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INTEGRAL MEANS OF ANALYTIC FUNCTIONS

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Abstract. If $0 < p < \infty$ and f is an analytic function in the unit disc $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, we set, as usual,

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p}, \qquad 0 < r < 1.$$

Given $p \in (0,\infty)$, we let $\mathscr{F}_p$ denote the space of those functions f which are analytic in Δ and satisfy $M_p(r,f') = \mathrm{O}(1/(1-r))$, as $r \to 1$. In this paper we obtain sharp estimates on the growth of the integral means $M_p(r,f)$, $f \in \mathscr{F}_p$.

1. Introduction and main results

Let Δ denote the unit disc $\{z \in \mathbf{C} : |z| < 1\}$. If $0 < r < 1$ and g is an analytic function in Δ, we set

$$I_p(r,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^p d\theta, \qquad 0 < p < \infty,$$
$$M_p(r,g) = I_p(r,g)^{1/p}, \qquad 0 < p < \infty,$$
$$M_\infty(r,g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \le \infty$ the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$\|g\|_{H^p} = \sup_{0<r<1} M_p(r,g) < \infty.$$

We refer to [4] for the theory of Hardy spaces.

It is well known that there is a close relation between the integral means of an analytic function and those of its derivative. A classical result of Hardy and Littlewood [8], [9] (see Theorem 5.5 of [4]) asserts that if $0 < p \le \infty$, $\alpha > 1$ and f is an analytic function in Δ, then

$$M_p(r,f') = \mathrm{O}\left(\frac{1}{(1-r)^\alpha}\right), \qquad \text{as } r \to 1, \tag{1}$$

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if and only

$$M_p(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^{\alpha-1}}\right), \qquad \text{as } r \to 1. \tag{2}$$

If $0 < \alpha < 1$ and f satisfies (1) then it follows that $f \in H^p$ (see Theorem 5.1 and Theorem 5.4 of [4] for the case $1 \le p \le \infty$ and Remark 1 below for the case $0 < p < 1$).

Now it remains to consider the case $\alpha = 1$. Studying this case is the main object of this paper.

Applying the continuous form of Minkowski's inequality, in the case $1 \le p < \infty$, and simply integration of the derivative, in the case $p = \infty$, yield:

If $1 \le p \le \infty$ and f is an analytic function in Δ which satisfies

$$M_p(r,f') = \mathrm{O}\left(\frac{1}{1-r}\right), \qquad \text{as } r \to 1, \tag{3}$$

then

$$M_p(r,f) = \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \tag{4}$$

This result is certainly sharp for $p = \infty$ and for $p = 1$. Indeed:

(i) $f(z) = \log\bigl(1/(1-z)\bigr)$, $z \in \Delta$, satisfies

$$M_\infty(r,f') \sim \frac{1}{1-r} \qquad \text{and} \qquad M_\infty(r,f) \sim \log\frac{1}{1-r}, \qquad \text{as } r \to 1.$$

(ii) The function $f(z) = 1/(1-z)$, $z \in \Delta$, satisfies

$$M_1(r,f') \sim \frac{1}{1-r} \qquad \text{and} \qquad M_1(r,f) \sim \log\frac{1}{1-r}, \qquad \text{as } r \to 1.$$

However, in Theorem 1 and Theorem 2 we obtain better estimates for $1 < p < \infty$ and we also obtain sharp estimates in the case $0 < p < 1$.

Theorem 1. (a) *If $2 < p < \infty$ and f is an analytic function in Δ such that*

$$M_p(r,f') = \mathrm{O}\left(\frac{1}{1-r}\right), \qquad \text{as } r \to 1, \tag{5}$$

then

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \qquad \text{as } r \to 1,\ \text{for all } \beta > \frac{1}{2}. \tag{6}$$

(b) *Furthermore, this result is sharp in the sense that there exists a function f, analytic in Δ, which satisfies (5) for every $p \in (2,\infty)$ and such that*

$$M_p(r,f) \asymp \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right) \qquad \text{as } r \to 1, \text{ for every } p \in (2,\infty).$$

Theorem 2. (a) *If $0 < p \le 2$ and f is an analytic function in Δ which satisfies (5) then*

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1. \tag{7}$$

(b) *Furthermore, this result is sharp: For every $p \in (0,2]$ the function $f(z) = 1/(1-z)^{1/p}$, $z \in \Delta$, satisfies (5) and*

$$M_p(r,f) \asymp \left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1. \tag{8}$$

