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COMPOSITION OPERATORS BETWEEN WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS ON BANACH SPACES

Domingo García, Manuel Maestre, and Pablo Sevilla-Peris

Universidad de Valencia, Departamento de Análisis Matemático
C/ Doctor Moliner 50, E-46100 Burjasot (Valencia), Spain
Domingo.Garcia@uv.es; Manuel.Maestre@uv.es; Pablo.Sevilla@uv.es

Abstract. In this paper we study composition operators between weighted spaces of holomorphic functions defined on the open unit ball of a Banach space. Necessary and sufficient conditions are given for composition operators to be compact. We show that new phenomena appear in the infinite-dimensional setting different from the ones of the finite-dimensional one.

1. Weights. Weighted spaces

The starting idea of composition operators is a simple and very natural question. Consider $\mathbf{D}$ the open unit disc of $\mathbf{C}$ and a holomorphic map $\phi\colon \mathbf{D} \longrightarrow \mathbf{D}$. If $f\colon \mathbf{D} \longrightarrow \mathbf{C}$ is a holomorphic function, we can compose $f \circ \phi$ and try to analyze what happens when we let the f vary; in other words we define an operator between spaces of holomorphic functions and we want to study what properties does this operator have (continuity, compactness, ...). This obviously depends on which are the spaces considered. First candidates are the Hardy spaces and a full study of the situation in this case can be found in [18]. In the last few years a lot of research has been done studying the behavior of operators between weighted spaces of holomorphic functions $H_w(B)$ whenever B is the unit disk of $\mathbf{C}$ or, more in general, an open subset of $\mathbf{C}^n$ (see below for definitions and notation). Among the operators between these spaces particular attention has been paid to composition operators. We refer to [4], [5], [7], [8], [9], [10], [15], [19] and particularly to the recent surveys [3], [6] for information about the subject. But also some interest has been given to the more general case where X is a Banach space and B_X is its open unit ball (see e.g. [1], [2], [13], [14], [17]). In this paper, strongly influenced by the work of Bonet, Domański, Lindström and Taskinen [8], we study composition operators between $H_w(B_X)$ and $H_v(B_Y)$ and we find that new phenomena appear in the infinite-dimensional setting different from the ones of the finite-dimensional one. In Section 2 we make an introductory

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study of composition operators. In Section 3 we study the compactness of a composition operator giving necessary and sufficient conditions for such an operator to be compact. Finally, in Section 4 we show that Hilbert spaces are a natural setting to extend [8, Theorem 2.3], a result that gives conditions on the weight v such that all composition operators from $H_v(B_X)$ into itself are continuous.

We fix the notation to be used in the rest of the article. Let X be a complex Banach space and B_X its open unit ball. Any continuous bounded mapping $v: B_X \to]0,\infty[$ is called a *weight*.

We denote by $H(B_X)$ the space of all holomorphic functions $f: B_X \longrightarrow \mathbf{C}$. A set $A \subset B_X$ is said to be B_X-bounded if there exists $0 < r < 1$ such that $A \subset rB_X$. The subspace of $H(B_X)$ of those functions that are bounded on the B_X-bounded sets is denoted by $H_b(B_X)$.

Following [8] and [17] we consider

$$H_v(B_X) = \Big\{ f \in H(B_X) : \|f\|_v = \sup_{x\in B_X} v(x)|f(x)| < \infty \Big\}.$$

With the norm $\|\cdot\|_v$, the space $H_v(B_X)$ is a Banach space. We denote $\mathfrak{B}_v$ the closed unit ball of $H_v(B_X)$. It is well known that in $H_v(B_X)$ the τ_v (norm) topology is finer than the τ_0 (compact-open) topology ([17, Section 3]) and that $\mathfrak{B}_v$ is τ_0-compact ([17, p. 349]).

Following [4], [6], we say that a weight is *radial* if $v(\lambda x) = v(x)$ for every $\lambda \in \mathbf{C}$ with $|\lambda| = 1$ and every $x \in B_X$.

A weight v satisfies *Condition I* if $\inf_{x\in rB_X} v(x) > 0$ for every $0 < r < 1$ ([14]). If v satisfies Condition I, then $H_v(B_X) \subseteq H_b(B_X)$ ([14, Proposition 2]). If X is finite-dimensional, then all weights on B_X satisfy Condition I. From now on, unless otherwise stated, every weight is assumed to satisfy Condition I.

Given any weight v, following [5], we consider an associated growth condition $u: B_X \longrightarrow]0,+\infty[$ defined by $u(x) = 1/v(x)$. With this new function we can rewrite

$$\mathfrak{B}_v = \big\{ f \in H_v(B_X) : |f| \le u \big\}.$$

From this, $\tilde{u}: B_X \longrightarrow]0,+\infty[$ is defined by

$$\tilde{u}(x) = \sup_{f\in\mathfrak{B}_v} |f(x)|$$

and a new associated weight $\tilde{v} = 1/\tilde{u}$. All these functions were defined by Bierstedt, Bonet and Taskinen for open subsets of $\mathbf{C}^n$ in [5]. In [5, Proposition 1.2], the following relations between weights for open sets on $\mathbf{C}^n$ are proved. The same arguments work for the unit ball of a Banach space.

Proposition 1.1 *Let X be a Banach space and v a weight defined on B_X. The following hold*:

(i) $0 < v \leq \tilde{v}$ *and* $\tilde{v}$ *is bounded and continuous; i.e.,* $\tilde{v}$ *is a weight.*

(ii) $\tilde{u}$ *(respectively* $\tilde{v}$*) is radial and decreasing or increasing whenever* u *(respectively* v*) is so.*

(iii) $\|f\|_v \leq 1 \Leftrightarrow \|f\|_{\tilde{v}} \leq 1$.

(iv) *For each* $x \in B_X$ *there exists* $f_x \in \mathfrak{B}_v$ *such that* $\tilde{u}(x) = |f_x(x)|$.

As an immediate consequence of (iii) we have

Corollary 1.2 ([5, Observation 1.12]). *If* X *is a Banach space and* v *is any weight defined on* B_X*, then* $H_v(B_X) = H_{\tilde{v}}(B_X)$ *holds isometrically.*

Since the constant function 1 belongs to $H_v(B_X)$ we have

$$\sup_{x \in B_X} v(x) = \|1\|_v = \|1\|_{\tilde{v}} = \sup_{x \in B_X} \tilde{v}(x).$$

Definition 1.3 ([19]). A weight v is said to be *essential* if there exists $C > 0$ such that $v(x) \leq \tilde{v}(x) \leq C\, v(x)$ for all $x \in B_X$.

We say that a weight v is *norm-radial* if $v(x) = v(y)$ for every x, y such that $\|x\| = \|y\|$. If v is norm-radial and non-increasing (with respect to the norm) then $\tilde{v}$ is also norm-radial. Indeed, if v is such a weight and $T\colon X \to X$ is a linear mapping, $T \neq 0$, with $\|T\| \leq 1$, then for any $f \in H_v(B_X)$ we can consider $f_T = f \circ T$. Then

$$\|f_T\|_v = \sup_{z \in B_X} v(z)\big|f\big(T(z)\big)\big| \leq \sup_{z \in B_X} v\big(T(z)\big)\big|f\big(T(z)\big)\big| \leq \|f\|_v.$$

Hence, for any $x \in B_X$ we have $\sup_{\|f\|_v \leq 1} |f(x)| \geq \sup_{\|f\|_v \leq 1} |f(T(x))|$. Now if $y \in B_X$ with $\|x\| = \|y\|$ we can take T such that $T(x) = y$ to get that $\tilde{v}(x) \leq \tilde{v}(y)$. The converse inequality is proved in the same way.

