Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 383–406

DISTORTION OF THE EXPONENT OF CONVERGENCE IN SPACE

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Abstract. In this paper we introduce and develop properties of the chordal exponent of convergence for the Poincaré series of a quasiconformal group acting discontinuously in $\overline{\mathbf{R}}^n$ so that we can establish effective bounds on the distortion of this exponent of convergence under quasiconformal conjugacy. We also relate this exponent of convergence to a geometric variant of the standard exponent of convergence, and in doing so we are able to extend previous results to the full class of discrete quasiconformal groups.

1. Introduction and main results

In our paper [BTT2] we analyze the distortion of the exponent of convergence of a discrete quasiconformal group under quasiconformal conjugacy in dimension 2. It is the purpose of this paper to generalize these results to dimensions $n > 2$, and also to generalize the class of discrete quasiconformal groups to which the analysis applies.

Recall that a *discrete K-quasiconformal group G* acting on $\overline{\mathbf{R}}^n$ is a discrete group of homeomorphisms of $\overline{\mathbf{R}}^n$, endowed with the chordal metric, each of which is a K-quasiconformal mapping. A discrete 1-quasiconformal group is called a *Kleinian group.*

We have been investigating the relationship between the Hausdorff dimension of the limit set and the exponent of convergence for the class of discrete quasiconformal groups in [BTT1], [BTT2], [BTT3], and [ABT]. Central to our considerations is that it is not known whether in dimensions $n \geq 3$ a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ possesses an extension to a *group* action preserving $\mathbf{H}^{n+1}$. Thus the standard definition of the exponent of convergence, as used in the study of Kleinian groups, must be adapted to our uses.

2000 Mathematics Subject Classification: Primary 30C62, 30F40, 30F45.

The first author was supported in part by NSF grants DMS 0070335 and DMS 0305704. The second author was supported in part by NSF grant DMS 0305634. The third author was supported in part by NSF grant DMS 0305704.

For discrete quasiconformal groups acting on $\overline{\mathbf{R}}^n$ having regular set one may define an exponent of convergence solely via the group's discontinuous action on its regular set by

$$\delta_{\mathrm{chord}}(G) = \inf\Big\{ s > 0 \;\Big|\; \sum_{g\in G} \mathrm{dist}_{\mathrm{chord}}\big(g(z_0), \Lambda(G)\big)^s < \infty \Big\} \tag{1.1}$$

for any $z_0 \in \Omega(G)$ (see Definition 2.6). We will fully develop the properties of this exponent of convergence. This is the optimal definition for the distortion problem.

If the limit set is all of $\overline{\mathbf{R}}^n$ this definition will no longer work. We show that every discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ has an extension to a *quasiconformal hyperbolic action* on $\mathbf{H}^{n+1}$. The point is that this action does not necessarily possess a group structure. However, by exploiting the discreteness of the underlying group on $\overline{\mathbf{R}}^n$, we show that the extended action is in fact geometrically tractable. In particular, it allows us to define a variant of the usual hyperbolic exponent of convergence $\delta_{\mathrm{hyp}}(G)$ (see Definition 2.4). We observe that the properties known from the Kleinian case mainly remain unchanged (see Lemma 2.5).

Both approaches to the exponent of convergence are quite natural. In fact, we show in Theorem 4.2 that for a non-elementary discrete quasiconformal group with non-empty regular set the hyperbolic exponent of convergence agrees with the chordal exponent:

Theorem. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ having non-empty regular set $\Omega(G) \subset \overline{\mathbf{R}}^n$ and so that $|\Lambda(G)| \geq 2$. Then*

$$\delta_{\mathrm{hyp}}(G) = \delta_{\mathrm{chord}}(G).$$

Note that this theorem has been established in the Kleinian group setting by Bishop and Jones [BJ2].

Remark. The assumption that $|\Lambda(G)| \geq 2$ is necessary in the above theorem as for example the group $\langle z \mapsto z+1 \rangle$, acting on $\mathbf{H}^2$, has hyperbolic exponent $\frac{1}{2}$ and chordal exponent 1.

The relationship between the conical limit set and the exponent of convergence of a discrete quasiconformal group is more complicated than for a Kleinian group. We show in [BTT1] that the Hausdorff dimension of the conical limit set of a discrete quasiconformal group acting on $\mathbf{H}^{n+1}$ (and extended naturally to $\overline{\mathbf{R}}^n$) is bounded above by its exponent of convergence, but the exponent of convergence can be *strictly larger* than the Hausdorff dimension of the conical limit set (see Example 4.1 in [BTT1].) We show that we can remove the assumption that the group act on $\mathbf{H}^{n+1}$ and only consider discrete quasiconformal groups acting on $\overline{\mathbf{R}}^n$ (see Theorem 5.1):

Theorem. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$. Then* $\delta(G) \geq \dim \Lambda_c(G)$. (Here, $\delta(G)$ is to be properly interpreted according to context, see the discussion in the beginning of Section 5.)

This result drives the question that is the central motivation for undertaking the project described in this paper: *Is there an upper bound on the exponent of convergence in terms of the Hausdorff dimension of the conical limit set and the quasiconformal dilatation of the group*? (See Conjecture 6.1 and in dimension $n = 2$, [BTT2] and [BTT4].) In this paper we restrict our focus to the easier analysis of the distortion of the exponent of convergence under quasiconformal conjugacy. The central result in this paper is Theorem 5.3.

Main Theorem. *For each $n \geq 2$ and each $K \geq 1$ there exits a constant $c > 0$, depending only on n and K, such that the following holds: Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set. Let $\varphi\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal homeomorphism, and set $H = \varphi G \varphi^{-1}$. Then*

$$\delta_{\text{chord}}(H) \leq \frac{(n+c)\delta_{\text{chord}}(G)}{c + \delta_{\text{chord}}(G)}.$$

The constant c comes from a theorem of Gehring (Theorem 5.6) on the integrability of the Jacobian of a quasiconformal mapping. In dimension 2 Astala proved [As] that $c = 2/(K-1)$, in higher dimensions it is conjectured that $c = n/(K^{1/(n-1)} - 1)$. In dimension 2 we proved the above result [BTT2] under the additional assumption that $\Lambda(G)$ be uniformly perfect.

Finally, we consider discrete quasiconformal groups acting on $\overline{\mathbf{R}}^n$ having a purely conical limit set and non-empty regular set. In our paper [ABT] we prove that *if the group has an extension to a group action preserving $\mathbf{H}^{n+1}$ then it has the Sullivan–Tukia property*, i.e. the exponent of convergence and the Hausdorff dimension of its limit set are both strictly less than n. We provide a new proof that removes the assumption that the group extend to $\mathbf{H}^{n+1}$ (see Theorem 5.5):

Theorem. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$, having non-empty regular set and having a purely conical limit set $\Lambda(G)$. Then* $\dim \Lambda(G) \leq \delta_{\text{chord}}(G) < n$.

2. Basic facts concerning discrete groups and their exponents of convergence

In this section we will compile a list of pertinent facts concerning discrete group actions on $\overline{\mathbf{R}}^n$. Recall that Gehring and Martin [GM] observed that discrete K-quasiconformal groups are in fact a sub-class of a larger category of discrete groups called *discrete convergence groups*. In particular, a discrete quasiconformal group G has the *convergence property*, i.e. for every sequence in G there exists a subsequence $\{g_j\}$ and two (not necessarily distinct) points $a, b \in \overline{\mathbf{R}}^n$ so that

$\{g_j(x)\}$ converges to a locally uniformly in $x \in \overline{\mathbf{R}}^n \setminus \{b\}$ and $\{g_j^{-1}(y)\}$ converges to b locally uniformly in $y \in \overline{\mathbf{R}}^n \setminus \{a\}$. Quasiconformal groups, and indeed the more general class of convergence groups, share many of the basic properties of Kleinian groups, e.g. the notion of a *limit set* $\Lambda(G)$ and a *regular set* $\Omega(G)$, the dynamical classification of group elements, etc. The *conical limit set* is the set of all points of approximation of G, i.e. the set of all points $x \in \overline{\mathbf{R}}^n$ for which there exists a sequence $\{g_k\}$ in G and two distinct points $a, b \in \overline{\mathbf{R}}^n$ so that $g_k(x) \to a$, and $g_k(y) \to b$ for all $y \in \overline{\mathbf{R}}^n \setminus \{x\}$. (See also Maskit [Mas].) For the basics on the dynamical action of a convergence group see [GM]; for an introduction to the dynamic and geometric properties of Kleinian groups see [Mas].