2. Proof of the main results

For simplicity, for $0 < p \le \infty$ we let $\mathscr{F}_p$ be the space of those functions f which are analytic in Δ and satisfy (3). Notice that $\mathscr{F}_\infty$ coincides with the space $\mathscr{B}$ of Bloch functions [1]. Also, since $M_p(r,g)$ increases with p, we have

$$\mathscr{B} = \mathscr{F}_\infty \subset \mathscr{F}_q \subset \mathscr{F}_p, \qquad 0 < p < q < \infty. \tag{9}$$

Let us recall that Clunie and MacGregor [3] and Makarov [11] proved the following result.

Theorem A. *If $f \in \mathscr{B}$ then*

$$M_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as } r \to 1,$$

for all $p \in (0,\infty)$.

Theorem A is sharp. Indeed, if

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}, \qquad z \in \Delta,$$

then $f \in \mathscr{B}$ (see e.g. Lemma 2.1 of [1]) and

$$M_p(r,f) \asymp \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as } r \to 1,$$

for all $p \in (0,\infty)$. Notice that, bearing in mind (9), this proves part (b) of Theorem 1.

Proof of Theorem 1(a). Take $p \in (2, \infty)$ and $f \in \mathscr{F}_p$. As noted above, we have

$$M_p(r,f) = \mathrm{O}\left(\log \frac{1}{1-r}\right), \qquad \text{as } r \to 1. \tag{10}$$

Using a result of Hardy [7] (see p. 126 of [12]), we see that

$$\frac{d}{dr}\left[rI_p'(r,f)\right] = \frac{p^2 r}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt,$$

which, since $I_p'(r,f) \geq 0$, implies

$$I_p''(r,f) \leq \frac{p^2}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt. \tag{11}$$

For $0 < r < 1$, we set

$$E_{1,1}(r) = \left\{ t \in [-\pi, \pi] : |f'(re^{it})| \leq \frac{|f(re^{it})|}{(1-r)\log \dfrac{1}{1-r}} \right\}, \tag{12}$$

and

$$E_{2,1}(r) = [-\pi, \pi] \setminus E_{1,1}(r). \tag{13}$$

Using (11), (12), (13) and bearing in mind that $p > 2$, we deduce that

$$\begin{aligned} I_p''(r,f) &\leq \frac{p^2}{2\pi} \int_{E_{1,1}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\quad + \frac{p^2}{2\pi} \int_{E_{2,1}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\leq p^2 \left[\frac{I_p(r,f)}{(1-r)^2 \left(\log \dfrac{1}{1-r}\right)^2} + \left((1-r)\log \frac{1}{1-r}\right)^{p-2} I_p(r,f') \right]. \end{aligned} \tag{14}$$

Using this, (5) and (10), we obtain

$$I_p''(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^2} \left(\log \frac{1}{1-r}\right)^{p-2} \right), \qquad \text{as } r \to 1,$$

which, integrating twice, gives

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p-1}\right), \qquad \text{as } r \to 1, \tag{15}$$

which is better than (10).

This process can be iterated. We define a sequence $\{\alpha_k\}_{k=1}^{\infty}$ inductively as follows:

$$\begin{cases} \alpha_1 = 1, \\ \alpha_{k+1} = \dfrac{1}{p}\left(p - 2\displaystyle\sum_{j=1}^{k}\left(\alpha_j - \frac{1}{2}\right)\right), \qquad k = 1, 2, \ldots. \end{cases} \tag{16}$$

We have

$$p\alpha_{k+1} = p\alpha_k - 2\left(\alpha_k - \tfrac{1}{2}\right) = (p-2)\alpha_k + 1, \qquad k \geq 1. \tag{17}$$

Arguing by induction we can easily see that

$$\alpha_k > \tfrac{1}{2}, \qquad k = 1, 2, \ldots, \tag{18}$$

which, together with (17), implies that the sequence $\{\alpha_k\}_{k=1}^{\infty}$ is decreasing and, hence, convergent. Using again (17), we deduce that

$$\lim_{k\to\infty} \alpha_k = \tfrac{1}{2}. \tag{19}$$

Let us remark that (15) can be written as

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_2}\right), \qquad \text{as } r \to 1.$$

We are going to prove inductively that

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_j}\right), \qquad \text{as } r \to 1, \tag{20}$$

for all j. Since $\alpha_j \downarrow \frac{1}{2}$, this will finish the proof.

