 0, |\mu|(E) > 0$ and $\varepsilon > 0$ such that

$$|\langle u_1, \dots, u_{n-1}\rangle| < \frac{1}{2\sqrt{(n-1)}}\,|u_n| \tag{9}$$

on

$$\Gamma = \left\{(y_1, \dots, y_n) \in \bigcup_{x\in E} \Gamma_x^\phi \cap,\ 0 < y_n < \varepsilon\right\}.$$

By Lemma 3.3 the function $u_n^2 - (n-1)\sum_{1\leq k<n} u_k^2$ is superharmonic on Γ and by (9) it is positive. Hence $u_n^2 - (n-1)\sum_{1\leq k<n} u_k^2$, and therefore (by (9)) u_n^2 is summable on the boundary of Γ with respect to the harmonic measure there.

Notice that Γ is a Lipschitz domain. The harmonic measure on $\partial\Gamma$ can be written as $w\,ds$ for some positive w. Since the angle ϕ is greater than $\frac{1}{4}\pi$, the density w satisfies

$$\int_{\partial\Gamma} w^{-1}\,ds < \infty.$$

(One will need to "smooth-out" the upper part of the boundary to make the integral over the whole $\partial\Gamma$ finite; we are actually only interested in the lower part of the boundary.) But now, by the Cauchy–Schwarz inequality,

$$\int_{\partial\Gamma} |u_n|\,ds \leq \left(\int_{\partial\Gamma} u_n^2 w\,ds\right)^{1/2}\left(\int_{\partial\Gamma} w^{-1}\,ds\right)^{1/2} < \infty.$$

By (9) this implies that $\int_{\partial\Gamma} |u|\, ds < \infty$ which contradicts Lemma 3.1.

Let now ϕ satisfy $0 < \phi \le \frac{1}{4}\pi$. Suppose that for some $x = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$ we have

$$|\langle u_1, \dots, u_{n-1}\rangle| = o(u_n) \tag{10}$$

as $z \to x$, $z \in \Gamma_x^\phi$ but

$$|\langle u_1, \dots, u_{n-1}\rangle| \ne o(u_n) \tag{11}$$

in $\Gamma_x^{3\pi/8}$ (the latter holds μ-a.e. by the first part of the proof). Suppose also that $u_n \to \infty$ in $\Gamma_x^{3\pi/8}$ (this holds a.e. with respect to the positive component of μ_s). Then there exist points $z_k = \langle x_1, \dots, x_{n-1}, y_k\rangle$, $y_k \to 0+$ such that in each ball $B_k = B\big(z_k, y_k \sin\frac{3}{8}\pi\big)$ (B_k is the maximal ball centered at z_k that lies in $\Gamma_x^{3\pi/8}$) there is a point w_k where

$$|\langle u_1(w_k), \dots, u_{n-1}(w_k)\rangle| > c|u_n(w_k)|$$

for some fixed $c > 0$. We can choose B_k and w_k so that $w_k \in (1-\varepsilon)B_k = B\big(z_k, (1-\varepsilon)y_k \sin\frac{3}{8}\pi\big)$ for some small $\varepsilon > 0$. Consider the rescaling maps from the unit ball $B = B(0,1)$ to B_k:

$$\xi_k(z) = z_k + z y_k \sin\frac{3\pi}{8}.$$

Put

$$v_k = \frac{1}{\max_{B_k} |\nabla u|} (\nabla u) \circ \xi_k.$$

Then $\{v_k\}$ is a sequence of gradients of harmonic functions in the unit ball B whose magnitude is bounded by 1. By the normal families argument, one can choose a subsequence v_{n_k} converging to a gradient v, $|v| < 1$ pointwise in the disk. Since at the origin the last coordinates of v_k are bounded away from zero, the last coordinate of v is bounded away from zero at the origin. Since the last coordinates of v_k are positive (recall that we assumed $u_n \to \infty$ in $\Gamma_x^{3\pi/8}$) the last coordinate is bounded away from zero at the origin and positive in B. Hence it is bounded away from zero in $(1-\varepsilon)B$. By the choice of w_k, for each k there is a point in $B\big(0, (1-\varepsilon)\big)$ where the magnitude of the first $n-1$ coordinates of v_k is large in comparison to the last coordinate. Thus the first $n-1$ coordinates of v cannot be all zero. But since $|\langle u_1, \dots, u_{n-1}\rangle| = o(u_n)$ in Γ_x^ϕ, the first $n-1$ coordinates of v are zero on a large part of the ball (a set of nonzero volume) and we have a contradiction. It is left to notice that by the first part of the proof (11) holds in $\Gamma_x^{3\pi/8}$ for μ_s-a.e. x and therefore (10) may not hold in Γ_x^ϕ except for a zero set of x with respect to the positive part of μ_s. Other parts of μ_s can be treated similarly. □

4. Applications

4.1. Measures in the plane. The goal of this subsection is to prove Theorem 1.4.

Let μ be a finite positive measure in $\mathbf{C}$ and let γ be a C^1 curve.

WLOG γ is a graph of a C^1 function f on the interval $[-1,1]$: $\gamma = \{x + if(x)\}$. Denote by $\Omega_\pm$ the open domains above and below γ, i.e. $\Omega_+ = \{x+iy \mid |x| \le 1,\ y > f(x)\}$ and $\Omega_- = \{x+iy \mid |x| \le 1,\ y < f(x)\}$.