Recall that it is not known whether every discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ extends to a *group* action preserving $\mathbf{H}^{n+1}$. It is, however, known that each K-quasiconformal mapping of $\overline{\mathbf{R}}^n$ extends (non-uniquely) to a K'-quasiconformal mapping preserving $\mathbf{H}^{n+1}$ [TV], where $K' = K'(n, K)$ only depends on the dimension n and the dilatation K.

Definition 2.1. A *K-quasiconformal hyperbolic action* G' on $\mathbf{H}^{n+1}$ is a collection of K-quasiconformal homeomorphisms preserving $\mathbf{H}^{n+1}$ so that:

(1) The collection of extensions of the elements G' to $\overline{\mathbf{R}}^n$ forms a discrete quasiconformal group G on $\overline{\mathbf{R}}^n$.

(2) The elements of G' and G correspond to each other in a one-to-one manner.

Thus we have:

Lemma 2.2. *A discrete quasiconformal group G acting on $\overline{\mathbf{R}}^n$ has an associated quasiconformal hyperbolic action G' on $\mathbf{H}^{n+1}$.*

In fact, G has associated to it *many* quasiconformal hyperbolic actions. However, each acts nicely on $\mathbf{H}^{n+1}$:

Lemma 2.3. *Each quasiconformal hyperbolic action acts discontinuously on $\mathbf{H}^{n+1}$.*

Proof of Lemma 2.3. Let G' be a quasiconformal hyperbolic action on $\mathbf{H}^{n+1}$ with associated boundary group G. The proof that G' acts discontinuously everywhere on $\mathbf{H}^{n+1}$ is by contradiction: assume that there exists a point $x_0 \in \mathbf{H}^{n+1}$, a small neighborhood U of x_0, and a sequence of distinct elements $\{g_j\} \in G'$ so that $g_j(U) \cap U \neq \emptyset$ for all j.

We consider the corresponding sequence $\{g_j\} \in G$. Then by the convergence property there exist points $a, b \in \overline{\mathbf{R}}^n$ and a subsequence $\{g_{j_k}\} \subseteq \{g_j\}$ so that $g_{j_k}(\,\cdot\,) \to a$ uniformly on compact subsets of $\overline{\mathbf{R}}^n \setminus \{b\}$. Choose a bi-infinite hyperbolic geodesic $\beta \in \mathbf{H}^{n+1}$ through x_0, so that both of the endpoints $\{c, d\}$ of β do not lie in the collection $\{a, b\}$. Then $g_{j_k}(c) \to a$ and $g_{j_k}(d) \to a$ as $k \to \infty$. Furthermore, since g_{j_k} is K-quasiconformal, we know that $g_{j_k}(\beta)$ is contained in a bounded hyperbolic neighborhood of the hyperbolic geodesic with endpoints $g_{j_k}(c)$ and $g_{j_k}(d)$ (see Lemma 3.4.2 in [E]). Thus we can conclude that for large enough k the image $g_{j_k}(\beta)$ lies in an arbitrarily small $(n+1)$-ball centered at a, and this implies that $g_{j_k}(U)\cap U=\emptyset$ for large enough k, a contradiction. □

Thus we can define a Poincaré series and an exponent of convergence δ_{hyp} that shares many of the properties that are known in the Kleinian case.

Definition 2.4. Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$, extend G to a quasiconformal hyperbolic action G' on $\mathbf{H}^{n+1}$. Then the s-*geometric Poincaré series* is

$$\sum_{g\in G'} e^{-s\varrho(x,g(y))}.$$

Furthermore, the *hyperbolic exponent of convergence* of G is

$$\delta_{\mathrm{hyp}}(G)=\inf\biggl\{s>0\;\bigg|\;\sum_{g\in G'} e^{-s\varrho(x,g(y))}<\infty\biggr\}.$$

The following lemma validates the previous definition:

Lemma 2.5. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$. Then the value of $\delta_{\mathrm{hyp}}(G)$ only depends on G, but not on the particular extension action G'. Furthermore, $\delta_{\mathrm{hyp}}(G)$ is also independent of the choice of base points x and y, and $0\le\delta_{\mathrm{hyp}}(G)\le n$.*

The proof of this lemma is a slight modification of the proofs of Theorem 3.3 and Lemma 2.3 in [BTT1]. Under additional assumptions we will sharpen this lemma in Theorem 5.5 and Corollary 5.2.

In considering distortion questions our analysis involves the area distortion of sets in $\overline{\mathbf{R}}^n$ under quasiconformal mappings. The optimal bounds are achieved in dimension n and so we are obliged to ignore the possible existence of an extension to $\mathbf{H}^{n+1}$ and develop an exponent of convergence in terms of the chordal metric q.

Definition 2.6. Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set. Let $z_0\in\Omega(G)$. Then

$$\delta_{\mathrm{chord}}(G)=\inf\biggl\{s>0\;\bigg|\;\sum_{g\in G}\mathrm{dist}_{\mathrm{chord}}\bigl(g(z_0),\Lambda(G)\bigr)^s<\infty\biggr\}$$

is the *chordal exponent of convergence* of G.

The value of the chordal exponent of convergence does not depend on the choice of $z_0\in\Omega(G)$:

Lemma 2.7. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set, and let z_0, w_0 be two points in $\Omega(G)$. Then there exists a constant C such that*

$$\begin{aligned}\frac{\operatorname{dist}_{\mathrm{chord}}\big(g(w_0),\Lambda(G)\big)}{C} &\le \operatorname{dist}_{\mathrm{chord}}\big(g(z_0),\Lambda(G)\big)\\ &\le C\operatorname{dist}_{\mathrm{chord}}\big(g(w_0),\Lambda(G)\big)\end{aligned} \tag{2.1}$$

holds for all $g\in G$. The constant C does not depend on $g\in G$ (but it does depend on z_0, w_0).

An immediate corollary is the above-mentioned base point independence for $\delta_{\mathrm{chord}}(G)$.

Corollary 2.8. *The chordal exponent of convergence is independent of the choice of the base point $z_0\in\Omega(G)$.*

Proof of Lemma 2.7. By conjugation with an isometry of the chordal metric we can assume without loss of generality that $\infty\in\Lambda(G)$. Observe that each $g\in G$ extends to a quasiconformal mapping of $\mathbf{H}^{n+1}$, by reflection we can then extend g to a quasiconformal mapping g' of $\overline{\mathbf{R}^{n+1}}$. By [TV] we can extend each $g\in G$ in this way and keep the dilatation of all g' bounded by, say, K. Note that the collection $\{g'\mid g\in G\}$ may no longer have a group structure. Let $W=\overline{\mathbf{R}^{n+1}}\setminus\Lambda(G)\subset\mathbf{R}^{n+1}$. Then each g' keeps W invariant. (In this proof we have to pass from $\Omega(G)$ to W to ensure that we have a *connected* open set to work with. If $\Omega(G)$ is connected then the passage to W is not necessary.)

Let now z_0 and w_0 be two points in $\Omega(G)$. Let H be the subset of all $g\in G$ so that

$$\begin{aligned}\operatorname{dist}_{\mathrm{chord}}\big(g(z_0),\Lambda(G)\big) &\le \operatorname{diam}_{\mathrm{chord}}\big(\Lambda(G)\big)/10 \quad\text{and}\\ \operatorname{dist}_{\mathrm{chord}}\big(g(w_0),\Lambda(G)\big) &\le \operatorname{diam}_{\mathrm{chord}}\big(\Lambda(G)\big)/10.\end{aligned}$$

Then $G\setminus H$ is finite. Let $g\in H$. Let $z^*\in\Lambda(G)$ be a point such that $\operatorname{dist}_{\mathrm{chord}}\big(g(z_0),\Lambda(G)\big)=q\big(g(z_0),z^*\big)$ (recall that q denotes the chordal metric). Let $z^{**}\in\Lambda(G)$ be a point that has chordal distance at least $\operatorname{diam}_{\mathrm{chord}}(\Lambda(G))/2$ from z^*. Using a chordal isometry φ we map z^{**} to ∞. Then

$$\operatorname{dist}_{\mathrm{chord}}(\varphi(z^*),\infty)=\operatorname{dist}_{\mathrm{chord}}(z^*,z^{**})\ge\operatorname{diam}_{\mathrm{chord}}\big(\Lambda(G)\big)/2, \tag{2.2}$$

and

$$\begin{aligned}\operatorname{dist}_{\mathrm{chord}}\big(\varphi\big(g(z_0)\big),\infty\big) &\ge \operatorname{dist}_{\mathrm{chord}}\big(\varphi(z^*),\infty\big)-q\big(\varphi(z^*),\varphi\big(g(z_0)\big)\big)\\ &\ge\big(\tfrac12-\tfrac1{10}\big)\operatorname{diam}_{\mathrm{chord}}\big(\Lambda(G)\big).\end{aligned} \tag{2.3}$$