Note that given a Banach space X such that for any two $x, y \in B_X$ with $\|x\| = \|y\|$ there exists a holomorphic isometry $T\colon B_X \longrightarrow B_X$ with $T(x) = y$ then any norm-radial weight v satisfies that $\tilde{v}$ is also norm-radial. This happens if X is a Hilbert space.

2. Composition operators

Let X, Y be Banach spaces. We denote by B_X, B_Y their open unit balls. Let $\phi\colon B_X \to B_Y$ be a holomorphic mapping. The *composition operator* associated to ϕ is defined by

$$C_\phi\colon H(B_Y) \longrightarrow H(B_X), \qquad f \rightsquigarrow C_\phi(f) = f \circ \phi.$$

C_ϕ is clearly linear and (τ_0, τ_0)-continuous. Given any two weights v, w we consider the restriction $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ whenever this is well defined. If $h\colon B_X \longrightarrow Y$ is bounded we denote as usual $\|h\|_\infty = \sup\big\{\|h(x)\| : \|x\| < 1\big\}$.

Remark 2.1. Let H, E be two Banach spaces of holomorphic functions whose topologies are stronger than the pointwise convergence topology. If $C_\phi\colon H \to E$ is well defined then, by the closed graph theorem, C_ϕ is continuous. *As a consequence, to find out if the composition operator C_ϕ is continuous it is enough to find out if C_ϕ is well defined.*

Proposition 2.2. *If there is some $0 < r < 1$ such that $\phi(B_X) \subseteq rB_Y$, then $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is well defined (and then continuous) for any two weights v with Condition* I *and w.*

Proof. Since $\phi(B_X) \subseteq rB_Y$, then for each $f \in H_v(B_Y)$ there is $K > 0$ such that $\sup_{y\in\phi(B_X)} |f(y)| \le K$. Hence

$$\sup_{x\in B_X} w(x)|C_\phi(f)(x)| = \sup_{x\in B_X} w(x)\big|f\big(\phi(x)\big)\big| \le \sup_{x\in B_X} w(x) \sup_{x\in B_X} \big|f\big(\phi(x)\big)\big| < \infty.$$

Therefore $C_\phi(f) \in H_w(B_X)$ and C_ϕ is well defined. □

The following proposition extends some of the results in [8, Proposition 2.1] (see also [7, Theorem 4]).

Proposition 2.3. *Let v, w be two weights satisfying Condition* I *and $\phi\colon B_X \longrightarrow B_Y$ holomorphic. Then the following are equivalent:*

(i) *$C_\phi\colon H_v(B_Y) \longrightarrow H_w(B_X)$ is well defined and continuous.*

(ii) $\sup_{x\in B_X}\big(w(x)/\tilde v(\phi(x))\big) < \infty$.

(iii) $\sup_{x\in B_X}\big(\widetilde w(x)/\tilde v(\phi(x))\big) < \infty$.

(iv) $\sup_{\|\phi(x)\|>r_0}\big(w(x)/\tilde v(\phi(x))\big) < \infty$ *for some $0 < r_0 < 1$.*

Proof. The implication (iii) $\Rightarrow$ (ii) is trivial, since $w \le \widetilde w$. Let us assume now (ii). Let $f \in H_v(B_Y)$; we have

$$w(x)\big|f\big(\phi(x)\big)\big| = \frac{w(x)}{\tilde v\big(\phi(x)\big)}\tilde v\big(\phi(x)\big)\big|f\big(\phi(x)\big)\big| \le M\|f\|_{\tilde v} = M\|f\|_v$$

for all x. Hence C_ϕ is continuous.

Suppose now that C_ϕ is continuous. If (iii) does not hold there exists $(x_n)_{n\in\mathbf{N}} \subseteq B_X$ such that

$$\lim_{n\to\infty}\frac{\widetilde w(x_n)}{\tilde v\big(\phi(x_n)\big)} = \infty.$$

For each $n \in \mathbf{N}$ we can take $f_n \in \mathfrak{B}_v$ so that $\big|f_n\big(\phi(x_n)\big)\big| = \tilde u\big(\phi(x_n)\big) = 1/\tilde v\big(\phi(x_n)\big)$. Hence

$$\big|f_n\big(\phi(x_n)\big)\big|\widetilde w(x_n) = \frac{\widetilde w(x_n)}{\tilde v\big(\phi(x_n)\big)}$$

which is a contradiction with the fact that $C_\phi(\mathfrak{B}_v)$ is bounded.

Clearly (ii) implies (iv). Conversely, if (iv) holds, let

$$M = \sup_{\|\phi(x)\| > r_0} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)}.$$

We take $x \in B_X$; if $\|\phi(x)\| > r_0$ then

$$w(x)\big|f\big(\phi(x)\big)\big| = \frac{w(x)}{\tilde{v}\big(\phi(x)\big)}\tilde{v}\big(\phi(x)\big)\big|f\big(\phi(x)\big)\big| \le M\|f\|_v.$$

If $\|\phi(x)\| \le r_0$, since f is bounded in $\overline{r_0 B_Y}$, we have

$$w(x)\big|f\big(\phi(x)\big)\big| \le \Big(\sup_{x \in B_X} w(x)\Big)\Big(\sup_{y \in \overline{r_0 B_Y}} |f(y)|\Big).$$

Joining both cases we have $\sup_{x \in B_X} w(x)\big|f\big(\phi(x)\big)\big| < \infty$ and $C_\phi(f) \in H_w(B_X)$ for all $f \in H_v(B_Y)$. By Remark 2.1, C_ϕ is continuous. □

Note that (i), (ii) and (iii) above are equivalent even if Condition I does not hold. On the other hand, as Example 2.4 below shows, Condition I is necessary to prove that (iv) implies (i).

Example 2.4. Let X be any infinite-dimensional Banach space and let $\phi(x) = x$ for every $x \in B_X$. By the Josefson–Nissenzweig theorem [11, Chapter XII] we can choose $(x_n^*)_n \subseteq X^*$ with $\|x_n^*\| = 2$ and weak-star converging to 0. We define $a(x) = 1 + \sum_{n=1}^{\infty} |x_n^*(x)|^n$ and $v(x) = 1/a(x)$.