We already know that (20) is true for $j = 1$ and $j = 2$. Suppose that it is true for $j = k$, that is, suppose that

$$I_p(r,f) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{p\alpha_k}\right), \qquad \text{as } r \to 1. \tag{21}$$

For $0 < r < 1$, we set

$$E_{1,k}(r) = \left\{ t \in [-\pi, \pi] : |f'(re^{it})| \leq \frac{|f(re^{it})|}{(1-r)\left(\log \frac{1}{1-r}\right)^{\alpha_k}} \right\}, \tag{22}$$

and

$$E_{2,k}(r) = [-\pi, \pi] \setminus E_{1,k}(r). \tag{23}$$

Using (11), (22) and (23), we obtain

$$\begin{aligned} I_p''(r,f) &\leq \frac{p^2}{2\pi} \int_{E_{1,k}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\quad + \frac{p^2}{2\pi} \int_{E_{2,k}(r)} |f(re^{it})|^{p-2} |f'(re^{it})|^2 \, dt \\ &\leq p^2 \left[\frac{I_p(r,f)}{(1-r)^2 \left(\log \frac{1}{1-r}\right)^{2\alpha_k}} \right. \\ &\quad \left. + (1-r)^{p-2} \left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k} I_p(r,f') \right]. \end{aligned} \tag{24}$$

Using (5), (21) and (24), we deduce that

$$I_p''(r,f) = \mathrm{O}\left(\frac{1}{(1-r)^2} \left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k} \right), \qquad \text{as } r \to 1,$$

and then, integrating twice and using (17), it follows that

$$\begin{aligned} I_p(r,f) &= \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{(p-2)\alpha_k + 1} \right) \\ &= \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{p\alpha_{k+1}} \right), \qquad \text{as } r \to 1. \end{aligned} \tag{25}$$

This is (20) for $j = k+1$. □

We do not know whether or not (6) can be substituted by

$$M_p(r,f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{1/2} \right), \qquad \text{as } r \to 1, \tag{26}$$

in Theorem 1(a). We know that any Bloch function satisfies (26) for any $p \in (2, \infty)$ but the question of determining whether or not the condition $f \in \mathscr{F}_p$, $2 < p < \infty$, is enough to conclude (26) remains open. It is worth noticing that no counterexample (if any) can be given by a power series with Hadamard gaps because of the following result.

Proposition 1. *Let f be an analytic function in Δ given by a power series with Hadamard gaps,*

$$f(z) = \sum_{j=1}^{\infty} a_j z^{n_j}, \qquad \text{with } \frac{n_{j+1}}{n_j} \geq \lambda > 1 \text{ for all } j\,. \tag{27}$$

Then the following conditions are equivalent.

(i) $f \in \mathscr{B}$.
(ii) $f \in \mathscr{F}_p$ *for some* $p \in (0, \infty)$.
(iii) $f \in \mathscr{F}_p$ *for all* $p \in (0, \infty)$.
(iv) $\sup_{j \geq 1} |a_j| < \infty$.

Proof. The implication (i) $\Rightarrow$ (ii) follows from (9). Bearing in mind that f' is also given by a power series with Hadamard gaps, (ii) $\Rightarrow$ (iii) follows using Theorem 8.20 in Chapter V of [14], Vol. I.

We turn now to prove that (iii) $\Rightarrow$ (iv). Hence, suppose that f is given by (27) and satisfies (iii). Then there exists a positive constant C such that

$$M_1(r, f') \leq C \frac{1}{1-r}, \qquad 0 < r < 1. \tag{28}$$

Now,

$$z f'(z) = \sum_{j=1}^{\infty} n_j a_j z^{n_j},$$

and then, using Cauchy's formula and (28), we obtain

$$n_j |a_j| = \left| \frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{z^{n_j}}\, dz \right| \leq \frac{M_1(r, f')}{r^{n_j - 1}} \leq \frac{C}{r^{n_j - 1}(1-r)}, \qquad 0 < r < 1.$$

Taking $r = 1 - 1/n_j$ we obtain (iv).