First, we will "move" the whole μ under the graph γ to be able to consider the holomorphic function $C\mu$ in Ω_+. This will allow us to apply complex methods like in Sections 3 and 4.

To do this, consider the map ϕ: $\Omega_+ \mapsto \Omega_-$, $\phi(x+iy) = x + i\big(f(x) - \big(y - f(x)\big)\big)$ that maps points in Ω_+ into points in Ω_- symmetric with respect to γ. Denote by ν the restriction of μ on γ. Let $\eta = \mu - \nu$ and denote by $\eta_\pm$ the restrictions of η on $\Omega_\pm$. Consider the measure $\phi(\eta_+)$ on Ω_-: $\big[\phi(\eta_+)\big](B) = \eta_+\big(\phi^{-1}(B)\big)$ for any Borel $B \subset \Omega_-$. Denote $\eta^* = \eta_- + \phi(\eta_+)$ and $\mu^* = \nu + \eta^*$.

Now $C\mu^*$ is a holomorphic function in Ω_+. If $z \in \gamma$ denote by α_z the arctangent of the slope of the tangent line at z. Then the point $z + ie^{\alpha_z}\varepsilon$ approaches z along the normal line from Ω_+.

Throughout this section, if $\mu \in M(\mathbf{C})$ is supported on a rectifiable curve, we denote by μ_{ac}, μ_s and μ_{sc} the absolutely continuous, singular and singular continuous parts of μ with respect to H^1 on the curve.

We want to proceed as follows. Suppose that $C\mu$ is finite μ-a.e. on γ. First we will show that $\operatorname{Re} e^{-\alpha_z} C_\varepsilon \mu^*(z + ie^{\alpha_z}\varepsilon)$, the analog of the conjugate Poisson integral from the line case considered in Section 2, grows slower than $\operatorname{Im} e^{-\alpha_z} C_\varepsilon \mu^*(z + ie^{\alpha_z}\varepsilon)$, the analog of the Poisson integral, as $\varepsilon \to 0+$ on ν_s-a.e. normal line, see Claims 4.1–4.6 below. Then, using methods similar to those from Sections 2 and 3, we will show that this is possible only if ν_s is discrete.

Claim 4.1. *For every $z \in \gamma$ there exists a finite constant C such that for any $\varepsilon > 0$*

$$\operatorname{Im} e^{-\alpha_z} C_\varepsilon \mu^*(z) \ge \operatorname{Im} e^{-\alpha_z} C_\varepsilon \mu(z) + C.$$

Proof. Let $z = 0 \in \gamma$ and assume that the tangent line to γ at 0 is horisontal. Then $e^{-\alpha_z} = 1$. For any $\delta > 0$, γ lies inside $\{|y| < \delta |x|\}$ near 0. WLOG we can assume that the whole γ lies there. Note that

$$\operatorname{Im} C\mu(w) = \int \frac{\operatorname{Im} w - \operatorname{Im} \xi}{|w - \xi|^2}\, d\mu(\xi).$$

The kernel of $\operatorname{Im} C_\varepsilon(0)$ is negative in the upper half-plane and positive in the lower half-plane. The part of η_+ that lies above $y = 3\delta|x|$ was mapped by ϕ from the upper to the lower half-plane, and therefore after replacing that part with its "image" under ϕ the integral could only increase. WLOG the part of η_+ under $y = 3\delta|x|$ is pure point. Notice that each point mass moves down under ϕ. If δ

is small enough, for any point mass that lies inside $\{|y| < 3\delta|x|\}$ such a motion increases its integral. $\square$

If $\mu \in M(\mathbf{C})$ we will denote by $\mathscr{P}\mu(z,\varepsilon)$ the integral

$$\mathscr{P}\mu(z,\varepsilon) = \int \frac{\varepsilon}{|\xi - z|^2 + \varepsilon^2}\, d\mu(\xi).$$

Let ψ be the map from $\operatorname{Clos}\Omega_-$ to the closed lower half-plane defined as $\psi\big(f(x) - iy\big) = -iy$ for every $y \ge 0$ (recall that γ is the graph of f). The map ψ projects γ on the real line and sends every curve $\gamma - iy$ into the horisontal segment $[-1 - iy, 1 - iy]$ below the real line. Denote by μ_p^* the measure $\psi(\mu^*)$, i.e. $\mu_p^*(B) = \mu^*\big(\psi^{-1}(B)\big)$. Similarly, let ν_p, η_p^* stand for the images of the corresponding measures.