Observe that since φ is a Möbius transformation, it extends to W. Furthermore, $W' = \varphi\big(g'(W)\big) = \varphi(W) \subset \mathbf{R}^{n+1}$ since $\infty \in \varphi(\Lambda(G))$. An application of Theorem 11.30 in [Vu] (here we need that W is a connected open subset of $\mathbf{R}^{n+1}$) to the mapping $\varphi \circ g'\colon W \to W'$ yields that

$$\frac{\big|\varphi\big(g(z_0)\big) - \varphi\big(g(w_0)\big)\big|}{\min\big\{\operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(z_0)\big), \varphi\big(\Lambda(G)\big)\big), \operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(w_0)\big), \varphi\big(\Lambda(G)\big)\big)\big\}} \\ \le M_{n,K}\left(\frac{|z_0 - w_0|}{\min\big\{\operatorname{dist}_{\mathrm{Euc}}\big(z_0, \Lambda(G)\big), \operatorname{dist}_{\mathrm{Euc}}\big(w_0, \Lambda(G)\big)\big\}}\right) =: M,$$

where $M_{n,K}\colon [0,\infty) \to [0,\infty)$ is a homeomorphism that only depends on n and K. Hence

$$\operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(w_0)\big), \varphi\big(\Lambda(G)\big)\big) \le (M+1)\operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(z_0)\big), \varphi\big(\Lambda(G)\big)\big)$$

and

$$\operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(z_0)\big), \varphi\big(\Lambda(G)\big)\big) \le (M+1)\operatorname{dist}_{\mathrm{Euc}}\big(\varphi\big(g(w_0)\big), \varphi\big(\Lambda(G)\big)\big),$$

where M depends on n, K, z_0, w_0, $\Lambda(G)$, but not on the specific element $g \in G$.

By our normalizations (2.2) and (2.3) we now have that the Euclidean and chordal distances are comparable, so that we obtain

$$\operatorname{dist}_{\mathrm{chord}}\big(\varphi\big(g(w_0)\big), \varphi\big(\Lambda(G)\big)\big) \le (\widetilde{M}+1)\operatorname{dist}_{\mathrm{chord}}\big(\varphi\big(g(z_0)\big), \varphi\big(\Lambda(G)\big)\big)$$

and

$$\operatorname{dist}_{\mathrm{chord}}\big(\varphi\big(g(z_0)\big), \varphi\big(\Lambda(G)\big)\big) \le (\widetilde{M}+1)\operatorname{dist}_{\mathrm{chord}}\big(\varphi\big(g(w_0)\big), \varphi\big(\Lambda(G)\big)\big),$$

where $\widetilde{M}$ is independent of g. Since φ is a chordal isometry, this implies that

$$\operatorname{dist}_{\mathrm{chord}}\big(g(w_0), \Lambda(G)\big) \le (\widetilde{M}+1)\operatorname{dist}_{\mathrm{chord}}\big(g(z_0), \Lambda(G)\big)$$

and

$$\operatorname{dist}_{\mathrm{chord}}\big(g(z_0), \Lambda(G)\big) \le (\widetilde{M}+1)\operatorname{dist}_{\mathrm{chord}}\big(g(w_0), \Lambda(G)\big),$$

for all $g \in H$.

Since $G \setminus H$ is finite, we can term by term replace $\widetilde{M}$ with a larger constant (if necessary) so that the last two inequalities hold for all $g \in G$. This proves the lemma. □

Remark 2.9. We will observe that $\delta_{\mathrm{chord}}(G) \le n$ by establishing that $\delta_{\mathrm{chord}}(G) = \delta_{\mathrm{hyp}}(G)$ (see Theorem 4.2).

3. Geometric facts about quasiconformal mappings

In this section we provide some basic facts concerning the distortion of the hyperbolic and quasihyperbolic metrics under quasiconformal mappings.

Recall that the *hyperbolic metric* ϱ on $\mathbf{B}^n$ is derived from the differential $2|dz|/(1-|z|^2)$. The first lemma describes how the hyperbolic distance is distorted under quasiconformal mappings preserving $\mathbf{B}^n$. This lemma is a special case of the more general Theorem 3.2 and Corollary 3.3 (see below).

Lemma 3.1. *For each $n \in \mathbf{N}$ and $K \geq 1$ there exists a homeomorphism $\Phi_{K,n}\colon [0,\infty) \to [0,\infty)$ so that any K-quasiconformal mapping g preserving $\mathbf{B}^n$ satisfies*

$$\varrho\bigl(g(x), g(y)\bigr) \leq \Phi_{K,n}\bigl(\varrho(x,y)\bigr)$$

for all $x, y \in \mathbf{B}^n$. Furthermore, there exists a constant $L_{K,n}$ depending only on n and K, so that

$$\frac{1}{L_{K,n}}\varrho(x,y) \leq \varrho\bigl(g(x), g(y)\bigr) \leq L_{K,n}\varrho(x,y)$$

holds for all $x, y \in \mathbf{B}^n$ with $\varrho(x,y) \geq 1$.

Next we record some results concerning the distortion of the quasihyperbolic metric under quasiconformal mappings. For a proper subdomain D of $\mathbf{R}^n$ we define the *quasihyperbolic metric* k_D on D by

$$k_D(x_1, x_2) = \inf_C \int_C \frac{1}{\operatorname{dist}(x, \partial D)}\, ds,$$

where the infimum is taken over all rectifiable arcs C joining x_1 and x_2 in D, and dist denotes the Euclidean distance. Many of the basic properties of this metric can be found in [GO]. In particular, (D, k_D) is a complete geodesic space. The following theorem is proved in [GO]:

Theorem 3.2 (Gehring–Osgood). *For each $n \in \mathbf{N}$ and $K \geq 1$ there exists a constant c only depending on n and K with the following property: If D and D' are proper subdomains of $\mathbf{R}^n$ and if f is a K-quasiconformal mapping of D onto D' then*

$$k_{D'}\bigl(f(x_1), f(x_1)\bigr) \leq c \max\bigl(k_D(x_1,x_2), k_D(x_1,x_2)^\alpha\bigr), \qquad \alpha = K^{1/(1-n)},$$

for all $x_1, x_2 \in D$.

In particular, this theorem implies that a quasiconformal mapping f as in the theorem is bi-Lipschitz “in the large”:

Corollary 3.3. *For each $n \in \mathbf{N}$, each $K \geq 1$ and each $a > 0$ there exists a constant $L > 1$ with the following property: If D and D' are proper subdomains of $\mathbf{R}^n$ and if f is a K-quasiconformal mapping of D onto D' then*

$$\frac{1}{L} k_D(x_1, x_2) \leq k_{D'}\big(f(x_1), f(x_2)\big) \leq L k_D(x_1, x_2)$$

for all $x_1, x_2 \in D$ with $k_D(x_1, x_2) \geq a$. Here, $L \to \infty$ as $a \to 0$.

Recall finally that a quasiconformal mapping of $\overline{\mathbf{R}}^n$ "quasi-preserves" the cross ratio

$$|a, b, c, d| = \frac{|a - c|\,|b - d|}{|a - d|\,|b - c|},$$

where a, b, c, d are four distinct points in $\overline{\mathbf{R}}^n$, and this quantity is appropriately interpreted if one of the four points is the point ∞. See for example [AVV].

Lemma 3.4. *For each $n \in \mathbf{N}, n \geq 2$ and each $K \geq 1$ there exists a homeomorphism $\eta_{K,n}\colon [0, \infty) \to [0, \infty)$ so that every K-quasiconformal mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ satisfies:*

$$\frac{1}{\eta_{K,n}(1/|a, b, c, d|)} \leq |f(a), f(b), f(c), f(d)| \leq \eta_{K,n}(|a, b, c, d|).$$

Note that $\eta_{1,n}(t) = t$, i.e. the cross ratio is invariant under Möbius transformations.

We are now ready to present our first lemma. It quantifies the relation between balls in the Euclidean versus the quasihyperbolic metric.

Lemma 3.5. *Let $D \subset \mathbf{R}^n$ be a proper subdomain, and let k_D be the quasihyperbolic metric in D. Then for each $z_0 \in D$ and each $M \geq 2$ we have that*

$$B_{k_D}\left(z_0, \frac{1}{M+1}\right) \subset B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right) \subset B_{k_D}\left(z_0, \frac{1}{M-1}\right).$$

Here, $B_{k_D}(z_0, r)$ and $B_{\mathrm{Euc}}(z_0, r)$ denote a ball with center z_0 and radius r in the quasihyperbolic and the Euclidean metric, respectively.