Finally, the implication (iv) $\Rightarrow$ (i) follows from Lemma 2.1 of [1]. □

Before embarking into the case $0 < p \leq 2$, let us introduce a new family of spaces. For $0 < p < \infty$, the space of Dirichlet type $\mathscr{D}^p_{p-1}$ consists of all functions f which are analytic in Δ and satisfy

$$\int_{\Delta} (1 - |z|^2)^{p-1} |f'(z)|^p \, dA(z) < \infty.$$

Here, $dA(z) = (1/\pi)\, dx\, dy$ denotes the normalized Lebesgue area measure in Δ. The spaces $\mathscr{D}^p_{p-1}$ are closely related to Hardy spaces. Indeed, a classical result of Littlewood and Paley [10] asserts that

$$H^p \subset \mathscr{D}^p_{p-1}, \qquad 2 \leq p < \infty. \tag{29}$$

On the other hand, we have

$$\mathscr{D}^p_{p-1} \subset H^p, \qquad 0 < p \le 2, \tag{30}$$

(see [10] and [13]). Notice that $\mathscr{D}^2_1 = H^2$. However, we remark that if $p \neq 2$ then $H^p \neq \mathscr{D}^p_{p-1}$. A number of results about the spaces $\mathscr{D}^p_{p-1}$ have been recently proved in [2] and [6]. In particular, estimates on the growth of the integral means of $\mathscr{D}^p_{p-1}$-functions have been obtained in [6]. We remark that

$$\mathscr{D}^p_{p-1} \subset \mathscr{F}_p, \qquad 0 < p < \infty. \tag{31}$$

Indeed, the condition $f \in \mathscr{D}^p_{p-1}$ is equivalent to saying that

$$\int_0^1 (1-r)^{p-1} I_p(r, f')\, dr < \infty, \tag{32}$$

and, taking into account that $I_p(r, f')$ is an increasing function of r, we easily see that (32) implies that $M_p(r, f') = \mathrm{o}\bigl(1/(1-r)\bigr)$, as $r \to 1$.

In view of (31), the results contained in this paper complement those of [6].

Our proof of Theorem 2 will be based on (30). Actually, we shall use the following result which follows from (30) by the closed graph theorem.

Theorem B. *If $0 < p \le 2$ then there exists a positive constant C_p, which only depends on p, such that*

$$\|f\|^p_{H^p} \le C_p \biggl(|f(0)|^p + \int_\Delta (1-|z|^2)^{p-1} |f'(z)|^p \, dA(z) \biggr), \tag{33}$$

for every $f \in \mathscr{D}^p_{p-1}$.

Proof of Theorem 2. Take $p \in (0, 2]$ and $f \in \mathscr{F}_p$. Assume, without loss of generality, that $f(0) = 0$. For $0 < r < 1$, set $f_r(z) = f(rz)$, $z \in \Delta$. In what follows we shall be using the convention that C will denote a positive constant which may depend on p and f but not on r or ϱ, and which is not necessarily the same at different occurrences. Applying (33) to f_r, $0 < r < 1$, and using that $f \in \mathscr{F}_p$, yield

$$\begin{aligned} I_p(r,f) &\le C \int_0^1 (1-\varrho)^{p-1} I_p(r\varrho, f')\, d\varrho \le C \int_0^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho \\ &= C \biggl(\int_0^r \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho + \int_r^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^p}\, d\varrho \biggr), \qquad 0 < r < 1. \end{aligned} \tag{34}$$

Since $r\varrho < \varrho$ and $r\varrho < r$, $0 < r, \varrho < 1$, (34) implies

$$\begin{aligned} I_p(r,f) &\le C \biggl(\int_0^r \frac{1}{1-\varrho}\, d\varrho + \frac{1}{(1-r)^p} \int_r^1 (1-\varrho)^{p-1}\, d\varrho \biggr) \\ &= \mathrm{O}\biggl(\log \frac{1}{1-r} \biggr), \qquad \text{as } r \to 1. \end{aligned} \tag{35}$$

This finishes the proof of part (a) of Theorem 2. Part (b) is clear. □

Remark 1. If $0 < p < 1$, $0 < \alpha < 1$, and f is an analytic function in Δ which satisfies (1) with $f(0) = 0$, then, arguing as in the proof of Theorem 2, we obtain

$$I_p(r, f) \le C \int_0^1 \frac{(1-\varrho)^{p-1}}{(1-r\varrho)^{\alpha p}}\, d\varrho \le C \int_0^1 (1-\varrho)^{p(1-\alpha)-1}\, d\varrho, \qquad 0 < r < 1,$$

and hence it follows that $f \in H^p$ as we asserted in Section 1.