Claim 4.2. *We have*

$$\operatorname{Im} C\eta_p^*(\psi(z) + i\varepsilon) = o\big(\mathscr{P}\nu(z,\varepsilon)\big) \text{ for } \nu_s\text{-a.e. } z \text{ as } \varepsilon \to 0+.$$

Proof. The function $\operatorname{Im} C\eta_p^*$ is a positive harmonic function in the upper half-plane. Therefore it is equal to $P\sigma$ for some positive measure σ on $\mathbf{R}$. The measure σ is absolutely continuous. Indeed, denote by η_ε the restriction of η_p^* on $\{\operatorname{Im} z < -\varepsilon\}$ and let σ_ε be the corresponding measure on $\mathbf{R}$: $\operatorname{Im} C\eta_\varepsilon = P\sigma_\varepsilon$. Then $\sigma_\varepsilon \to \sigma$ in norm. Since all σ_ε are absolutely continuous, so is σ. Therefore by Lemma 4.8

$$\operatorname{Im} C\eta_p^*(x + i\varepsilon) = P\sigma(x + i\varepsilon) = o\big(P\nu_p(x + i\varepsilon)\big) \text{ for } (\nu_p)_s\text{-a.e. } x \text{ as } \varepsilon \to 0+.$$

It is left to notice that for any $z \in \gamma$ there exists $C > 0$ such that $\mathscr{P}\nu(z,\varepsilon) > CP\nu_p(x + i\varepsilon)$. $\square$

Let $\xi \in \gamma$ and $\delta > 0$. Near ξ, γ lies in $\Delta_\xi = e^{\alpha_\xi}\{|\operatorname{Im}(z-\xi)| < \delta|\operatorname{Re}(z-\xi)|\}$. Our next claim shows that the part of μ^* that lies outside of Δ_ξ has little influence on the asymptotics of $e^{\alpha_\xi} \operatorname{Im} C_\varepsilon \mu^* \xi$.

Claim 4.3. *For any $\delta > 0$ and $\xi \in \gamma$ denote by η_ξ^* the restriction of η^* on $\mathbf{C} \setminus \Delta_\xi$. Then for ν_s-a.e. ξ*

$$e^{\alpha_\xi} \operatorname{Im} C_\varepsilon \eta^* \xi = o\big(\mathscr{P}\mu^*(\xi,\varepsilon)\big).$$

Proof. By comparing the kernels one can notice that

$$|e^{\alpha_\xi} \operatorname{Im} C_\varepsilon \eta^* \xi| < C \operatorname{Im} C\eta_p^*\big(\psi(\xi) + i\varepsilon\big).$$

Now the statement follows from the previous claim. $\square$

Now we show that $\operatorname{Im} e^{-\alpha_z} C\mu^*(w)$ grows fast as w approaches z non-tangentially from Ω_+ for ν_s-a.e. z. For the rest of this subsection for any $\xi \in \gamma$, $0 < \phi < \frac{1}{2}\pi$ we denote by $\Gamma_\phi(\xi)$ the non-tangential sector $\{\operatorname{Im} e^{\alpha_\xi}(z-\xi)/|z-\xi| > \sin\phi\}$. Note that near ξ the sector $\Gamma_\phi(\xi)$ lies entirely in Ω_+.

Claim 4.4. *For any* $0 < \phi < \frac{1}{2}\pi$

$$\frac{1}{L}\mathscr{P}\mu^*(0,\varepsilon) + C \le \operatorname{Im} e^{-\alpha_z} C\mu^*(w) \le L\mathscr{P}\mu^*(0,\varepsilon) + C$$

as $w \to z$, $w \in \Gamma_\phi(z)$, $|w-z| = \varepsilon$ *for some positive* L *for* ν_s-a.e. $z \in \gamma$.

Proof. Again we can assume that $z = 0 \in \gamma$, the tangent line to γ at 0 is horisontal and γ lies in $\{|y| < \delta|x|\}$, where δ is so small that γ does not intersect the sector $\Gamma_\phi(0)$. Let D be a large positive constant. Simple calculations show that for all $\xi \in \{|y| < \delta|x|\}$, $|\xi| > D\varepsilon$ we have

$$\frac{-\operatorname{Im}\xi}{|\xi|^2} + C_1\frac{\varepsilon}{\varepsilon^2 + |\xi|^2} \le \frac{\operatorname{Im} w - \operatorname{Im}\xi}{|w-\xi|^2} \le \frac{-\operatorname{Im}\xi}{|\xi|^2} + C_2\frac{\varepsilon}{\varepsilon^2 + |\xi|^2}$$

for some $C_1, C_2 > 0$ (if δ is small and D is large enough). The part of the measure outside of $\{|y| < \delta|x|\}$ can be ignored by the previous claim. Therefore

$$\begin{aligned}\operatorname{Im} C\mu^*(w) &= \int_{|\xi|>D\varepsilon} \frac{\operatorname{Im} w - \operatorname{Im}\xi}{|w-\xi|^2}\, d\mu^*(xi) + \int_{|\xi|\le D\varepsilon} \frac{\operatorname{Im} w - \operatorname{Im}\xi}{|w-\xi|^2}\, d\mu^*(xi) \\ &\asymp \int_{|\xi|\le D\varepsilon} \frac{\operatorname{Im} w - \operatorname{Im}\xi}{|w-\xi|^2}\, d\mu^*(xi) + \int_{|\xi|\ge D\varepsilon} \frac{\varepsilon}{\varepsilon^2 + |\xi|^2}\, d\mu^*(xi) + \operatorname{Im} C_{D\varepsilon}\mu^*(0).\end{aligned}$$

Since μ^* is concentrated under $y = \delta|x|$

$$\int_{|\xi|\le D\varepsilon} \frac{\operatorname{Im} w - \operatorname{Im}\xi}{|w-\xi|^2}\, d\mu^*(xi) + \int_{|\xi|\ge D\varepsilon} \frac{\varepsilon}{\varepsilon^2 + |\xi|^2}\, d\mu^*(xi) \asymp \mathscr{P}\mu^*(0,\varepsilon).$$

Now recall that $C\mu(0)$ is finite ν-a.e. and apply Claim 4.3. □

Next we estimate the "conjugate Poisson part", $\operatorname{Re} e^{-\alpha_z} C\mu^*(z)$.