For every $b > \frac{1}{2}$, $\sup_{\|x\|=b} a(x) = +\infty$; indeed, since $\|x_n^*\| = 2$, for each $n \ge 2$ there is $x_n \in X$ such that $\|x_n\| = 1$ and $|x_n^*(x_n)| > 2 - 1/n$. Let $y_n = bx_n$. Then

$$a(y_n) > |x_n^*(y_n)|^n = \big(b|x_n^*(x_n)|\big)^n > \left(b\left(2 - \frac{1}{n}\right)\right)^n$$

for all $n \ge 2$. Therefore $\sup_{\|x\|=b} a(x) = \infty$ for every $b > \frac{1}{2}$; hence $\inf_{\|x\|=b} v(x) = 0$ and v does not satisfy Condition I. We fix $\frac{1}{2} < c < d < \frac{3}{4}$ and we consider continuous mappings φ, ψ: $[0,1] \to [0,1]$ such that

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \le c, \\ > 0 & \text{if } c < |t| < \frac{3}{4}, \\ 0 & \text{if } \frac{3}{4} \le |t|, \end{cases} \qquad \psi(t) = \begin{cases} 0 & \text{if } |t| \le d, \\ > 0 & \text{if } d < |t| < \frac{3}{4}, \\ 1 & \text{if } \frac{3}{4} \le |t|. \end{cases}$$

We define now

$$w(x) = \psi(\|x\|)\frac{1}{a(x)} + \varphi(\|x\|).$$

Clearly, if $\|x\| \leq c$, then $w(x) = 1$ and if $\|x\| \geq \frac{3}{4}$ then $w(x) = 1/a(x)$. Hence, for each $b \geq \frac{3}{4}$ we have

$$\sup_{\|\phi(x)\|>b} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} \leq \sup_{\|\phi(x)\|>b} \frac{w(x)}{v\big(\phi(x)\big)} = \sup_{\|x\|>b} \frac{w(x)}{v(x)} = 1.$$

Let us see that $C_\phi\colon H_v(B_X) \to H_w(B_X)$ is not well defined. We have that $f(x) = \sum_{n=1}^{\infty} x_n^*(x)^n$ is an entire function on X (see ([12, p. 157]) and

$$\|f\|_v = \sup_{x \in B_X} \frac{|\sum_{n=1}^{\infty} x_n^*(x)^n|}{1 + \sum_{n=1}^{\infty} |x_n^*(x)|^n} \leq \sup_{x \in B_X} \frac{\sum_{n=1}^{\infty} |x_n^*(x)|^n}{1 + \sum_{n=1}^{\infty} |x_n^*(x)|^n} < 1.$$

Hence $f \in H_v(B_X)$. But, by the maximum modulus theorem and the Cauchy inequality, if $\frac{1}{2} < b < c$ then

$$\sup_{\|x\|=b} w(x)|f(x)| = \sup_{\|x\|=b} |f(x)| = \sup_{\|x\| \leq b} |f(x)| \geq \sup_{\|x\| \leq b} \left| \sum_{n=1}^{\infty} x_n^*(x)^n \right| = \infty.$$

This obviously implies $C_\phi f = f \notin H_w(B_X)$ and C_ϕ is not well defined.

3. Compactness

We now study conditions for the operators C_ϕ to be compact. The proof of the next result is easily adapted from those of [18, Section 2.4] and [8, Lemma 3.1] and will be used several times.

Lemma 3.1. *Let $C_\phi\colon H_v(B_Y) \longrightarrow H_w(B_X)$ be continuous. Then the following are equivalent*:

(i) *C_ϕ is compact.*

(ii) *Each bounded sequence $(f_n)_n \subseteq H_v(B_Y)$ such that $f_n \xrightarrow{\tau_0} 0$ satisfies that $\|C_\phi f_n\|_w \longrightarrow 0$.*

Many authors, when working in the finite-dimensional setting, consider weights with the property that $\lim_{\|x\| \to 1^-} w(x) = 0$ (e.g. $w(x) = g(\|x\|)$ with $g\colon [0,1] \to [0, +\infty[$ continuous and non-increasing such that $g(0) = 1$ and $g(1) = 0$, [15], [10]). This kind of weights are also considered in [8]. A characterization of compactness is given in [8, Theorem 3.3]. Strongly inspired by that we have the following.

Proposition 3.2. *Let v, w be weights with $\lim_{\|x\| \to 1^-} w(x) = 0$ and $\phi\colon B_X \longrightarrow B_Y$. Then, $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is compact if and only if*

$$\lim_{\|x\| \to 1^-} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} = 0 \qquad (1)$$

and

(2) $\phi(rB_X)$ *is relatively compact for every* $0 < r < 1$.

Proof. Let us begin by assuming that C_ϕ is compact. Suppose that there is r_0 such that $\phi(r_0 B_X)$ is not relatively compact, then there is $(x_n)_n \subseteq r_0 B_X$ and $\varepsilon > 0$ with $\|\phi(x_n) - \phi(x_m)\| > \varepsilon$ for every $n \neq m$. For each pair (n,m), $n \neq m$, we choose $y^*_{nm} \in Y^*$ with $\|y^*_{nm}\| = 1$ such that

$$\left|y^*_{nm}\big(\phi(x_n)\big) - y^*_{nm}\big(\phi(x_m)\big)\right| \geq \varepsilon.$$

We have $(y^*_{nm}) \subset H_v(B_Y)$ and $\|y^*_{nm}\|_v \leq \|v\|_\infty$ for all $n \neq m$.

The adjoint operator of C_ϕ, C^t_ϕ: $H_w(B_X)^* \to H_v(B_Y)^*$ is also compact. For each $x \in B_X$ we denote by δ_x the evaluation functional. We denote by $\|\cdot\|^*_w$ the dual norm in $H_w(B_X)^*$; that is $\|\gamma\|^*_w = \sup_{\|f\|_w \leq 1} |\gamma(f)|$ for $\gamma \in H_w(B_X)^*$. Clearly $\|\delta_x\|^*_w = 1/\widetilde{w}(x)$. Since w satisfies Condition I so does $\widetilde{w}$ and $\{\delta_x : x \in r_0 B_X\}$ is bounded in $H_w(B_X)^*$. Then $\{C^t_\phi(\delta_x) : x \in r_0 B_X\} = \{\delta_{\phi(x)} : x \in r_0 B_X\}$ is relatively compact in $H_v(B_Y)^*$. On the other hand,

$$\varepsilon \leq \left|\delta_{\phi(x_n)}(y^*_{nm}) - \delta_{\phi(x_m)}(y^*_{nm})\right| \leq \|\delta_{\phi(x_n)} - \delta_{\phi(x_m)}\|^*_v \|y^*_{nm}\|_v$$

for every $n \neq m$. Hence, for all $n \neq m$, $\|\delta_{\phi(x_n)} - \delta_{\phi(x_m)}\|^*_v \geq \varepsilon/\|v\|_\infty$. This is a contradiction.

Let us suppose that $w(x)/\tilde{v}(\phi(x))$ does not converge to 0 when $\|x\| \to 1^-$. Then there is a sequence $(x_n)_n \subseteq B_X$ with $\lim_n \|x_n\| = 1$ and $c > 0$ such that $w(x_n) \geq c\tilde{v}\big(\phi(x_n)\big)$ for all $n \in \mathbf{N}$. Using Proposition 1.1, for each $n \in \mathbf{N}$ we can choose $f_n \in \mathfrak{B}_v$ such that $\left|f_n\big(\phi(x_n)\big)\right| = 1/\tilde{v}\big(\phi(x_n)\big)$.

Suppose that there exists $0 < r_0 < 1$ such that $\|\phi(x_n)\| \leq r_0$ for every n. Since v satisfies Condition I, so does $\tilde{v}$ and $M = \inf_{y \in r_0 B_Y} \tilde{v}(y) > 0$. Then

$$w(x_n) \geq c\tilde{v}\big(\phi(x_n)\big) \geq cM > 0.$$

But this contradicts the fact that $\lim_n w(x_n) = 0$. Hence we can extract a subsequence of $(x_n)_n$ (that we denote the same) so that $\lim_n \|\phi(x_n)\| = 1$.