Proof of Lemma 3.5. Let $M \geq 2$ and let $z_0 \in D$.

(1) To prove the first inclusion let

$$z \in \partial B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right), \quad \text{i.e.} \quad |z - z_0| = \frac{\operatorname{dist}(z_0, \partial D)}{M}.$$

Then there exists a geodesic C in the quasihyperbolic metric so that

$$k_D(z_0, z) = \int_C \frac{1}{\operatorname{dist}(w, \partial D)}\, ds.$$

Since quasihyperbolic geodesics are 1-convex with respect to balls ([Mar, Theorem 2.2]) we know that

$$C \subset \overline{B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right)}.$$

Hence, for any $w \in C$, we have that

$$\begin{aligned}\operatorname{dist}(w, \partial D) &\le \operatorname{dist}(z_0, \partial D) + |w - z_0| \\ &\le \operatorname{dist}(z_0, \partial D) + \frac{\operatorname{dist}(z_0, \partial D)}{M} = \frac{M+1}{M}\operatorname{dist}(z_0, \partial D).\end{aligned}$$

Thus

$$k_D(z_0, z) \ge \frac{M}{M+1}\frac{1}{\operatorname{dist}(z_0, \partial D)}\int_C ds \ge \frac{M}{M+1}\frac{|z - z_0|}{\operatorname{dist}(z_0, \partial D)} = \frac{1}{M+1}.$$

Since the sphere $\partial B_{\mathrm{Euc}}\bigl(z_0, \operatorname{dist}(z_0, \partial D)/M\bigr)$ separates $\mathbf{R}^n$ we conclude that

$$B_{k_D}\left(z_0, \frac{1}{M+1}\right) \subset B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right).$$

(2) To prove the second inclusion, let

$$z \in B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right), \quad \text{i.e.} \quad |z - z_0| \le \frac{\operatorname{dist}(z_0, \partial D)}{M}.$$

Then for any w on the line segment $[z_0, z]$ that connects z_0 to z we have that

$$\begin{aligned}\operatorname{dist}(w, \partial D) &\ge \operatorname{dist}(z_0, \partial D) - |z_0 - w| \\ &\ge \operatorname{dist}(z_0, \partial D) - \frac{\operatorname{dist}(z_0, \partial D)}{M} = \frac{M-1}{M}\operatorname{dist}(z_0, \partial D).\end{aligned}$$

Thus we have for the quasihyperbolic distance:

$$\begin{aligned}k_D(z_0, z) &\le \int_{[z_0,z]} \frac{1}{\operatorname{dist}(w, \partial D)}\, ds \le \frac{M}{M-1}\frac{1}{\operatorname{dist}(z_0, \partial D)}\int_{[z_0,z]} ds \\ &= \frac{M}{M-1}\frac{|z_0 - z|}{\operatorname{dist}(z_0, \partial D)} \le \frac{1}{M-1}.\end{aligned}$$

Hence we have shown that

$$B_{\mathrm{Euc}}\left(z_0, \frac{\operatorname{dist}(z_0, \partial D)}{M}\right) \subset B_{k_D}\left(z_0, \frac{1}{M-1}\right). \quad \square$$

Lemma 3.6 *Let $D \subset \mathbf{R}^n$ be a proper subdomain, and let $\varphi\colon D \to \mathbf{R}^n$ be a K-quasiconformal mapping, let $D' = \varphi(D)$. Denote by k_D and $k_{D'}$ the quasihyperbolic metrics in D, D', respectively. Then for each $r_0 > 0$ there exists a constant L depending only on r_0 and K so that for all $r \geq r_0$ and all $z_0 \in D$ we have that*

$$B_{k_{D'}}\left(\varphi(z_0), \frac{r}{L}\right) \subset \varphi\bigl(B_{k_D}(z_0, r)\bigr) \subset B_{k_{D'}}\bigl(\varphi(z_0), Lr\bigr).$$

Proof. Let $r_0 > 0$. Then, following Corollary 3.3 there exists a constant L depending only on r_0 and K so that

$$\frac{1}{L} k_D(v, w) \leq k_{D'}\bigl(\varphi(v), \varphi(w)\bigr) \leq L k_D(v, w)$$

holds for all $v, w \in D$ with $k_D(v, w) \geq r_0$. Let now $r \geq r_0$ and let $z_0 \in D$.

(1) To show the first inclusion let $w \in B_{k_{D'}}\bigl(\varphi(z_0), r/L\bigr)$ and suppose that $w \notin \varphi\bigl(B_{k_D}(z_0, r)\bigr)$. Then $\varphi^{-1}(w) \notin B_{k_D}(z_0, r)$ and so $k_D\bigl(\varphi^{-1}(w), z_0\bigr) \geq r$. But since $r \geq r_0$ this implies that $k_{D'}\bigl(w, \varphi(z_0)\bigr) \geq r/L$, and this is a contradiction.

(2) To show the second inclusion let $z \in \partial B_{k_D}(z_0, r)$, i.e. $k_D(z_0, z) = r$. Then $k_{D'}\bigl(\varphi(z_0), \varphi(z)\bigr) \leq Lr$, i.e.

$$\varphi(z) \in \overline{B_{k_{D'}}\bigl(\varphi(z_0), Lr\bigr)}.$$

But this implies that

$$\varphi\bigl(B_{k_D}(z_0, r)\bigr) \subset B_{k_{D'}}\bigl(\varphi(z_0), Lr\bigr). \quad \square$$

Remark 3.7. Setting $\Psi_{K,n}(t) = c_{K,n} \max(t, t^{\alpha})$, where $\alpha = K^{1/(1-n)}$ and $c_{K,n}$ is the constant from Theorem 3.2 one similarly obtains that under the hypothesis of Lemma 3.6 we have that

$$\varphi\Bigl(B_{k_D}\bigl(z_0, \psi_{K,n}^{-1}(t)\bigr)\Bigr) \subset B_{k_{D'}}\bigl(\varphi(z_0), t\bigr) \subset \varphi\Bigl(B_{k_D}\bigl(z_0, \psi_{K,n}(t)\bigr)\Bigr)$$

holds for all $t > 0$.

4. The equality of the chordal and hyperbolic exponents

The space $\mathscr{X} = \overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n \setminus$ diagonals can be identified with the collection of mutually distinct triples of points in $\overline{\mathbf{R}}^n$. For a point $(a,b,c) \in \mathscr{X}$, denote by p the projection map

$$p\colon \mathscr{X} \to \mathbf{H}^{n+1}$$

that maps c to the point ζ on the hyperbolic geodesic $(a,b) \in \mathbf{H}^{n+1}$ so that the hyperbolic geodesic determined by c and ζ meets (a,b) orthogonally.

The action of the Möbius group acting on $\overline{\mathbf{R}}^n$ extends via the diagonal extension to an action on $\overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n \times \overline{\mathbf{R}}^n \setminus$ diagonals; observe that the action of a Möbius transformation commutes with the projection map p, that is

$$p\big(\gamma(a), \gamma(b), \gamma(c)\big) = \gamma\big(p(a,b,c)\big). \tag{4.1}$$

(Of course the action of γ on the right side of the above equation is that of the isometric action of γ on $\mathbf{H}^{n+1}$.)

Equation (4.1) is no longer true for quasiconformal mappings, however it is not too false either (see Lemma C2 in [T4] for example):

Lemma 4.1. *For each $n \in \mathbf{N}$ and for each $K \geq 1$ there exists a constant $C_{K,n}$ so that for any K-quasiconformal mapping $g\colon \mathbf{H}^{n+1} \to \mathbf{H}^{n+1}$ (naturally extended to $\overline{\mathbf{R}}^n$) and any triple (a,b,c) of mutually distinct points in $\overline{\mathbf{R}}^n$ we have that*

$$\varrho\big(g\big(p(a,b,c)\big), p\big(\big(g(a), g(b), g(c)\big)\big)\big) \leq C_{K,n}.$$

The results in Section 3 and Lemma 4.1 now enable us to show that both the chordal and the hyperbolic exponent of convergence agree for discrete quasiconformal groups with non-empty regular set and large enough limit set, acting on $\overline{\mathbf{R}}^n$. The analogous result for Kleinian groups was proved by Bishop and Jones in [BJ2].