Claim 4.5. *We have*

$$\operatorname{Re} e^{-\alpha_z} C_\varepsilon\mu^*(z) = \operatorname{Re} e^{-\alpha_z} C_\varepsilon\mu(z) + o\big(\mathscr{P}(0,\varepsilon)\mu^*(z)\big) \text{ as } \varepsilon \to 0+ \text{ for } \mu_s^*\text{-a.e. } z \in \gamma.$$

Proof. Again we can assume that $z = 0 \in \gamma$, the tangent line to γ at 0 is horisontal and γ lies in $\{|y| < \delta|x|\}$. Note that

$$\operatorname{Re} C_\varepsilon\mu^*(w) = \int_{|\xi|>\varepsilon} \frac{\operatorname{Re} w - \operatorname{Re}\xi}{|w-\xi|^2}\, d\mu^*(\xi).$$

To prove the statement we need to compare kernels of $\operatorname{Re} C_\varepsilon \eta_+(0)$ and $\operatorname{Re} C_\varepsilon \eta_+^*(0)$ at the points $x + i\big(f(x) + y\big)$ and $x + i\big(f(x) - y\big)$ correspondingly. Simple calculations show that, since $|f(x)| < \delta |x|$,

$$\left| \frac{x}{|x + if(x) + iy|^2} - \frac{x}{|x + if(x) - iy|^2} \right| < C \frac{2y}{x^2 + 4y^2}.$$

Since the right-hand side is the kernel for the Poisson integral of the "projected" measure $P\psi(\eta_-^*)$ at the point $\psi\big(x + i\big(f(x) - y\big)\big) = x - iy$, this inequality and Claim 4.2 imply

$$\begin{aligned} |\operatorname{Re} C_\varepsilon \mu(0) - \operatorname{Re} C_\varepsilon \mu^*(0)| &= |\operatorname{Re} C_\varepsilon \eta_+(0) - \operatorname{Re} C_\varepsilon \eta_+^*(0)| \\ &< C \operatorname{Im} C\psi(\eta_-^*)(x + iy) = o\big(\mathscr{P}\mu^*(0, \varepsilon)\big). \ \square \end{aligned}$$

Claim 4.6. *Suppose that $z \in \gamma$ and $C\mu(z)$ is finite. Then*

$$\operatorname{Re} e^{-\alpha_z} C\mu^*(z + ie^{\alpha_z}\varepsilon) = o\big(\mathscr{P}\mu^*(z, \varepsilon)\big) \text{ as } \varepsilon \to 0+ \text{ for } \mu_s^*\text{-a.e. } z \in \gamma.$$

Proof. Again we can assume that $z = 0$, the tangent line to γ at 0 is horisontal and γ lies in $\{|y| < \delta|x|\}$.

Denote

$$h_\alpha(z) = \begin{cases} \dfrac{\operatorname{Re} z}{|z|^2} & \text{on } \{|z| > \alpha\}, \\ 0 & \text{on } \{|z| \le \alpha\}. \end{cases}$$

Notice that there exists a linear combination of such functions

$$\sum a_n h_{\alpha_n}, \quad \sum a_n = 1, \quad \alpha_n \le 10\varepsilon$$

which approximates the kernel of $\operatorname{Re} C\mu^*(i\varepsilon)$ on $B(0, 10\varepsilon) \cap \{|y| < \delta|x|\}$:

$$\left| \frac{-\operatorname{Re} \xi}{|i\varepsilon - \xi|^2} - \sum a_n h_{\alpha_n}(\xi) \right| < C \frac{\delta}{\varepsilon}$$

for some absolute constant C. Outside of $B(0, 10\varepsilon) \cap \{|y| < \delta|x|\}$ the condition $\sum a_n = 1$ will automatically imply

$$\left| \frac{-\operatorname{Re} \xi}{|i\varepsilon - \xi|^2} - \sum a_n h_{\alpha_n}(\xi) \right| < C\delta \frac{\varepsilon}{\varepsilon^2 + |\xi|^2}$$

for any $\xi \in \{|y| < \delta|x|\}$ and

$$\left| \frac{-\operatorname{Re} \xi}{|i\varepsilon - \xi|^2} - \sum a_n h_{\alpha_n}(\xi) \right| < C \frac{\varepsilon}{\varepsilon^2 + |\xi|^2}$$

for other $\xi \notin B(0, 10\varepsilon)$. Integrating the last three estimates with respect to $\mu^* = \nu + \eta^*$ we obtain

$$\left| \operatorname{Re} C\mu^*(i\varepsilon) - \sum a_n C_{\alpha_n} \mu^*(0) \right| < C\big(\delta \mathscr{P}\nu(0,\varepsilon) + \mathscr{P}\eta^*(0,\varepsilon)\big).$$