We can assume that $\|\phi(x_n)\| > \sqrt[n]{1 - 1/n}$ for every n. We choose $y^*_n \in Y^*$ with $\|y^*_n\| = 1$ such that $\left|y^*_n\big(\phi(x_n)\big)\right| > \sqrt[n]{1 - 1/n}$ and we define $g_n(y) = y^*_n(y)^n f_n(y)$, for all $y \in B_Y$. We have

$$\sup_{y \in B_Y} v(y)|y^*_n(y)|^n |f_n(y)| \leq \sup_{y \in B_Y} v(y)\|y\|^n |f_n(y)| \leq \|f_n\|_v \leq 1.$$

Hence $(g_n)_n \subseteq H_v(B_Y)$ and it is bounded. Since $\mathfrak{B}_v$ is τ_0-bounded ([17]), given any compact set $K \subseteq B_Y$ there exists $M > 0$ such that $\sup_{y \in K} |f_n(y)| \leq M$ for all $n \in \mathbf{N}$. Since K is compact, $K \subseteq rB_Y$ for some $0 < r < 1$; hence

$$\sup_{y \in K} |g_n(y)| = \sup_{y \in K} |y^*_n(y)|^n |f_n(y)| \leq M \sup_{y \in K} \|y\|^n \leq M\ r^n.$$

Thus, $(g_n)_n \subseteq H_v(B_Y)$ is bounded and τ_0 convergent to 0. By Lemma 3.1, the sequence $\|C_\phi(g_n)\|_w$ must tend to 0. On the other hand we have, for every $n \in \mathbf{N}$,

$$\|C_\phi(g_n)\|_w \geq w(x_n)\big|g_n\big(\phi(x_n)\big)\big| = \frac{w(x_n)}{\tilde{v}\big(\phi(x_n)\big)}\big|y_n^*\big(\phi(x_n)\big)\big|^n > c\frac{n-1}{n}.$$

This gives a contradiction and implies (1).

Assume now that (1) holds and $\phi(rB_X)$ is relatively compact for every r. We begin by showing that C_ϕ is continuous. By hypothesis there is $0 < r_0 < 1$ such that

$$\sup_{\|x\|>r_0} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} \leq 1.$$

Since $\phi(r_0B_X)$ is relatively compact there is $M > 0$ such that $0 < M \leq \tilde{v}\big(\phi(x)\big)$ for all $\|x\| \leq r_0$. Therefore

$$\sup_{\|x\|\leq r_0} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} \leq \frac{1}{M} \sup_{\|x\|\leq r_0} w(x) < \infty.$$

This gives

$$\sup_{x\in B_X} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} < \infty$$

and C_ϕ is continuous.

Let us suppose that C_ϕ is not compact. From Lemma 3.1, there is a τ_0-null sequence $(f_n)_n \subseteq \mathfrak{B}_v$ such that $\big(\|C_\phi(f_n)\|_w\big)_n$ does not converge to 0. Going to a subsequence if necessary we can assume that there is $\lambda > 0$ such that

$$\sup_{x\in B_X} w(x)\big|f_n\big(\phi(x)\big)\big| = \|C_\phi(f_n)\|_w > \lambda > 0$$

for all $n \in \mathbf{N}$. We choose $(x_n)_n \subseteq B_X$ with $w(x_n)\big|f_n\big(\phi(x_n)\big)\big| \geq \lambda$ for all n and let us suppose that $(x_n)_n$ has a subsequence $(x_{n_k})_k$ such that $\lim_k \|x_{n_k}\| = 1$. Given any $\varepsilon > 0$ there is k_1 such that

$$w(x_{n_k}) \leq \varepsilon\tilde{v}\big(\phi(x_{n_k})\big)$$

for all $k \geq k_1$. Hence

$$\lambda \leq w(x_{n_k})\big|f_{n_k}\big(\phi(x_{n_k})\big)\big| \leq \varepsilon\tilde{v}\big(\phi(x_{n_k})\big)\big|f_{n_k}\big(\phi(x_{n_k})\big)\big| \leq \varepsilon\|f_{n_k}\|_v < \varepsilon.$$

This contradicts the fact that $\lambda > 0$. Therefore there exists $0 < s < 1$ such that $\|x_n\| \leq s$ for every n. Since $\phi\big((x_n)_n\big) \subseteq \phi(sB_X)$ which is relatively compact, given $\varepsilon > 0$ and $M = \sup_{x\in B_X} w(x)$, there exists n_2 such that, for $n \geq n_2$

$$\sup_{y\in\phi(sB_X)} |f_n(y)| < \frac{\varepsilon}{M}.$$

Therefore $\big|f_n\big(\phi(x_n)\big)\big| < \varepsilon/M$ for all $n \geq n_2$ and

$$\lambda \leq w(x_n)\big|f_n\big(\phi(x_n)\big)\big| < \varepsilon.$$

This again gives a contradiction and finally shows that C_ϕ is compact. □

Nevertheless many weights do not satisfy this condition on the limit (see [5], [8]). We are now interested in the study of the compactness of C_ϕ with general weights. So far, two different situations have been considered. First, the finite-dimensional case with general weights was studied in [8]. In this case the condition that $\overline{\phi(rB_X)}$ is compact and contained in B_Y is trivial. For the infinite-dimensional case, only composition operators between $H^\infty(B_Y)$ and $H^\infty(B_X)$ (i.e. $v(x) = w(x) = 1$) have been studied in [1] and [13]. There, it is proved that $C_\phi\colon H^\infty(B_Y) \to H^\infty(B_X)$ is compact if and only if $\phi(B_X) \subseteq sB_Y$ for some $0 < s < 1$ and $\phi(B_X)$ is relatively compact.

In [8, Theorem 3.3] a characterization of the compactness of a composition operator is obtained for general weights when $X = Y = \mathbf{C}$. This characterization is given in terms of an analytical condition (see (3) below). Proposition 3.2 shows that some topological condition is also needed if we want to have a characterization whenever X and Y are general Banach spaces.

Theorem 3.3. *Let v, w be weights with Condition* I *and $\phi\colon B_X \to B_Y$ a holomorphic mapping. Then the following hold.*

(a) *If $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is compact then $\phi(rB_X)$ is relatively compact for every $0 < r < 1$.*

(b) *Suppose that $\|\phi\|_\infty < 1$. If $\phi(B_X)$ is relatively compact, then $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is compact.*

(c) *Suppose that $\|\phi\|_\infty = 1$.* (i) *If $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is compact, then*

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} = 0. \tag{3}$$

(ii) *If $\phi(B_X) \cap rB_Y$ is relatively compact for every $0 < r < 1$ and*

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} = 0$$

then $C_\phi\colon H_v(B_Y) \to H_w(B_X)$ is compact.

Proof. (a) Note that in Proposition 3.2 when we proved that if C_ϕ is compact then $\phi(rB_X)$ is relatively compact for every $0 < r < 1$ we did not use the fact that $\lim_{\|x\|\to 1^-} w(x) = 0$. Therefore, this implication remains true for any weight w.