Theorem 4.2. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set $\Omega(G)$ and so that $|\Lambda(G)| \geq 2$. Then*

$$\delta_{\mathrm{hyp}}(G) = \delta_{\mathrm{chord}}(G). \tag{4.2}$$

Proof. Suppose first that G is non-elementary. Let G' be a quasiconformal hyperbolic action on $\mathbf{H}^{n+1}$ associated with G. Let $z_0 \in \Omega(G)$. By conjugation with a Möbius transformation we may assume that $\infty \in \Omega(G)$, and that the orbit of z_0 is bounded in $\mathbf{R}^n$, i.e. $\infty \notin G(z_0)$. With these normalizations, we can replace the chordal distance $\mathrm{dist}_{\mathrm{chord}}$ with the Euclidean distance $\mathrm{dist}_{\mathrm{Euc}}$ in the definition of the chordal exponent of convergence (Definition 2.6) without affecting the convergence behavior of the sum. Furthermore, in the definition of the hyperbolic exponent of convergence (Definition 2.4) we choose $x = y =$

$j = (0,\dots,0,1) \in \mathbf{H}^{n+1}$ and then can replace $\varrho\big(j, g(j)\big)$ with $\big(-\log \mathrm{Im}\big(g(j)\big)\big)$ without affecting the convergence properties. Here, for a point $x \in \mathbf{H}^{n+1}$, we write $\mathrm{Im}(x)$ for the $(n+1)$st coordinate of x. We often write $x = z + tj$, where $t = \mathrm{Im}(x)$, and $z \in \mathbf{R}^n$. We also refer to the point z in $\mathbf{R}^n$ given by the first n coordinates of x as $\mathrm{Re}(x)$. Thus (4.2) becomes

(4.3)

$$\inf\Big\{s>0 \;\Big|\; \sum_{g\in G'} \big(\mathrm{Im}\, g(j)\big)^s < \infty\Big\} = \inf\Big\{s>0 \;\Big|\; \sum_{g\in G} \mathrm{dist}_{\mathrm{Euc}}\big(g(z_0), \Lambda(G)\big)^s < \infty\Big\}.$$

Clearly, j can be replaced by any other $x \in \mathbf{H}^{n+1}$.

Choose $r>0$ small enough so that the Euclidean ball $D = B(z_0, r)$ in $\mathbf{R}^n$ of radius r centered at z_0 is contained in $\Omega(G)$ and its orbit under G is bounded in $\mathbf{R}^n$.

Let $x = z_0 + rj \in \mathbf{H}^{n+1}$. Pick (and fix) $a,b,c \in \Lambda(G)$ mutually distinct, and recall that $C = p(a,b,c) \in \mathbf{H}^{n+1}$ is the projection of c onto the hyperbolic geodesic with endpoints a and b.

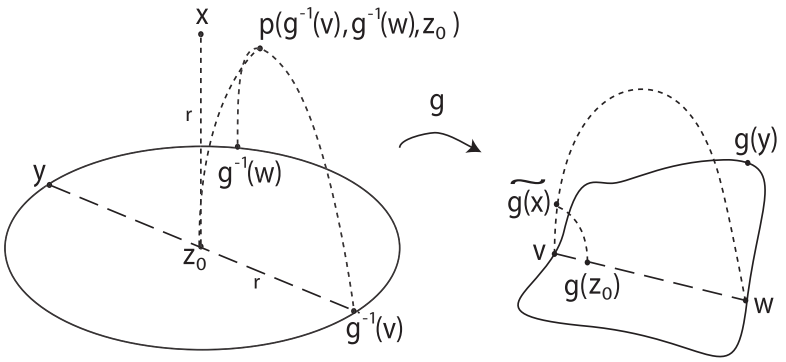

Figure 1.

Let $v \in \partial g(D)$ be a point with minimal Euclidean distance to $g(z_0)$. Let w be the closest point on $\partial g(D)$ that lies on the ray emanating from v and passing through $g(z_0)$. Define $\tau g(x) = p\big(v, w, g(z_0)\big)$, see Figure 1. We will show that $\tau g(x)$ is hyperbolically close to $g(x)$, where "close" is independent of the mapping $g \in G$. To do so, let y be the point on $\partial B(z_0, r)$ that is diametrically opposite to $g^{-1}(v)$. Then

$$|g^{-1}(v), z_0, g^{-1}(w), y| = \frac{|g^{-1}(v) - g^{-1}(w)|\,|z_0 - y|}{|g^{-1}(v) - y|\,|z_0 - g^{-1}(w)|} = |g^{-1}(v) - g^{-1}(w)|\frac{1}{2r}.$$

On the other hand, using Lemma 3.4 we obtain that

$$|g^{-1}(v), z_0, g^{-1}(w), y| \ge \frac{1}{\eta_{K,n}\big(1/|v, g(z_0), w, g(y)|\big)},$$

and

$$\begin{aligned}|v, g(z_0), w, g(y)| &= \frac{|v-w|}{|g(z_0)-w|}\frac{|g(z_0)-g(y)}{|v-g(y)|} \\ &\geq 1\cdot\frac{|g(z_0)-g(y)|}{|v-g(z_0)|+|g(z_0)-g(y)|} \\ &= \frac{1}{\dfrac{|v-g(z_0)|}{|g(z_0)-g(y)|}+1} \geq \frac{1}{1+1} = \frac{1}{2}.\end{aligned}$$

Hence $|g^{-1}(v), z_0, g^{-1}(w), y| \geq 1/\eta_{K,n}(2)$ and so $|g^{-1}(v) - g^{-1}(w)| \geq 2r/\eta_{K,n}(2)$. Thus $\varrho\bigl(x, p(g^{-1}(v), g^{-1}(w), z_0)\bigr) \leq D_{K,n}$, where $D_{K,n}$ is a constant that only depends on K and n, but not on g or r. Hence Lemma 3.1 implies that

$$\varrho\bigl(g(x), g\bigl(p(g^{-1}(v), g^{-1}(w), z_0)\bigr)\bigr) \leq \Phi_{K,n}(D_{K,n}).$$

Now, since $\widetilde{g(x)} = p\bigl(v, w, g(z_0)\bigr) = p\bigl(g\bigl(g^{-1}(v)\bigr), g\bigl(g^{-1}(w)\bigr), g(z_0)\bigr)$ we obtain from Lemma 4.1 that

$$\varrho(\widetilde{g(x)}\,, g\bigl(p\bigl(g^{-1}(v), g^{-1}(w), z_0\bigr)\bigr)\bigr) \leq C_{K,n},$$

so that we can conclude:

$$\begin{aligned}\varrho\bigl(g(x), \widetilde{g(x)}\,\bigr) &\leq \varrho\bigl(g(x), g\bigl(p(g^{-1}(v), g^{-1}(w), z_0)\bigr)\bigr) \\ &\quad + \varrho\bigl(g\bigl(p(g^{-1}(v), g^{-1}(w), z_0)\bigr), \widetilde{g(x)}\,\bigr) \\ &\leq \Phi_{K,n}(D_{K,n}) + C_{K,n}.\end{aligned}$$

Thus, in (4.3) we can replace $\operatorname{Im} g(j)$ with $\operatorname{Im}\widetilde{g(x)}$ and hence have to show that

(4.4)
$$\inf\Bigl\{s>0 \Bigm| \sum_{g\in G}\bigl(\operatorname{Im}\widetilde{g(x)}\,\bigr)^s < \infty\Bigr\} = \inf\Bigl\{s>0 \Bigm| \sum_{g\in G}\operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)^s < \infty\Bigr\}.$$

In order to prove (4.4) we compare $\bigl(\operatorname{Im}\widetilde{g(x)}\,\bigr)$ and $\operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)$ to each other.

(1) We first show that "$\leq$" holds in (4.4). Note that $|g(z_0) - v| \leq \operatorname{Im}\widetilde{g(x)} \leq 2|g(z_0) - v|$. Since by choice of v we have that $|g(z_0) - v| \leq \operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)$ this implies that

$$\operatorname{Im}\widetilde{g(x)} \leq 2\operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)$$

for all $g \in G$, and this proves that

$$\inf\Bigl\{s>0 \Bigm| \sum_{g\in G}\bigl(\operatorname{Im}\widetilde{g(x)}\,\bigr)^s < \infty\Bigr\} \leq \inf\Bigl\{s>0 \Bigm| \sum_{g\in G}\operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)^s < \infty\Bigr\}.$$

(2) Next we show that "$\geq$" holds in (4.4). First observe that $\operatorname{Im}\widetilde{g(x)} \geq |v - g(z_0)| \geq \bigl|g(z_0) - \operatorname{Re}\bigl(\widetilde{g(x)}\bigr)\bigr|$. Recall that $C = p(a,b,c)$, where a, b, c are three mutually distinct points in $\Lambda(G)$. Let $M = \varrho(x,C)$, and define $\widetilde{g(C)} = p\bigl(g(a), g(b), g(c)\bigr)$. Then $\varrho\bigl(g(C), \widetilde{g(C)}\bigr) \leq C_{K,n}$ by Lemma 4.1 and thus

$$\begin{aligned}\varrho\bigl(\widetilde{g(C)}\,, \widetilde{g(x)}\bigr) &\leq \varrho\bigl(\widetilde{g(C)}\,, g(C)\bigr) + \varrho\bigl(g(C), g(x)\bigr) + \varrho\bigl(\widetilde{g(x)}\,, g(x)\bigr)\\ &\leq 2C_{K,n} + \Phi_{K,n}(M).\end{aligned}$$