Moreover, by the properties of both kernels, for any $0 < s < 1$ we will have the estimate

$$\begin{aligned}\left| \operatorname{Re} C\mu^*(is\varepsilon) - \sum a_n C_{s\alpha_n} \mu^*(0) \right| &< C\big(\delta \mathscr{P}\nu(0,s\varepsilon) + \mathscr{P}\eta^*(0,s\varepsilon)\big) \\ &= C\delta \mathscr{P}\nu(0,s\varepsilon) + o\big(\mathscr{P}\nu(0,s\varepsilon)\big) \\ &= C\delta \mathscr{P}\mu^*(0,s\varepsilon) + o\big(\mathscr{P}\mu^*(0,s\varepsilon)\big)\end{aligned}$$

by Lemma 4.8. The statement now follows from the fact that $C_{s\alpha_n}\mu^*(0) = o\big(\mathscr{P}\mu^*(0,s\varepsilon)\big)$ by the last claim and that δ can be chosen arbitrarily small near 0. □

Let us summarize the above claims: We obtained the measure $\mu^* = \nu + \eta^*$, where ν is supported on the C^1-graph γ and η^* lies under the graph (in Ω_-). We know that for ν_s-a.e. $\xi \in \gamma$, $\operatorname{Re} e^{\alpha_z} C\mu^*(z) = o\big(\operatorname{Im} e^{\alpha_z} C\mu^*(z)\big)$ as z approaches ξ along the normal line from above. We have to show that then ν_s must be discrete.

Following the algorithm of Section 3, we first choose a large ϕ and assume that there exists a set $E \subset \gamma$ such that $\nu_s(E) > 0$ and $\operatorname{Re} e^{\alpha_z} C\mu^*(z) = o\big(\operatorname{Im} e^{\alpha_z} C\mu^*(z)\big)$ as $z \to \xi$, $z \in \Gamma_\phi(\xi)$ for every $\xi \in E$. WLOG all α_ξ for $\xi \in E$ are smaller than a fixed δ. Then there exists $\varepsilon_0 > 0$ and $E' \subset E$, $\nu_s(E') > 0$ such that $\operatorname{Re} C\mu^*(z) < \frac{1}{2} \operatorname{Im} C\mu^*(z)$ for any $z \in \Gamma_\phi(\xi)$, $\xi \in E'$, $|z - \xi| < \varepsilon_0$. Consider a positive sequence $\{\varepsilon_k\}_{k=1}^\infty$ monotonously decreasing to zero, $\varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$. Define

$$\Gamma_k = \left\{ z \mid z \in \bigcup_{\xi \in E'} \Gamma_\phi(\xi),\ \varepsilon_k < \operatorname{Im} e^{\alpha_\xi}(z - \xi) < \varepsilon_0 \right\}$$

and $\Gamma = \cup \Gamma_k$. Since $\operatorname{Re} C\mu^*(z) < \frac{1}{2} \operatorname{Im} C\mu^*(z)$ in Γ_k,

$$\operatorname{Re}\big(C\mu^*(z)\big)^2 = \big(\operatorname{Re} C\mu^*(z)\big)^2 - \big(\operatorname{Im} C\mu^*(z)\big)^2$$

is a negative harmonic function in Γ_k. Therefore it is summable with respect to the harmonic measure on $\partial\Gamma_k$. Since $\operatorname{Re} C\mu^*(z) < \frac{1}{2} \operatorname{Im} C\mu^*(z)$, $\big(\operatorname{Im} C\mu^*(z)\big)^2$ is summable with respect to the harmonic measure on $\partial\Gamma_k$. Each Γ_k is a Lipschitz domain. Let $\zeta \in \Gamma_1$. If $\phi > \frac{1}{4}\pi + 10\delta$ the Lipschitz constant for the boundary of $\partial\Gamma_k$ is large enough so that the density w_k of the harmonic measure on $\partial\Gamma_k$ with respect to ζ satisfies

$$\int_{\partial\Gamma_k} w_k^{-1}\, ds < C < \infty$$

for all k. (Again, one needs to make the "upper" part of $\partial\Gamma$ smooth to have this, which can always be done; but on the lower part, which we are mostly interested in, the integral converges as it is.) Then by the Cauchy–Schwarz inequality

$$\int_{\partial\Gamma_k} |\operatorname{Im} C\mu^*(z)| \le \left(\int_{\partial\Gamma_k} \operatorname{Im} C\mu^*(z) w_k \, ds\right)^{1/2} \left(\int_{\partial\Gamma_k} w_k^{-1} \, ds\right)^{1/2} < \infty$$

and therefore $\operatorname{Im} C\mu^*(z)$ is summable on $\partial\Gamma_k$ with respect to the arclength. Denote by $h_k, k = 0, 1, 2, \ldots$ the summable function on $[-1, 1]$ obtained by "projection" of the values of $\operatorname{Im} C\mu^*$ from $\partial\Gamma_k$:

$$h_k(x) = \begin{cases} \max_{x+iy\in\Gamma_k} |\operatorname{Im} C\mu^*(x+iy)| & \text{if } x+iy \in \partial\Gamma_k \text{ for some } y\ , \\ 0 & \text{otherwise.} \end{cases}$$