(b) By assumption there is $0 < s < 1$ such that $\phi(B_X) \subseteq sB_Y$. If $\phi(B_X)$ is relatively compact then

$$\sup_{x\in B_X} \frac{1}{\tilde{v}\big(\phi(x)\big)} \le \sup_{y\in\overline{\phi(B_X)}} \frac{1}{\tilde{v}(y)} < \infty.$$

Hence

$$\sup_{x\in B_X} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} < \infty.$$

By Proposition 2.3, C_ϕ is continuous.

Let $(f_n)_n \subseteq H_v(B_Y)$ be bounded and τ_0-convergent to 0. We take $\varepsilon > 0$. Let us write $M = \sup_{x\in B_X} w(x) < \infty$. We choose $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$,

$$\sup_{y\in\phi(B_X)} |f_n(y)| < \frac{\varepsilon}{M}.$$

Hence, for $n \geq n_0$, $\|C_\phi f_n\|_w = \sup_{x\in B_X} w(x)\big|f_n\big(\phi(x)\big)\big| \leq M \sup_{y\in\phi(B_X)} |f_n(y)| < \varepsilon$. By Lemma 3.1, C_ϕ is compact.

(c) Let us suppose now that $\|\phi\|_\infty = 1$. Let C_ϕ be compact and assume that

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} \neq 0.$$

So we can find $(r_n)_n \subseteq]0,1[$ with $\lim_n r_n = 1$ and $c > 0$ so that, for all $n \in \mathbf{N}$,

$$\sup_{\|\phi(x)\|>r_n} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} > c.$$

From this we get a sequence $(x_n)_n \subseteq B_X$ with $\|\phi(x_n)\| > r_n$ and $w(x_n) \geq c\tilde{v}\big(\phi(x_n)\big)$ for all $n \in \mathbf{N}$. Without loss of generality we can assume that $r_n > \sqrt[n]{1-1/n}$. Applying Proposition 1.1, for each $n \in \mathbf{N}$, we can choose $f_n \in \mathfrak{B}_v$ satisfying $\big|f_n\big(\phi(x_n)\big)\big| = 1/\tilde{v}\big(\phi(x_n)\big)$. We take $y_n^* \in Y^*$ such that $\|y_n^*\| = 1$ and $\big|y_n^*\big(\phi(x_n)\big)\big| > r_n$ and we define $g_n(y) = y_n^*(y)^n f_n(y)$. Proceeding now as in Proposition 3.2 we obtain the contradiction we are looking for. Hence $\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} w(x)/\tilde{v}\big(\phi(x)\big) = 0$.

Now we assume that (3) holds and $\phi(B_X) \cap rB_Y$ is relatively compact for every $0 < r < 1$. By (3), given $\varepsilon > 0$, there is r_0 such that, for every $r_0 < r < 1$,

$$\sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} < \varepsilon. \tag{4}$$

By Proposition 2.3(iv), C_ϕ is continuous. From (4), $w(x) < \varepsilon\tilde{v}\big(\phi(x)\big)$ for all $\|\phi(x)\| > r_0$. Suppose that C_ϕ is not compact. By Lemma 3.1 there exists $(f_n)_n \subseteq \mathfrak{B}_v$ τ_0-convergent to 0 such that $(\|C_\phi f_n\|_w)_n$ does not converge to 0. Going to a subsequence if necessary, we can find $\lambda > 0$ such that $\|C_\phi f_n\|_w > \lambda$ for all n. Let $(x_n)_n \subseteq B_X$ with $w(x_n)\big|f_n\big(\phi(x_n)\big)\big| \geq \lambda$ for all n. If $(x_n)_n$ has a subsequence $(x_{n_k})_k$ such that $\lim_k \|\phi(x_{n_k})\| = 1$, then there exists $k_1 \in \mathbf{N}$ with $\|\phi(x_{n_k})\| > r_0$ for all $k \geq k_1$. So, for $k \geq k_1$, $w(x_{n_k}) < \varepsilon\tilde{v}\big(\phi(x_{n_k})\big)$. Therefore

$$\lambda \leq w(x_{n_k})\big|f_{n_k}\big(\phi(x_{n_k})\big)\big| < \varepsilon\tilde{v}\big(\phi(x_{n_k})\big)\big|f_{n_k}\big(\phi(x_{n_k})\big)\big| \leq \varepsilon\|f_{n_k}\|_{\tilde{v}} = \varepsilon\|f_{n_k}\|_v \leq \varepsilon.$$

Hence $\lambda \leq \varepsilon$ for every $\varepsilon > 0$. This leads to a contradiction.

Thus there exists $0 < s < 1$ satisfying $\|\phi(x_n)\| < s$ for all n. So, $\big(\phi(x_n)\big)_n \subseteq \phi(B_X) \cap sB_Y$ which is relatively compact. Let $M = \sup_{x \in B_X} w(x)$, given any $\varepsilon > 0$ there is $n_2 \in \mathbf{N}$ such that for all $n \geq n_2$

$$\sup_{y \in \phi(B_X) \cap sB_Y} |f_n(y)| < \frac{\varepsilon}{M}.$$

Hence, if $n \geq n_2$, then $\big|f_n\big(\phi(x_n)\big)\big| < \varepsilon/M$. As a consequence, if $n \geq n_2$ we have $\lambda \leq w(x_n)\big|f_n\big(\phi(x_n)\big)\big| < \varepsilon$. Thus $\lambda \leq \varepsilon$ for all $\varepsilon > 0$. This leads to a contradiction that shows that C_ϕ is compact. □

If we want to get better results for part (b) of above theorem we need to add conditions on the weight w.

Proposition 3.4. *Let v be a weight on B_Y and ϕ: $B_X \to B_Y$ a holomorphic mapping. Then C_ϕ: $H_v(B_Y) \to H^\infty(B_X)$ is compact if and only if $\phi(B_X)$ is relatively compact and $\|\phi\|_\infty < 1$.*

Proof. If C_ϕ is compact, we take the canonical injection i: $H^\infty(B_Y) \to H_v(B_Y)$. Composing $i \circ C_\phi$ we get a compact composition operator from $H^\infty(B_Y)$ into $H^\infty(B_X)$. By [1, Proposition 3], $\phi(B_X)$ is relatively compact and $\|\phi\|_\infty < 1$. The other implication is a particular case of Theorem 3.3(b). □

Corollary 3.5 *Let v, w be weights such that w is norm-radial and ϕ: $B_X \to B_Y$ a holomorphic mapping.*

(a) *If $w(x)$ converges to 0 as $\|x\| \to 1-$ then C_ϕ: $H_v(B_Y) \to H_w(B_X)$ is compact if and only if $\phi(rB_X)$ is relatively compact for every $0 < r < 1$ and $\lim_{\|x\| \to 1-} w(x)/\tilde{v}\big(\phi(x)\big) = 0$.*

(b) *If $w(x)$ does not converge to 0 as $\|x\| \to 1-$ then C_ϕ: $H_v(B_Y) \to H_w(B_X)$ is compact if and only if $\phi(B_X)$ is relatively compact and $\|\phi\|_\infty < 1$.*

Proof. Part (a) is a particular case of Proposition 3.2.