Hence there exists a constant A that only depends on M, n, K, so that the Euclidean ball of radius $A \cdot \operatorname{Im}\widetilde{g(x)}$, centered at $\operatorname{Re}\widetilde{g(x)}$, must contain at least one of the points $g(a)$, $g(b)$, $g(c)$. This implies that $\operatorname{dist}_{\mathrm{Euc}}\bigl(\operatorname{Re}\widetilde{g(x)}\,, \Lambda(G)\bigr) \leq A \cdot \operatorname{Im}\widetilde{g(x)}$. Hence

$$\begin{aligned}\operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr) &\leq |g(z_0) - \operatorname{Re}\widetilde{g(x)}\,| + \operatorname{dist}_{\mathrm{Euc}}\bigl(\operatorname{Re}\widetilde{g(x)}\,, \Lambda(G)\bigr)\\ &\leq \operatorname{Im}\widetilde{g(x)} + A \cdot \operatorname{Im}\widetilde{g(x)}\,.\end{aligned}$$

Thus

$$\inf\Bigl\{s > 0 \Bigm| \sum_{g \in G} \bigl(\operatorname{Im}\widetilde{g(x)}\,\bigr)^s < \infty\Bigr\} \geq \inf\Bigl\{s > 0 \Bigm| \sum_{g \in G} \operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0), \Lambda(G)\bigr)^s < \infty\Bigr\}.$$

If G is elementary, then by our assumptions we have that $\Lambda(G)$ contains exactly 2 points. One can modify the above argument and still show that $\delta_{\mathrm{chord}}(G) = \delta_{\mathrm{hyp}}(G)$. □

5. The chordal exponent of convergence and its distortion

The results in the previous section show that we can define the exponent of convergence of a non-elementary discrete quasiconformal group with non-empty regular set, acting on $\overline{\mathbf{R}}^n$, even if the group action does not extend to $\mathbf{H}^{n+1}$, in two different ways that lead to the same result. In what follows, we will use the symbol $\delta(G)$ to refer to an appropriate choice of exponent of convergence, i.e. $\delta(G) = \delta_{\mathrm{hyp}}(G)$ if G is elementary or if $\Omega(G) = \emptyset$, and otherwise, $\delta(G)$ is interpreted hyperbolically or chordally, in whichever way is most convenient.

We show in [BTT1] that the Hausdorff dimension of the conical limit set of a discrete quasiconformal group acting on $\mathbf{H}^{n+1}$ (and extended naturally to $\overline{\mathbf{R}}^n$) is bounded above by its exponent of convergence, but the exponent of convergence can be strictly larger than the Hausdorff dimension of the conical limit set. In fact, using a standard argument (see for example [BJ1], [N]) which generalizes to quasiconformal hyperbolic actions (see Definition 2.1) one can see that *the Hausdorff dimension of the conical limit set of a discrete quasiconformal group is bounded above by its hyperbolic exponent of convergence.* Thus we have:

Theorem 5.1. *Let G be a discrete quasiconformal group acting on $\overline{\mathbf{R}}^n$. Then $\delta(G) \geq \dim \Lambda_c(G)$.*

Establishing this fact allows us to observe that $\delta(G)$ shares another fundamental property with $\delta(\Gamma)$ where Γ is Kleinian.

Corollary 5.2. *Let G be a discrete non-elementary quasiconformal group. Then $\delta(G) > 0$.*

Proof of Corollary 5.2. For any such G we can find a non-elementary subgroup $H \subseteq G$ so that H has purely conical limit set and non-empty regular set. It is a result of Bonfert-Taylor and Martin [BTM] that $\Lambda(H)$ is uniformly perfect and so $\dim\bigl(\Lambda(H)\bigr) > 0$ (see [JV].) The conical limit set of the full group G contains the conical limit set of H, and so $\dim\bigl(\Lambda_c(G)\bigr) \geq \dim\bigl(\Lambda_c(H)\bigr)$. The conclusion is now immediate from Theorem 5.1. □

As stated in the introduction, our formulation of the exponent of convergence in the chordal metric was undertaken in order to establish the following distortion theorem:

Theorem 5.3. *Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set. Let $\varphi\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal homeomorphism, and set $H = \varphi G \varphi^{-1}$. Then*

$$\delta(H) \leq \frac{(n+c)\delta(G)}{c+\delta(G)},$$

where $c > 0$ is the constant from Gehring's Theorem 5.5 (see below).

Remark 5.4. Astala showed that in dimension $n = 2$ the sharp bound for c is $c = 2/(K-1)$. Thus under the assumptions of the theorem, in dimension 2 we obtain that

$$\delta(H) \leq \frac{2K\delta(G)}{2+(K-1)\delta(G)}.$$

This was proved in [BTT2] under the additional assumption that $\Lambda(G)$ be uniformly perfect.

Finally, we consider discrete quasiconformal groups acting on $\overline{\mathbf{R}}^n$ with purely conical limit set and non-empty regular set. In our paper [ABT] we prove that if the group extends to a group acting on $\mathbf{H}^{n+1}$ then the exponent of convergence of the group is strictly less than n, compare to Theorems 1.2 and 1.3 in [ABT]. Here we provide a new proof not requiring the assumption that the group extend to $\mathbf{H}^{n+1}$:

Theorem 5.5. *Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with purely conical limit set and non-empty regular set. Then $\delta(G) < n$.*

We now prove the results from this section.

In order to prove Theorem 5.3 we recall results of Gehring [G] and Astala [As] concerning the integrability of the partial derivatives of a quasiconformal mapping.

Let D be a domain in $\mathbf{R}^n$ and let $f\colon D \to \mathbf{R}^n$ be a homeomorphism. Let

$$L_f(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} \quad \text{and} \quad J_f(x) = \limsup_{r\to 0} m\left(\frac{f\bigl(B(x,r)\bigr)}{m\bigl(B(x,r)\bigr)}\right)$$

be the *maximum stretching* and the *generalized Jacobian* for f at $x\in D$, respectively, where $B(x,r)$ denotes the open n-ball of radius r about x and m denotes Lebesgue measure in $\mathbf{R}^n$. These functions are nonnegative and measurable in D, and $J_f(x) \le L_f(x)^n$. Lebesgue's theorem implies that $\int_E J_f\,dm \le m\bigl(f(E)\bigr) < \infty$ for each compact set $E\subset D$, and hence J_f is locally L^1-integrable in D. If f is also K-quasiconformal in D, then $L_f(x)^n \le K J_f(x)$ almost everywhere in D, and thus L_f is locally L^n-integrable in D.

Gehring shows ([G, Theorem 1]):

Theorem 5.6. *Let $D\subset \mathbf{R}^n$ be a domain and let $f\colon D\to\mathbf{R}^n$ be K-quasiconformal. Then L_f is locally L^p-integrable in D for each $p\in[n,n+c)$, where c is a positive constant which depends only on K and n.*

Analyzing the radial stretch mapping near the origin one observes that necessarily $c \le n/(K^{1/(n-1)}-1)$, and it is conjectured that this upper bound for c is sharp. In dimension $n=2$ Astala shows in [As] that indeed $c = 2/(K-1)$.

We will need a localized version of the exponent of convergence. In [BTT3] we localize the definition of the hyperbolic exponent of convergence; here we will do the same in terms of the chordal exponent of convergence (Definition 2.6).

Definition 5.7. Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set. Let $z_0\in\Omega(G)$. For $x\in\overline{\mathbf{R}}^n$ and $r>0$ define

$$\delta_x^r(G) := \inf\Biggl\{ s>0 \;\Bigg|\; \sum_{g\in G: g(z_0)\in B_{\mathrm{chord}}(x,r)} \operatorname{dist}_{\mathrm{chord}}\bigl(g(z_0),\Lambda(G)\bigr)^s < \infty \Biggr\}.$$

Furthermore, define the *local exponent of convergence of G at x* to be

$$\delta_x(G) := \lim_{r\to 0} \delta_x^r(G).$$

Remark 5.8. Note that for fixed $x\in\overline{\mathbf{R}}^n$ the quantity $\delta_x^r(G)$ is non-increasing as $r\searrow 0$, and hence the above limit exists. Furthermore the definition of the local exponent of convergence is independent of the choice of z_0. (This can be seen via an argument similar to the proof of Lemma 2.7.)