Since the domains Γ_k "converge" to Γ_0, one can show that

$$\int_{\partial\Gamma_k \setminus \partial\Gamma} |\operatorname{Im} C\mu^*(z)| \, ds \to 0$$

as $k \to \infty$. Indeed, since $\left|\operatorname{Im}\big((C\mu^*(z))^2\big)\right| + 1 > \left|\operatorname{Re}\big((C\mu^*(z))^2\big)\right|$ the function $\big(C\mu^*(z)\big)^2$ is an $H^1(w_k ds)$-function in Γ_k. In particular we have

$$\int_{\partial\Gamma_k} \operatorname{Im}\big((C\mu^*)^2\big) w_k \, ds = \operatorname{Im}\big((C\mu^*(\zeta))^2\big).$$

Let us fix k. If $l > k$ is large enough

$$\int_{\partial\Gamma\cap\partial\Gamma_k} \operatorname{Im}\big((C\mu^*)^2\big) w_l \, ds$$

is close to

$$\int_{\partial\Gamma\cap\partial\Gamma_k} \operatorname{Im}\big((C\mu^*)^2\big) w \, ds$$

which, in its turn, for large enough k is close to $\operatorname{Im}\big((C\mu^*(\zeta))^2\big)$. This means that

$$\int_{\partial\Gamma_l \setminus \partial\Gamma} \operatorname{Im}\big((C\mu^*)^2\big) w_l \, ds$$

is close to 0 for large l. Therefore

$$\begin{aligned} \int_{\partial\Gamma_l\setminus\partial\Gamma} |\operatorname{Im} C\mu^*(z)| &\le \left(\int_{\partial\Gamma_l\setminus\partial\Gamma} \operatorname{Im} C\mu^*(z) w_l \, ds\right)^{1/2} \left(\int_{\partial\Gamma_l} w_l^{-1} \, ds\right)^{1/2} \\ &\le C\left(\int_{\partial\Gamma_l\setminus\partial\Gamma_0} \operatorname{Im} C\mu^*(z) w_l \, ds\right)^{1/2} \to 0. \end{aligned}$$

Therefore the sequence h_k converges in $L^1[-1,1]$. This means that there exists a subsequence h_{n_k} that has a summable majorant $H \in L^1[-1,1]$: $|h_{n_k}| < H$ for all k.

Let again $\nu_p = \psi(\nu)$ be the projection of ν on $[-1,1]$: $\nu_p(B) = \nu(\{x+iy \mid x \in B\})$. By Claim 4.4 for ν_s-a.e. ξ

$$\operatorname{Im}\big(e^{\alpha_z} C\mu^*(z)\big) \asymp P\nu_p\big(x + e^{-\alpha_z}(z-\xi)\big)$$

as $z \to \xi$, $z \in \Gamma_\phi(\xi)$, where $x = \psi(\xi)$. Using this relation one can show, that $PH(z) \geq CP\nu_p(z)$ for $z \in \Gamma_\phi(x)$, $\operatorname{Im} z = \varepsilon_{n_k}$, $k = 1, 2, \ldots$ for $(\nu_p)_s$-a.e. $x \in \psi(E')$. But this contradicts Lemma 3.2 since $H\,dx \perp (\nu_p)_s$.

Therefore, the set of such $\xi \in \gamma$ for which

$$\operatorname{Re} e^{-\alpha_\xi} C\mu^*(z) = o\big(\operatorname{Im} e^{-\alpha_\xi} C\mu^*(z)\big) \tag{12}$$

as $z \to \xi$, $z \in \Gamma_\phi(\xi)$ for large ϕ has to be a zero-set with respect to ν_s.

Using the normal families argument like in Section 3 one can pass from large ϕ to arbitrary sectors and show that there is only a zero set of points ξ with respect to ν_s such that for some $\phi = \phi(\xi) > 0$, (12) holds as $z \to \xi$, $z \in \Gamma_\phi(\xi)$.

Now we have to make the last step from sectors to normal lines. Again our argument will be analogous to Section 2.

Definition 4.7. We will call $E \subset \gamma$ porous if for any $\xi \in E$ and for any $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that $E \cap B(\xi, 100\delta) \setminus B(\xi, \delta) = \emptyset$.

Now, like in Lemma 2.3, suppose that there is a set $E \subset \gamma$, $\nu_{sc}(E) > 0$ such that $\mathscr{P}(z, \varepsilon) \to \infty$ and

$$|\operatorname{Re} e^{-\alpha_z} C\mu^*(z + i\varepsilon e^{\alpha_z})| < h(\varepsilon)|\operatorname{Im} e^{-\alpha_z} C\mu^*(z + i\varepsilon e^{\alpha_z})|$$

with some uniform function $h > 0$, $h(\varepsilon) \to 0$ as $\varepsilon \to 0+$ for every $z \in E$.

We can repeat the proof of Lemma 2.3 almost word by word to show that E is porous. Indeed, based on the fact that (12) cannot hold in a non-zero set of sectors with respect to ν_s, for ν_s-a.e. $z \in E$ and any $\varepsilon > 0$ we can find a ball B centered on the normal line at $z + i1000\delta e^{\alpha_z}$ of the radius 200δ, where $2000\delta < \varepsilon$, such that the directional derivative of $\operatorname{Re} e^{-\alpha_z} C\mu^*$ in the direction perpendicular to the normal line is large in B in comparison to $\operatorname{Im} e^{-\alpha_z} C\mu^*$. Similarly to the proof of Lemma 2.3, this means that those points on γ for which the corresponding normal lines hit $B \setminus \frac{1}{200}B$ cannot belong to E (note that $\alpha_\xi \to \alpha_z$ as $\xi \to z$ since γ is a C^1 curve). But all normal lines going through the points from $\big(B\big(z, 100\delta - o(\delta)\big) \setminus B\big(z, \delta + o(\delta)\big)\big) \cap \gamma$ will hit $B \setminus \frac{1}{200}B$.