If $w(x)$ does not converge to 0 as $\|x\| \to 1-$ then there exist $\varepsilon > 0$ and a sequence $(r_n) \subset (0,1)$ convergent to 1 such that $w(x) > \varepsilon$ for all $x \in X$ with $\|x\| = r_n$ and all $n \in \mathbf{N}$. Given $x \in B_X$ we consider n such that $\|x\| < r_n$, by the maximum modulus theorem, we have

$$|f(x)| \leq \max_{|\lambda| = r_n^{-1}} |f(\lambda x)| \leq \frac{1}{\varepsilon} \max_{|\lambda| = r_n^{-1}} w(\lambda x)|f(\lambda x)| \leq \frac{1}{\varepsilon}\|f\|_w.$$

Then $\|f\|_\infty \leq (1/\varepsilon)\|f\|_w \leq (1/\varepsilon)\|w\|_\infty \|f\|_\infty$ for all $f \in H_w(B_X)$. Thus $H_w(B_X)$ and $H^\infty(B_X)$ coincide algebraically and topologically. Now Proposition 3.4 gives the conclusion. □

Let us point out that if Y is finite-dimensional then, trivially, $\phi(B_X)$ is always relatively compact and hence we have the following corollary, which is exactly [8, Theorem 3.3] whenever $X = Y = \mathbf{C}$.

Corollary 3.6. *Let Y be a finite-dimensional Banach space and X a complex Banach space. Let v, w be weights and ϕ: $B_X \to B_Y$ a holomorphic mapping.*

(a) *If $\|\phi\|_\infty < 1$, then C_ϕ: $H_v(B_Y) \to H_w(B_X)$ is compact.*

(b) *If $\|\phi\|_\infty = 1$, then C_ϕ: $H_v(B_Y) \to H_w(B_X)$ is compact if and only if*

$$\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} \frac{w(x)}{\tilde{v}\big(\phi(x)\big)} = 0.$$

After these corollaries it is natural to ask if the converse of (a), (b) and (c)(i) or (c)(ii) in Theorem 3.3 hold in general. The following two examples show that the answer is in the negative in all cases.

Example 3.7. There is a holomorphic mapping ϕ: $B_{l_p} \to B_{l_p}$ and weights v, w on B_{l_p} so that $\|\phi\|_\infty = 1$, ϕ satisfies condition (3) and $\phi(rB_{l_p})$ is relatively compact for $0 < r < 1$, but C_ϕ is not compact $H_v(B_{l_p}) \to H_w(B_{l_p})$. In addition, here $\phi(B_{l_p}) \cap rB_{l_p}$ is not relatively compact for any $0 < r < 1$. This shows that the converse of (c)(i) in Theorem 3.3 does hold in general. Take $X = Y = l_p$ with $1 < p < \infty$ and define ϕ: $B_{l_p} \to B_{l_p}$ by $\phi\big((x_n)_n\big) = (x_n^n)_n$. This is a holomorphic mapping such that $\phi(B_{l_p})$ is not relatively compact but $\phi(rB_{l_p})$ is so for every $0 < r < 1$. Take $(x_n)_n \subseteq l_p$ such that $\|x_n\|_p \le r$ for every $n \in \mathbf{N}$. A standard diagonal method allows us to obtain a subsequence $(x_{n_m})_m$ of $(x_n)_n$ such that $\big(x_{n_m}(k)\big)_m$ converges for every k.

The sequence $\big(\phi(x_{n_m})\big)_m$ converges in l_p. Indeed, as $\phi(x_{n_m}) = \big(x_{n_m}(k)^k\big)_k$ and $|x_{n_m}(k)^k| \le r^k$ for every k and m, given $\varepsilon > 0$, we can choose k_0 such that

$$\Big(\sum_{k=k_0+1}^{\infty} r^{kp}\Big)^{1/p} < \frac{\varepsilon}{4}.$$

We denote, for each m, $y_m = \big(x_{n_m}(k)^k\big)_{k\le k_0}$ and $z_m = \big(x_{n_m}(k)^k\big)_{k>k_0}$. We have a pointwise convergent sequence $(y_m)_m$ in $\mathbf{C}^{k_0}$, thus it converges in the $\|\cdot\|_p$-norm of $\mathbf{C}^{k_0}$. Let m_0 be such that $\|y_{m_1} - y_{m_2}\|_{l_p^{k_0}} < \frac{1}{2}\varepsilon$ for every $m_1, m_2 \ge m_0$. Thus

$$\begin{aligned}\|\phi(x_{n_{m_1}}) - \phi(x_{n_{m_2}})\|_{l_p}^p &= \|y_{m_1} - y_{m_2}\|_{l_p^{k_0}}^p + \|z_{m_1} - z_{m_2}\|_{l_p}^p \\ &< \Big(\frac{\varepsilon}{2}\Big)^p + \sum_{k=k_0}^{\infty} (r^k + r^k)^p < \varepsilon^p.\end{aligned}$$

Hence $\big(\phi(x_{n_m})\big)_m$ is convergent and $\phi(rB_{l_p})$ is relatively compact.

We define $v(x) = 1 - \|x\|$ and $w(x) = \big(1 - \|\phi(x)\|\big)^2$. We have $\|\phi\|_\infty = 1$ and

$$\frac{w(x)}{\tilde{v}\big(\phi(x)\big)} \le \frac{w(x)}{v\big(\phi(x)\big)} = 1 - \|\phi(x)\|,$$

hence $\lim_{r\to 1^-}\sup_{\|\phi(x)\|>r} w(x)/\tilde{v}\big(\phi(x)\big)=0$. We denote by $(e_n^*)_n$ the canonical basis of l_q. The sequence $(e_n^*)_n$ is clearly bounded in $H_v(B_{l_p})$. On the other hand, for every $n\neq m$,

$$\begin{aligned}\|C_\phi e_n^* - C_\phi e_m^*\|_w &= \sup_{x\in B_{l_p}} w(x)\big|e_n^*\big(\phi(x)\big)-e_m^*\big(\phi(x)\big)\big|\\ &\geq w\Big(\frac{1}{\sqrt[n]{2}}e_n\Big)\Big|e_n^*\Big(\phi\Big(\frac{1}{\sqrt[n]{2}}e_n\Big)\Big)-e_m^*\Big(\phi\Big(\frac{1}{\sqrt[n]{2}}e_n\Big)\Big)\Big|\\ &= \big(1-\big\|\tfrac12 e_n\big\|\big)^2\big|e_n^*\big(\tfrac12 e_n\big)-e_m^*\big(\tfrac12 e_n\big)\big| = \tfrac18.\end{aligned}$$

Hence $(C_\phi e_n^*)_n$ does not have any convergent subsequence and the operator is not compact. Now, as a consequence of Theorem 3.3(c)(ii), there exists $0<r<1$ such that $\phi(B_{l_p})\cap rB_{l_p}$ is not relatively compact. Actually $\phi(B_{l_p})\cap rB_{l_p}$ is not relatively compact for any $0<r<1$. Indeed, fix $0<r<1$ and let $x_n=\sqrt[n]{r/2}\,e_n$. Then $\|\phi(x_n)\|=\big\|\frac12 re_n\big\|=\frac12 r$ and $\big(\phi(x_n)\big)_n\subseteq\phi(B_{l_p})\cap rB_{l_p}$. On the other hand, for every $n\neq m$,

$$\|\phi(x_n)-\phi(x_m)\|=\big\|\tfrac12 re_n-\tfrac12 re_m\big\|=2^{(1-p)/p}r.$$