3. Univalent functions in the classes $\mathscr{F}_p$

A complex-valued function defined in Δ is said to be univalent if it is analytic and one-to-one there. We refer to [5] and [12] for the theory of these functions. Throughout the paper, $\mathscr{U}$ will stand for the class of all univalent functions in Δ. The aim of this section is studying the growth of the integral means $M_p(r, f)$ of functions $f \in \mathscr{U} \cap \mathscr{F}_p$.

It is well known that $\mathscr{U} \subset H^p$, if $0 < p < \frac{1}{2}$ (see e.g. Theorem 3.16 of [4]). Hence, if $0 < p < \frac{1}{2}$ and $f \in \mathscr{U}$, then $M_p(r, f) = \mathrm{O}(1)$, as $r \to 1$. We can prove the following result for $p \ge \frac{1}{2}$.

Theorem 3. *If $\frac{1}{2} \le p < \infty$ and $f \in \mathscr{U} \cap \mathscr{F}_p$, then*

$$M_p(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r} \right)^{1/p} \right), \qquad \text{as } r \to 1.$$

The following proposition will be used in the proof of Theorem 3.

Proposition 2. *If $\frac{1}{2} \le p < \infty$ and $f \in \mathscr{F}_p$, then*

$$M_\infty(r, f) = \mathrm{O}\left(\left(\frac{1}{1-r} \right)^{1/p} \right), \qquad \text{as } r \to 1. \tag{36}$$

Proof. Arguing as in the proof of Theorem 5.9 of [4] and using Minkowski's inequality in continuous form (notice that $2p \ge 1$), we see that there exists a constant C which only depends on p such that

$$\begin{aligned} M_\infty(r, f) &\le C \frac{M_{2p}\left(\frac{1}{2}(1+r), f\right)}{(1-r)^{1/2p}} \\ &\le C \left[\frac{|f(0)| + \int_0^{(1+r)/2} M_{2p}(s, f')\, ds}{(1-r)^{1/2p}} \right], \qquad 0 < r < 1. \end{aligned} \tag{37}$$

Reasoning again as in the proof of Theorem 5.9 of [4] and using that $f \in \mathscr{F}_p$, we obtain that

$$M_{2p}(s, f') \le C \frac{M_p\left(\frac{1}{2}(1+s), f'\right)}{(1-s)^{1/2p}} \le C \frac{1}{(1-s)^{1+1/2p}}, \qquad 0 < s < 1. \tag{38}$$

Putting together (37) and (38) we deduce that

$$M_\infty(r,f) = \mathrm{O}\left(\frac{\int_0^{(1+r)/2}(1-s)^{-1-1/2p}\,ds}{(1-r)^{1/2p}}\right) = \mathrm{O}\left(\left(\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1.$$

This finishes the proof. □

Proof of Theorem 3. Let p and f be as in Theorem 3. Assume, without loss of generality that $f \in S$, that is $f(0) = 0$ and $f'(0) = 1$. It follows from (11) that

$$rI_p'(r,f) \le C_p M_\infty^p(r,f),$$

(see p. 127 of [12]) and then, using (36), we deduce that

$$\begin{aligned} I_p(r,f) &\le I_p\left(\frac{1}{2},f\right) + 2C_p\int_{1/2}^r M_\infty^p(s,f)\,ds \le I_p\left(\frac{1}{2},f\right) + 2C_p\int_{1/2}^r \frac{1}{1-s}\,ds \\ &= \mathrm{O}\left(\log\frac{1}{1-r}\right), \qquad \text{as } r \to 1. \ \square \end{aligned}$$

Remark 2. Notice that if $\frac{1}{2} \le p < \infty$ then the function

$$f(z) = 1/(1-z)^{1/p}, \qquad z \in \Delta,$$

belongs to $\mathscr{U} \cap \mathscr{F}_p$ and satisfies

$$M_p(r,f) \asymp \left(\left(\log\frac{1}{1-r}\right)^{1/p}\right), \qquad \text{as } r \to 1.$$

Hence, Theorem 3 is sharp.

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