It is easy to see that $\delta_x(G)=0$ for $x\notin\Lambda(G)$ and furthermore that $\delta(G) = \max_{x\in\Lambda(G)} \delta_x(G)$.

Proof the Theorem 5.3. By composition with a Möbius transformation (which does not effect the exponent of convergence) we may assume that $\varphi(\infty)=\infty$, that $\infty\in\Lambda(G)$, and that there exists $\zeta_0\in\Lambda(G)$ with $\zeta_0\neq\infty$ so that $\delta_{\zeta_0}(G)=\delta(G)$, and $\xi_0\in\Lambda(H)$, $\xi_0\neq\infty$, with $\delta_{\xi_0}(H)=\delta(H)$. Choose $R>0$ large enough so that $|\zeta_0|<\frac{1}{2}R$, $|\xi_0|<\frac{1}{2}R$, $|\varphi(\zeta_0)|<\frac{1}{2}R$, $|\varphi^{-1}(\xi_0)|<\frac{1}{2}R$. Choose $z_0\in\Omega(H)$, and let $w_0=\varphi^{-1}(z_0)$. Then, since $\delta_{\xi_0}(H)=\delta(H)$, we have that

$$\delta(H)=\inf\left\{s>0\;\middle|\;\sum_{h\in H:|h(z_0)|<R}\operatorname{dist}_{\mathrm{Euc}}\bigl(h(z_0),\Lambda(H)\bigr)^s<\infty\right\}.$$

Note that we can use Euclidean distance instead of chordal distance since we only consider those $h\in H$ for which $|h(z_0)|<R$. In the following we shall write $A\sim B$ for two quantities A and B if there exists a constant C that only depends on n and K and possibly the group G (but not on the particular element $g\in G$ under consideration) so that $A/C\le B\le CA$.

In what follows, let $M\ge 2$ be a constant (whose exact value will be determined later) that depends on the group G. Let $h\in H$ with $|h(z_0)|<R$, let $g=\varphi^{-1}h\varphi\in G$. For an open set $U\subset\mathbf{R}^n$ and $z\in U$, $r>0$, denote by $B_{k_U}(z,r)$ the quasihyperbolic ball in the component of U that contains z that is centered at z and has radius r. Then using Lemma 3.5 and Remark 3.7 we obtain:

$$\begin{aligned}\operatorname{dist}_{\mathrm{Euc}}(h(z_0),\Lambda(H))&\sim\left(\operatorname{vol}_{\mathrm{Euc}}B_{\mathrm{Euc}}\left(h(z_0),\frac{\operatorname{dist}_{\mathrm{Euc}}\bigl(h(z_0),\Lambda(H)\bigr)}{M}\right)\right)^{1/n}\\&\sim\left(\operatorname{vol}_{\mathrm{Euc}}B_{k_{\Omega(H)}}\left(h(z_0),\frac{1}{M}\right)\right)^{1/n}\\&=\left(\operatorname{vol}_{\mathrm{Euc}}B_{k_{\Omega(H)}}\left(\varphi g(w_0),\frac{1}{M}\right)\right)^{1/n}\\&\le\left(\operatorname{vol}_{\mathrm{Euc}}\varphi\left(B_{k_{\Omega(G)}}\left(g(w_0),\Psi_{k,n}\left(\frac{1}{M}\right)\right)\right)\right)^{1/n}.\end{aligned}$$

Here, $\operatorname{vol}_{\mathrm{Euc}}$ denotes Euclidean volume.

Enumerate $\{g\in G\mid|\varphi g(w_0)|<R\}=\{g_i\mid i\in\mathbf{N}\}$, and write

$$B_i=B_{k_{\Omega(G)}}\left(g_i(w_0),\Psi_{K,n}\left(\frac{1}{M}\right)\right).$$

Then there is a constant C that only depends on n, K, M, so that

$$
\begin{aligned}
\sum_{h\in H:|h(z_0)|<R} &\operatorname{dist}_{\mathrm{Euc}}\big(h(z_0),\Lambda(H)\big)^s \le C^s \sum_i \big(\operatorname{vol}_{\mathrm{Euc}}\varphi(B_i)\big)^{s/n} \\
&= C^s \sum_i \Big(\int_{B_i} J_\varphi \, dm\Big)^{s/n} \\
&\le C^s \sum_i \Big[\Big(\int_{B_i} J_\varphi^{p_0}\, dm\Big)^{1/p_0} \cdot \Big(\int_{B_i} 1^{q_0}\, dm\Big)^{1/q_0}\Big]^{s/n}, \text{ where } \frac{1}{p_0}+\frac{1}{q_0}=1 \\
&= C^s \sum_i \Big[\Big(\int_{B_i} J_\varphi^{p_0}\, dm\Big)^{s/(np_0)} \cdot (\operatorname{vol}_{\mathrm{Euc}} B_i)^{s/(nq_0)}\Big] \\
&\le C^s \Big(\sum_i \int_{B_i} J_\varphi^{p_0}\, dm\Big)^{s/(np_0)} \cdot \Big(\sum_i (\operatorname{vol}_{\mathrm{Euc}} B_i)^{sp_0/(q_0(np_0-s))}\Big)^{(np_0-s)/(np_0)}. \qquad (5.1)
\end{aligned}
$$

By discontinuity of the action of G on $\Omega(G)$ there exists a constant $L>0$ so that $k_{\Omega(G)}\big(g(w_0),\tilde g(w_0)\big)\ge L$ holds for all $g,\tilde g\in G$ distinct. (Otherwise there are $g_j,\tilde g_j\in G$ distinct so that $k_{\Omega(G)}\big(g_j(w_0),\tilde g_j(w_0)\big)\to 0$ as $j\to\infty$, but then $k_{\Omega(G)}\big(w_0,g_j^{-1}(\tilde g_j(w_0))\big)\to 0$ as well by Theorem 3.2, and this contradicts discontinuity of the action of G near w_0.)

By choosing M large enough so that $\Psi_{K,n}(1/M)<\frac12 L$ we thus obtain that the balls $\{B_i,\ i\in\mathbf N\}$ are all disjoint. Furthermore, $\big\{g(w_0)\ \big|\ |\varphi(g(w_0))|<R\big\}$ is contained in the compact (in $\mathbf R^n$) set $\varphi^{-1}(\{|z|\le R\})$, and so there exists a compact set $F\subset\mathbf R^n$ so that $B_i\subset F$ for all $i\in\mathbf N$. Hence

$$\sum_i \int_{B_i} J_\varphi^{p_0}\, dm \le \int_F J_\varphi^{p_0}\, dm,$$

and this last integral is finite whenever $p_0<(n+c)/n$ since $J_\varphi\le L_\varphi^n$, and L_φ is locally L^p-integrable for $p\in[n,n+c)$ by Theorem 5.6. Hence

$$\Big(\sum_i \int_{B_i} J_\varphi^{p_0}\, dm\Big)^{s/np_0} < \infty \qquad (5.2)$$

whenever $p_0<(n+c)/n$.

We can assume that M was chosen large enough so that $\Psi_{K,n}(1/M)\le\frac13$. Hence using Lemma 3.5 we obtain:

$$
\begin{aligned}
\operatorname{vol}_{\mathrm{Euc}} B_i &\le \operatorname{vol}_{\mathrm{Euc}} B_{k_{\Omega(G)}}\big(g_i(w_0),\tfrac13\big) \\
&\le \operatorname{vol}_{\mathrm{Euc}} B_{\mathrm{Euc}}\big(g_i(w_0),\tfrac12\operatorname{dist}\big(g_i(w_0),\Lambda(G)\big)\big) \sim \operatorname{dist}\big(g_i(w_0),\Lambda(G)\big)^n.
\end{aligned}
$$

Thus

$$\sum_i (\mathrm{vol}_{\mathrm{Euc}}\, B_i)^{sp_0/(q_0(np_0-s))} \lesssim \sum_i \mathrm{dist}\big(g_i(w_0), \Lambda(G)\big)^{nsp_0/(q_0(np_0-s))}, \tag{5.3}$$

and this last sum is finite if $nsp_0/(q_0(np_0-s)) > \delta(G)$. From (5.2) and (5.3) we obtain that (5.1) is finite if

$$p_0 < \frac{n+c}{n} \quad \text{and} \quad \frac{nsp_0}{q_0(np_0-s)} > \delta(G).$$