Yet another version of the Lebesgue theorem that we will use is presented in the following statement:

Lemma 4.8. *Let $\mu, \nu \in M(\mathbf{R}^2)$, $\mu = f\nu$ where $f \in L^1(|\nu|)$. Then*

$$\lim_{\varepsilon\to 0} \frac{\mathscr{P}\mu(z,\varepsilon)}{\mathscr{P}\nu(z,\varepsilon)} = f(z)$$

as $\varepsilon \to 0+$ for μ-a.e. z. In particular the limit is 0 ν-a.e. if and only if $\mu \perp \nu$.

Now suppose that (12) holds as $z \to \xi$ along a normal line from Ω_+ for ν_{sc}-a.e. $\xi \in \gamma$. Then we can choose a set E, non-zero with respect to ν_{sc}, where that relation holds with a uniform "o" on the right-hand side. As we established above, E has to be porous. By approximating kernels, like in Claim 4.6, we can show that one can replace $\operatorname{Re} e^{-\alpha_\xi} C\mu^*(z)$, $z = \xi + ie^{\alpha_\xi}\varepsilon$ with $\operatorname{Re} e^{-\alpha_\xi} C_\varepsilon\mu^*(\xi)$. The new relation

$$|\operatorname{Re} e^{-\alpha_\xi} C_\varepsilon \mu^*(\xi)| < g(\varepsilon)|\operatorname{Im} e^{-\alpha_\xi} C\mu^*(\xi + i\varepsilon e^{\alpha_z})| \tag{13}$$

will still hold with some uniform function $g > 0$, $g(\varepsilon) \to 0$ as $\varepsilon \to 0+$ for ν_s-a.e. ξ. Denote by ν^E the restriction of ν on E. Then by Lemma 4.8

$$\mathscr{P}(\nu - \nu^E)(\xi,\varepsilon) = o\big(\mathscr{P}\nu(\xi,\varepsilon)\big) \tag{14}$$

for ν_{sc}-a.e. $\xi \in E$. Let $\xi \in E$ be a point where (13) and (14) hold. WLOG $\xi = 0$, the tangent line at 0 is horizontal and $\gamma \subset \{|y| < c|x|\}$ for some small $0 < c < 1$.

Since E is porous we can choose a small δ such that $E \cap \big(B(0,100\delta) \setminus B(0,\delta)\big) = \emptyset$. Let z_1 and z_2 be the points in $\gamma \cap B(0\delta)$ with the smallest and the biggest real parts correspondingly for which (13) holds. Denote $\Delta = |z_1 - z_2|$. (Note, that we can assume that $\Delta > 0$, i.e. $z_1 \neq z_2$. If that was not true, 0 would be an isolated point of E; but ν_{sc}-a.e. point of E is not isolated.)

We can estimate the difference between the kernels of $\operatorname{Re} e^{-\alpha_{z_1}} C_{(2\Delta)}\mu^*(z_1)$ and $\operatorname{Re} e^{-\alpha_{z_2}} C_{(2\Delta)}\mu^*(z_2)$ as follows: It is less than $-C_1\Delta/(|z|^2+\Delta^2)$ on $\gamma \setminus B(0,100\delta)$. Its absolute value is bounded by $C_2\Delta/(|z|^2+\Delta^2)$ on $\mathbf{C} \setminus B(0,\delta)$ for some $C_{1,2} > 0$. Finally, it is 0 on $B(0,\delta)$. Therefore

$$\begin{aligned}
&\operatorname{Re} e^{-\alpha_{z_1}} C_{(2\Delta)}\mu^*(z_1) - \operatorname{Re} e^{-\alpha_{z_2}} C_{(2\Delta)}\mu^*(z_2) \\
&\quad < -C_1 \int_{\gamma\setminus B(0,100\delta)} \frac{\Delta}{|z|^2+\Delta^2} d\mu^*(z) + C_2 \int_{\gamma\cap(B(0,100\delta)\setminus B(0,\delta))} \frac{\Delta}{|z|^2+\Delta^2}\, d\mu^*(z) \\
&\qquad + C_2 \int_{\mathbf{C}\setminus\gamma} \frac{\Delta}{|z|^2+\Delta^2}\, d\mu^*(z).
\end{aligned}$$