Example 3.8. Let $1<p<\infty$. We give now a holomorphic mapping $\phi\colon B_{l_p}\to B_{l_p}$ with $\|\phi\|_\infty=1$ and weights v, w on B_{l_p} satisfying condition (3) such that $C_\phi\colon H_v(B_{l_p})\to H_w(B_{l_p})$ is compact but $\phi(B_X)\cap rB_X$ is not relatively compact for any $0<r<1$. Let $\phi\colon B_{l_p}\to B_{l_p}$ be defined by $\phi\big((x_n)_n\big)=(x_n^n)_n$. By Example 3.7 we have that ϕ is a holomorphic mapping such that $\phi(rB_X)$ is relatively compact for every $0<r<1$ but $\phi(B_X)\cap rB_X$ is not. We take the weights $v(x)=1-\|x\|$ and $w(x)=(1-\|x\|)\big(1-\|\phi(x)\|\big)$. Clearly both v and w satisfy Condition I and

$$\frac{w(x)}{v\big(\phi(x)\big)}=1-\|x\|.$$

This tends to 0 as $\|x\|\to 1^-$. Applying Proposition 3.2, $C_\phi\colon H_v(B_X)\to H_w(B_X)$ is compact.

There is a holomorphic mapping $\phi\colon B_{l_p}\to B_{l_p}$ with $\|\phi\|_\infty<1$ and weights v, w on B_{l_p} so that $C_\phi\colon H_v(B_{l_p})\to H_w(B_{l_p})$ is compact even though $\phi(B_{l_p})$ is not relatively compact. Indeed, we define $\phi\colon B_{l_p}\to B_{l_p}$ by $\phi\big((x_n)_n\big)=2^{-1}(x_n^n)_n$, we have that $\|\phi\|_\infty=2^{-1}$, $\phi(B_X)$ is not relatively compact and, for the weights $v(x)=1-\|x\|$ and $w(x)=(1-\|x\|)\big(1-\|\phi(x)\|\big)$, C_ϕ is compact. Hence, whenever $\|\phi\|_\infty<1$, the hypothesis of being $\phi(B_X)$ relatively compact is a sufficient but not necessary condition for C_ϕ to be compact.

Remark 3.9. If $\phi\colon B_X\to B_X$ is holomorphic and satisfies that $\phi(B_X)\cap rB_X$ is relatively compact for every $0<r<1$ then $\phi(rB_X)$ is relatively compact for every $0<r<1$. Indeed, if $\|\phi\|_\infty<1$ then our assumption implies that

$\phi(B_X)$ is relatively compact. If $\|\phi\|_\infty = 1$ then we define the weights $v(x) = 1 - \|x\|$ and $w(x) = \big(1 - \|\phi(x)\|\big)^2$. Clearly, $\lim_{r\to 1^-} \sup_{\|\phi(x)\|>r} w(x)/v\big(\phi(x)\big) = 0$. Applying Theorem 3.3(c)(ii) we have that C_ϕ: $H_v(B_X) \to H_w(B_X)$ is compact. By Theorem 3.3(a) we obtain the claim.

An open problem is the following. If we assume the analytical condition (3), does there exist an intermediate topological condition between the ones in Theorem 3.3(c)(i) and (ii) that give a characterization of the compactness of C_ϕ?

Finally, the next example shows that to assume $\phi(B_X) \cap rB_X$ to be relatively compact for every $0 < r < 1$ is a strictly weaker condition than to assume $\phi(B_X)$ to be relatively compact.

Example 3.10. We give now an example of a holomorphic mapping ψ: $B_{l_p} \to B_{l_p}$ such that $\psi(B_{l_p}) \cap rB_{l_p}$ is relatively compact for every $0 < r < 1$ but $\psi(B_{l_p})$ is not. Take l_p with $1 \le p < \infty$ and define ψ by $\psi\big((x_n)_n\big) = \big(x_1, (x_1^n x_n)_{n\ge 2}\big)$. Clearly $\psi(B_{l_p})$ is not relatively compact. On the other hand, $\psi(B_{l_p}) \cap rB_{l_p}$ is relatively compact for every $0 < r < 1$. Its proof is analogous to the one given in Example 3.7 since $(x_1^n x_n) \in \psi(B_{l_p}) \cap rB_{l_p}$ implies that $|x_1| < r$ and, from this, $|x_1^n x_n| < r^n$ for every $n \in \mathbf{N}$.

In Remark 3.9 we have obtained a purely topological result by using weights and composition operators in the case $\|\phi\|_\infty = 1$. A strengthening of this topological result can nevertheless be obtained directly simply by adapting to our setting some known results for entire mappings due to Aron and Schottenlocher ([2]). We present here those adapted results.

Definition 3.11. A holomorphic mapping f: $B_X \to Y$ is called *compact* in $x \in B_X$ if there is a neighborhood of x, V_x, such that $f(V_x)$ is relatively compact in Y. A mapping f is said to be compact if it is compact in x for every $x \in B_X$.

If f is a holomorphic mapping in an open set U and $x \in U$, we denote by $\sum_{n=0}^\infty P^n f(x)$ the Taylor series expansion of f at x. The next lemma was obtained for entire functions by Aron and Schottenlocher in [2]. Their proof, except for trivial natural changes, works also for holomorphic functions on any balanced and convex open set U.

Lemma 3.12 ([2, Proposition 3.4]). *Let f: $U \to Y$ be a holomorphic function. The following conditions are equivalent.*

(i) *f is compact.*

(ii) *For all $x \in U$ and all $n \in \mathbf{N}$, $P^n f(x)$ is compact.*

(iii) *For all $n \in \mathbf{N}$, $P^n f(0)$ is compact.*

(iv) *There is a 0-neighborhood V_0 in U such that $f(V_0)$ is relatively compact.*

Proposition 3.13. *Let X be a Banach space and f: $B_X \to B_Y$ a compact holomorphic mapping. Then $f(rB_X)$ is relatively compact for every $0 < r < 1$.*

Proof. Given r, we choose $1/r > s > 1$. By the maximum modulus principle we have, for each $y \in rB_X$,

$$\|P^n f(0)(y)\| \leq \frac{1}{s^n}\|f\|_{rsB_X}.$$

This implies that $\sum_{n=0}^{\infty} P^n f(0)$ converges uniformly and absolutely on rB_X.