Using that

$$\frac{1}{p_0} + \frac{1}{q_0} = 1$$

we thus obtain that (5.1) is finite if

$$s > \frac{(n+c)\delta(G)}{c+\delta(G)}.$$

This proves that

$$\delta(H) \leq \frac{(n+c)\delta(G)}{c+\delta(G)}. \square$$

Finally, for the proof of Theorem 5.5 we need the following lemma, the proof of which makes use of the assumption that the limit set be purely conical:

Lemma 5.9 (Corollary 3.2 in [ABT]). *Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with purely conical limit set and non-empty regular set. Then $\Lambda(G)$ is uniformly porous in the following sense. There exists an $\varepsilon > 0$ such that if B is a chordal ball in $\overline{\mathbf{R}}^n$ of chordal radius r then B contains a chordal ball of chordal radius εr that does not intersect Λ.*

Proof of Theorem 5.5. In this proof we work with the chordal exponent of convergence, and thus omit the subscript $_{\mathrm{chord}}$. We can normalize so that $\infty \in \Lambda(G)$, and so that there exists $x \in \Lambda(G)$, $x \neq \infty$, so that $\delta_x(G) = \delta(G)$, see Definition 5.7. Let $z_0 \in \Omega(G)$ and define

$$\widetilde{G} = \{g \in G \mid g(z_0) \in Q_1(x)\},$$

where $Q_1(x)$ is a n-cube of side length 1, centered at x. Then

$$\delta(G) = \inf\left\{ s > 0 \,\middle|\, \sum_{g \in \widetilde{G}} \mathrm{dist}_{\mathrm{Euc}}\big(g(z_0),\ \Lambda(G)\big)^s < \infty \right\}.$$

Let $A_k = \big\{z \in Q_1(x) \mid e^{-(k+1)} \leq \mathrm{dist}_{\mathrm{Euc}}\big(z, \Lambda(G)\big) < e^{-k}\big\}$, and let $\#A_k$ be the number of elements $g \in G$ for which $g(z_0) \in A_k$.

Claim 1. *We have*

$$\limsup_{k\to\infty} \frac{\log \# A_k}{k} = \delta(G).$$

To prove Claim 1, let $r = \limsup_{k\to\infty} \log \# A_k / k$, and suppose first that $r < \delta(G)$. Choose s, t so that $r < s < t < \delta(G)$. Then $\# A_k \le e^{ks}$ for all but finitely many k, and hence there is a finite constant C so that

$$\begin{aligned}\sum_{g\in\widetilde{G}} \operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0),\Lambda(G)\bigr)^t &= \sum_k \sum_{g:g(z_0)\in A_k} \operatorname{dist}_{\mathrm{Euc}}\bigl(g(z_0),\Lambda(G)\bigr)^t \\ &\le \sum_k \# A_k (e^{-k})^t \le \sum_k e^{ks}e^{-kt} + C < \infty.\end{aligned}$$

This implies that $\delta(G) \le t < \delta(G)$, a contradiction. Similarly, one obtains a contradiction from the assumption that $r > \delta(G)$, and this proves Claim 1.

The proof of the Claim 2 below follows easily from Theorem 3.2 and the fact that G acts discontinuously on $\Omega(G)$.

Claim 2. *There exists a constant $C > 0$ so that*

$$k_{\Omega(G)}\bigl(g(z_0), h(z_0)\bigr) \ge C$$

for all $g, h \in G$ with $g \ne h$. Here, we define that $k_{\Omega(G)}(z,w) = \infty$, if z and w are in distinct components of $\Omega(G)$.

Our goal is to use Claim 1 in order to estimate $\delta(G)$, and we thus need to bound $\# A_k$ from above. Claim 2 implies that there exists a constant $M > 0$ so that $\# A_k \le M \cdot \mathrm{vol}_{k_{\Omega(G)}}(A_k)$, where $\mathrm{vol}_{k_{\Omega(G)}}$ denotes quasihyperbolic volume. Furthermore,

$$\begin{aligned}\mathrm{vol}_{k_{\Omega(G)}}(A_k) &= \int_{A_k} \left(\frac{1}{\operatorname{dist}(\,\cdot\,,\Lambda(G))}\right)^n dm \\ &\le \int_{A_k} (e^{k+1})^n \, dm = e^{n(k+1)} \cdot \mathrm{vol}_{\mathrm{Euc}}(A_k).\end{aligned}$$

Hence

$$\# A_k \le M e^{n(k+1)} \cdot \mathrm{vol}_{\mathrm{Euc}}(A_k). \tag{5.4}$$

In order to estimate $\mathrm{vol}_{\mathrm{Euc}}(A_k)$, we use Lemma 5.9 to assert the existence of an integer q so that the following is true: If Q is a (Euclidean) cube contained in $Q_1(x)$, and we divide Q into q^n sub-cubes of equal (Euclidean) side length, then at least one of these sub-cubes does not intersect $\Lambda(G)$.

For a cube Q in $\mathbf{R}^n$, denote by $3 \cdot Q$ the cube that has three times the side length of Q and is centered around Q. Then by replacing the integer q with $3q$ we can assume that the following holds: if Q is a cube contained in $Q_1(x)$, and we divide Q into q^n sub-cubes of equal side length, then at least one of these sub-cubes $\widetilde{Q}$ has the property that $3 \cdot \widetilde{Q}$ does not intersect $\Lambda(G)$. This implies that if the side length of this particular cube $\widetilde{Q}$ is at least e^{-k}, then $\widetilde{Q}$ cannot meet A_k.

Inductively we obtain that $Q_1(x)$ contains $(q^n - 1)^{i-1}$ sub-cubes of side length $1/q^i$ none of which meet A_k as long as $q^{-i} \geq e^{-k}$, and all these sub-cubes are disjoint. Hence

$$\mathrm{vol}_{\mathrm{Euc}}(A_k) \leq \mathrm{vol}_{\mathrm{Euc}}\big(Q_1(x)\big) - \sum_{i=1}^{I_k} (q^{-i})^n (q^n - 1)^{i-1},$$

where $I_k \in \mathbf{N}$ with $(k/\log q) - 1 < I_k \leq k/\log q$. Thus

$$\mathrm{vol}_{\mathrm{Euc}}(A_k) \leq 1 - q^{-n} \sum_{i=0}^{I_k - 1} \left(\frac{q^n - 1}{q^n}\right)^i = \left(\frac{q^n - 1}{q^n}\right)^{I_k} \leq \left(\frac{q^n - 1}{q^n}\right)^{(k/\log q) - 1}.$$

This together with (5.4) implies that

$$\#A_k \leq M e^{n(k+1)} \left(\frac{q^n - 1}{q^n}\right)^{k/\log q}.$$

Hence

$$\frac{\log \#A_k}{k} \leq \frac{\log M}{k} + n + \frac{n}{k} + \frac{1}{\log q} \log\left(\frac{q^n - 1}{q^n}\right) - \frac{1}{k} \log \frac{q^n - 1}{q^n},$$

and since $\log\big((q^n - 1)/q^n\big) < 0$ and $\log q > 0$ we obtain that

$$\limsup_{k \to \infty} \frac{\log \#A_k}{k} \leq n + \frac{1}{\log q} \log \frac{q^n - 1}{q^n} < n.$$

Claim 1 now implies that $\delta(G) < n$. □

Remark 5.10 (1) Theorem 5.5 as well as Lemma 5.9 should be true in the more general setting of discrete quasiconformal groups whose limit sets consist entirely of conical limit points and bounded parabolic points in the sense of Bowditch [Bow] and Tukia [T5].

(2) One could prove Theorem 5.5 using the hyperbolic exponent of convergence δ_{hyp} and quasiconformal hyperbolic actions. However, the proof given above is considerably shorter.

6. Conjecture

Given the relationship between the exponent of convergence and the Hausdorff dimension of the conical limit set it is desirable to find an upper bound on the exponent of convergence in terms of the Hausdorff dimension of the conical limit set and the dilatation K of the group.

Conjecture 6.1. *Let G be a discrete non-elementary quasiconformal group acting on $\overline{\mathbf{R}}^n$ with non-empty regular set. Then*

$$\delta_{\mathrm{chord}}(G) \leq \frac{(n+c)\dim\Lambda_c(G)}{c+\dim\Lambda_c(G)},$$

where

$$c = \frac{n}{K^{1/(n-1)}-1}.$$

A variant of this inequality was established in dimension $n=2$ in [BTT2] using the 2-dimensional analog of Theorem 5.3. The approach used in dimension 2 does not extend to higher dimensions because of the existence of discrete quasiconformal groups that are not conjugate to Möbius groups ([FS], [T3], see also [S1], [T1]). Recent progress has been made towards developing sharp bounds in dimension 2 in [BTT4].

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Received 30 December 2003