Note that the left-hand side is small by the absolute value in comparison to $\mathscr{P}\mu^*(0,\Delta)$ because (13) holds at z_1 and z_2. By Lemma 4.8 $\mathscr{P}\mu^*(\xi,\Delta) \sim \mathscr{P}\mu^*|_\gamma(\xi,\Delta)$ at ν_s-a.e. ξ. WLOG our point 0 is one of such ξ's. Then the third summand in the right-hand side is small in comparison to $\mathscr{P}\mu^*(0,\Delta)$ as well. The second summand is small in comparison to $\mathscr{P}\mu^*(0,\Delta)$ because of (14). Therefore

$$\int_{\gamma\setminus B(0,100\delta)} \frac{\Delta}{|z|^2+\Delta^2}\,d\mu^*(z)$$

is small in comparison to $\mathscr{P}\mu^*(0,\Delta)$. Then

$$\begin{aligned}\mathscr{P}\mu^*(0,\Delta) \sim \int_{\gamma\setminus B(0,100\delta)} \frac{\Delta}{|z|^2+\Delta^2}d\mu^*(z) &+ \int_{\gamma\cap(B(0,100\delta)\setminus B(0,\delta))} \frac{\Delta}{|z|^2+\Delta^2}\,d\mu^*(z)\\ &+ \int_{\gamma\cap B(0,100\delta)} \frac{\Delta}{|z|^2+\Delta^2}\,d\mu^*(z).\end{aligned}$$

As we have just shown, the first integral on the right-hand side is small. So is the second integral by (14). Therefore

$$\mathscr{P}\mu^*(0,\Delta) \sim \int_{\gamma\cap B(0,100\delta)} \frac{\Delta}{|z|^2+\Delta^2}\,d\mu^*(z) \le C_3\frac{\mu^*(B(0,\delta))}{\delta}.$$

At the same time, looking at kernels of $\operatorname{Re} e^{-\alpha_{z_1}} C_\Delta\mu^*(z_1)$ and $\operatorname{Re} e^{-\alpha_{z_2}} C_\Delta\mu^*(z_2)$, we get

$$\begin{aligned}&\operatorname{Re} e^{-\alpha_{z_1}} C_\Delta\mu^*(z_1) - \operatorname{Re} e^{-\alpha_{z_2}} C_\Delta\mu^*(z_2)\\ &\quad\ge \operatorname{Re} e^{-\alpha_{z_1}} C_{(2\Delta)}\mu^*(z_1) - \operatorname{Re} e^{-\alpha_{z_2}} C_{(2\Delta)}\mu^*(z_2) + C_4\frac{\mu^*(B(0,\delta))}{\delta} + o\big(\mathscr{P}\mu^*(0,\Delta)\big)\\ &\quad\ge C_5\mathscr{P}\mu^*(0,\Delta).\end{aligned}$$

This contradicts the fact that each $\operatorname{Re} e^{-\alpha_{z_k}} C_\Delta\mu^*(z_k)$, $k=1,2$ is small in comparison to $\operatorname{Im} e^{-\alpha_{z_1}} C_\Delta\mu^*(z_1)$, which in its turn is larger than $C_6\mathscr{P}\mu^*(0,\Delta)$ by Claim 4.4.

This finishes the proof of Theorem 1.4.

Note that instead of the existence of $C\mu$ a.e. on γ we only used the fact that the relation $C_\varepsilon\mu(\xi) = o\big(\mathscr{P}\mu(\xi,\varepsilon)\big)$ holds μ_s-a.e. on γ. This gives a slightly stronger version of Theorem 1.4 as mentioned in the introduction. The requirement that μ is positive is not crucial as well.

4.2. Radial inner functions. If $\mu\in M(\mathbf{T})$ we denote by $P\mu(z)$ and $Q\mu(z)$ its Poisson and conjugate Poisson integrals in the unit disk:

$$P\mu(z) = \int_{\mathbf{T}} \frac{1-|z|^2}{|z-\xi|^2}\,d\mu(\xi) \qquad\text{and}\qquad Q\mu(z) = \int_{\mathbf{T}} \frac{2\operatorname{Im} z\bar\xi}{|z-\xi|^2}\,d\mu(\xi).$$

Proof of Theorem 1.6. If θ is an inner function in the unit disk, consider the family of Clark measures, i.e. positive singular measures $\{\mu_\alpha\}_{\alpha\in\mathbf{T}}$ on $\mathbf{T}$ uniquely defined by the equation

$$P\mu_\alpha(z) = \operatorname{Re}\frac{\alpha+\theta}{\alpha-\theta}$$

(see, for instance, [2], [8] or [6] for more information on such families of measures). One of the basic properties of μ_α is the formula

$$\int \mu_\alpha(E)\,dm(\alpha) = m(E), \tag{15}$$

where E is any Borel subset of $\mathbf{T}$ and m is the normalized Lebesgue measure on $\mathbf{T}$, see [1]. In particular, a set of Lebesgue measure zero on $\mathbf{T}$ will have measure zero with respect to μ_α for a.e. $\alpha \in \mathbf{T}$. Hence if θ is radial near $\Sigma \subset \mathbf{T}$, then for a.e. $\alpha \in \mathbf{T}$, μ_α-a.e. radius that ends in Σ is mapped by θ into a curve, tangent to a radius. The definition of μ_α together with elementary calculations show that this property can be translated into the relation

$$Q\mu_\alpha(r\xi) = o\bigl(P\mu_\alpha(r\xi)\bigr)$$

as $r \to 1-$ for μ_α-a.e. $\xi \in \Sigma$. Hence, by Theorem 1.1 almost all μ_α are pure point on Σ. Another well-known property of the Clark measures says that $\mu_\alpha(\{\xi\}) > 0$ if and only if $\theta(\xi) = \alpha$ and θ has a non-tangential derivative at ξ (see, for instance, [8]). Utilizing again (15), one can see that θ has angular derivatives a.e. on $\mathbf{T}$. □

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