Given $\varepsilon > 0$ let k_0 be such that

$$\sum_{n=k_0+1}^{\infty} \frac{1}{s^n} < \frac{\varepsilon}{3\|f\|_{rsB_X}}.$$

Let $g_k := \sum_{n=0}^{k} P^n f(0)$; we have

$$\|f(y) - g_{k_0}(y)\| \leq \sum_{n=k_0+1}^{\infty} \|P^n f(0)(y)\| \leq \sum_{n=k_0+1}^{\infty} \frac{1}{s^n}\|f\|_{rsB_X} < \frac{\varepsilon}{3}$$

for every $y \in rB_X$. Now, as g_{k_0} is a compact polynomial, $g_{k_0}(rB_X)$ is relatively compact. Thus, there are $\{y_1, \ldots, y_m\}$ such that for each $y \in rB_X$, there exists y_{j_0} satisfying $\|g_{k_0}(y) - g_{k_0}(y_{j_0})\| < \frac{1}{3}\varepsilon$; hence

$$\|f(y) - g_{k_0}(y_{j_0})\| \leq \|f(y) - g_{k_0}(y)\| + \|g_{k_0}(y) - g_{k_0}(y_{j_0})\| < \varepsilon,$$

i.e. $f(rB_X)$ is a precompact set. □

4. A result for Hilbert spaces

The result in [8, Theorem 2.3] gives conditions on a weight v such that all composition operators from $H_v(\mathbf{D})$ into itself are continuous. The proof of this is based on the Schwarz lemma and on the decomposition of every holomorphic mapping ϕ from $\mathbf{D}$ into $\mathbf{D}$ as $\phi = \psi \circ \alpha_p$, where $\psi \in H(\mathbf{D}, \mathbf{D})$ with $\psi(0) = 0$ and α_p is a Möbius transform. This cannot be repeated for functions defined on the unit ball of an arbitrary Banach space. However, Renaud showed in [16] that a very close situation holds for Hilbert spaces.

Let B_H be the open unit ball of a Hilbert space H with a scalar product $(\cdot \mid \cdot)$. For each $a \in B_H$ a linear mapping $\Gamma(a)\colon B_H \longrightarrow B_H$ is defined by

$$\Gamma(a)(x) = \frac{1}{1+\nu(a)}(x \mid a)a + \nu(a)x,$$

where $\nu(a) = \sqrt{1 - \|a\|^2}$. Using this mapping an automorphism of B_H, $\alpha_a\colon B_H \to B_H$ is defined as

$$\alpha_a(x) = \Gamma(a)\left(\frac{x-a}{1-(x \mid a)}\right).$$

These are the Möbius transforms for Hilbert spaces defined by Renaud in [16], where a deep study can be found. Each one of them is holomorphic, and satisfies $\alpha_a(a)=0,\ \alpha_a(0)=-a,\ \alpha_a^{-1}=\alpha_{-a}$.

The following version of the Schwarz lemma for Banach spaces is well known. It is proved by applying the classical Schwarz lemma to the family of functions $\big\{\big[\lambda\mapsto x^*\circ f(\lambda x/\|x\|)\big] : x^*\in X^*,\ \|x^*\|\le 1,\ 0<\|x\|<1\big\}$.

Let X, Y be two Banach spaces and B_X, B_Y their open unit balls. Let $f\colon B_X\longrightarrow B_Y$ be holomorphic such that $f(0)=0$. Then for all $x\in B_X$

$$\|f(x)\|_Y\le\|x\|_X.$$

Let $h\colon[0,1]\to]0,\infty[$ be continuous and non-increasing. Given any Banach space X a weight v can be defined on B_X by putting $v(x)=h(\|x\|)$. Note that a weight defined in this way is clearly norm-radial. With such weights we have the following result.

Theorem 4.1. *Let X be any Banach space and H a Hilbert space. Let $h\colon[0,1]\to]0,\infty[$ be continuous and non-increasing and consider the weights defined by h on B_X and B_H, both denoted by v. Then the following are equivalent:*

(i) *$C_\phi\colon H_v(B_H)\longrightarrow H_v(B_X)$ is continuous for all holomorphic $\phi\colon B_X\to B_H$.*

(ii) *Each $(x_n)_{n\in\mathbf{N}}\subseteq B_H$ such that $\|x_n\|=1-2^{-n}$ satisfies*

$$\inf_{n\in\mathbf{N}}\frac{\tilde v(x_{n+1})}{\tilde v(x_n)}>0. \tag{5}$$

Proof. We will follow the pattern of the proof given in [8, Theorem 2.3] with the natural modifications using now Renaud's Möbius mappings and a suitable version of Schwarz' lemma. We present the details for the sake of completeness.

First of all if $\phi(0)=0$, by the general version of the Schwarz lemma, we have $\|\phi(x)\|_H\le\|x\|_X$ and C_ϕ is continuous. Now for each $a\in B_H$ we have $\alpha_a\colon B_H\to B_H$. If every C_{α_a} is continuous then all C_ϕ are continuous. Indeed, given ϕ, let $a=\phi(0)$ and define $\psi=\alpha_a\circ\phi$. Then $\psi(0)=0$ and $C_\phi=C_\psi\circ C_{\alpha_{-a}}$ is continuous. Therefore it is enough to prove that $C_{\alpha_a}\colon H_v(B_H)\to H_v(B_H)$ is continuous for all $a\in B_H$ if and only if v satisfies (5).

Let us begin by assuming that all C_{α_a} are continuous. By Proposition 2.3, for each $a\in B_H$ we can find $M_a>0$ such that $\tilde v(x)\le M_a\tilde v\big(\alpha_a(x)\big)$ for all $x\in B_H$. We also know that

$$\sup_{\|x\|=r}\|\alpha_a(x)\|=\frac{\|a\|+r}{1+r\|a\|}$$

and it is attained at

$$x_0=\frac{-r}{\|a\|}a$$

(see [16, (9′)]). Since v is norm-radial and non-increasing so is also $\tilde v$ and

$$\tilde v(x) \le M_a \tilde v\left(\alpha_a\left(\frac{-r}{\|a\|}a\right)\right)$$

for every $x \in B_H$ with $\|x\| = r$. We define a new function by $l(r) = \tilde v(x)$ with $\|x\| = 1 - r$. Let now $s = 1 - r$, then for $s < \frac{1}{2}$ we have

$$l\left(s\frac{1-\|a\|}{1+\|a\|}\right) \le l\left(1 - \frac{\|a\| + (1-s)}{1+(1-s)\|a\|}\right) \le l\left(s\frac{1-\|a\|}{1+\frac{1}{2}\|a\|}\right). \tag{6}$$

Taking $\|a\| = \frac{2}{5}$ and using the second inequality in (6) we can find M_a and $s_0 > 0$ such that $l(s) \le M_a l\left(\frac{1}{2}s\right)$ for $0 < s < s_0$. This implies (5).

Let us assume now that (5) holds. We define a function l exactly in the same way as we did before. Then there are $M > 0$ and $0 < t_0 \le \frac{1}{2}$ such that $l(t) \le M l\left(\frac{1}{2}t\right)$ for all $t < t_0$. Given any $c > 0$ we can choose $n \in \mathbf{N}$ with $c < 2^n$. If $t < t_0$, then $l(t) \le M^n l(t/c)$. Take $c = (1 + \|a\|)/(1 - \|a\|)$ and use the first inequality in (6) to get that for each $a \in B_H$ there exists $K_a > 0$ such that

$$l(t) \le K_a l\left(1 - \frac{\|a\| + (1-t)}{1+(1-t)\|a\|}\right)$$

for $t < t_0$. With this, for any fixed $a \in B_H$, we can find a constant $M_a > 0$ such that for $0 < r < 1$ and $\|x\| = r$,

$$\tilde v(x) \le M_a l\left(1 - \frac{\|a\| + r}{1 + r\|a\|}\right) \le M_a \tilde v\big(\alpha_a(x)\big).$$

Applying Proposition 2.3, C_{α_a} is continuous. □

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