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WEAKLY COMPACT WEDGE OPERATORS ON KÖTHE ECHELON SPACES

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Abstract. We study wedge operators defined on spaces of operators between Köthe echelon or co-echelon spaces of order $1 < p < \infty$. In this case the wedge operator, defined by $T \to LTR$ for non-zero operators L and R, maps bounded sets into relatively weakly compact sets if and only if the operator L or the operator R maps bounded sets into relatively compact sets. This is an extension of a result of Saksman and Tylli for the sequence space l_p. The corresponding result for operators mapping a neighbourhood into a relatively weakly compact set does not hold, as an example shows.

1. Introduction and notation

A continuous, linear operator $T \in L(X,Y)$ between Banach spaces is weakly compact if it maps the closed unit ball of X into a weakly relatively compact subset of Y. There are two possible extensions of this concept in case the continuous linear operator $T \in L(E,F)$ is defined between locally convex spaces E and F. As in [5], we say that T is *reflexive* if it maps bounded sets into weakly relatively compact sets, and it is called *weakly compact* (as in [10, 42.2]) if there is a 0-neighborhood U in E such that $T(U)$ is weakly relatively compact in F. Here we complete our study [4] of reflexive and weakly compact wedge operators defined on spaces of operators between locally convex spaces, analyzing the case of echelon or co-echelon Köthe spaces and extending results due to Saksman and Tylli [15].

Let E_1, E_2, E_3, E_4 be complete locally convex spaces. The wedge operator of $R \in L(E_1,E_2)$ and $L \in L(E_3,E_4)$ is defined by $R \wedge L$: $L(E_2,E_3) \to L(E_1,E_4)$, $T \to LTR$. The compactness of the wedge operator in case each E_i is a Banach space was studied by Vala [17]. He proved that, if R and L are non-zero, $R \wedge L$ is compact if and only if R and L are compact. Apiola [1] and Geue [7] presented

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a complete generalization of Vala's result to the locally convex setting for the nowadays so-called *Montel* operators which are those that map bounded sets into relatively compact sets. The case of compact operators between locally convex spaces was treated by Wrobel [19]. An operator is called *compact* if it maps a 0-neighborhood into a relatively compact set. Weakly compact wedge operators in the Banach space case were studied by Saksman and Tylli [15], [16], Racher [14] and Lindström, Schlüchtermann [11]. Saksman and Tylli proved in [15, 2.9] that if R and L are weakly compact operators between Banach spaces and if R or L is compact, then the operator $R \wedge L$ is weakly compact. The extensions of this result to reflexive and weakly compact operators between locally convex spaces were obtained by the authors in [4, 2.8, 2.9, 2.14, 2.15]. Consequences about reflexive and weakly compact composition operators on weighted spaces of vector valued holomorphic functions on the disc were presented in [4, Theorem 4.3]. In the case the spaces E_i are all Fréchet, or all complete barrelled (DF)-spaces, if $L\colon E_3 \to E_4$ and $R^t\colon (E_2)'_b \to (E_1)'_b$ are reflexive and if L or R^t is Montel, then $R \wedge L$ is reflexive. The work of Saksman and Tylli [15] already ensures that this result is not optimal: by [15, 2.11], if E is a Banach space which is an $\mathscr{L}_1$-space or a $\mathscr{L}_\infty$-space, $E_i = E$ for each i and R and L are non-zero, then $R \wedge L$ is weakly compact on $L(E)$ if and only if L and R are weakly compact. On the other hand, they show in [15, 3.2, 3.3] (see also [16]) that $R \wedge L$ is weakly compact on $L(l_p)$, $1 < p < \infty$, if and only if L or R is compact. We extend this result below for reflexive operators in the case of Köthe echelon and co-echelon spaces of order $1 < p < \infty$ in Theorems 7 and 8 respectively. The cases of Köthe echelon spaces of order one or zero is also treated in Propositions 1 and 6. Example 9 shows that the corresponding results for weakly compact operators does not hold. The sequence space representations of function spaces due to Valdivia and Vogt, see e.g. [18, Chapter 3], yield immediate consequences of our results.

We use standard notation for functional analysis and locally convex spaces [9], [10], [12]. The family of all closed absolutely convex 0-neighborhoods of a locally convex space E is denoted by $\mathscr{U}_0(E)$, the family of all closed absolutely convex bounded subsets of E by $\mathscr{B}(E)$, and the family of all continuous seminorms on E by $cs(E)$. The transpose of an operator T is denoted by T^t. The space of all (continuous, linear) operators between the locally convex spaces E and F is denoted by $L(E,F)$, and we write $L_b(E,F)$ when this space is endowed with the topology of uniform convergence on the bounded subsets of E. A basis of 0-neighborhoods in $L_b(E,F)$ is given by $W(B,V) := \{f \in L(E,F) \mid f(B) \subset V\}$, as B runs in $\mathscr{B}(E)$ and V in $\mathscr{U}_0(F)$. We briefly recall the notation and essential facts about Köthe echelon and co-echelon spaces, see [2], [12].

A Köthe matrix $A = (a_n)_{n\in\mathbf{N}}$ is an increasing sequence of strictly positive functions on $\mathbf{N}$. Corresponding to each Köthe matrix $A = (a_n)_n$ and $1 \le p < \infty$, we associate the spaces

$$\lambda_p(A) = \Big\{ x = \big(x(i)\big)_{i \in \mathbf{N}} \in \mathbf{C}^{\mathbf{N}} \text{ (or } \mathbf{R}^{\mathbf{N}});$$
$$\text{for all } n \in \mathbf{N}: \ q_n^p(x) = \Big(\sum_i \big(a_n(i)|x(i)|\big)^p \Big)^{1/p} < \infty \Big\},$$
$$\lambda_\infty(A) = \Big\{ x = \big(x(i)\big)_{i \in \mathbf{N}} \in \mathbf{C}^{\mathbf{N}} \text{ (or } \mathbf{R}^{\mathbf{N}});$$
$$\text{for all } n \in \mathbf{N}: \ q_n^\infty(x) = \sup_i a_n(i)|x(i)| < \infty \Big\},$$
$$\lambda_0(A) = \big\{ x = \big(x(i)\big)_{i \in I\mathbf{N}} \in \mathbf{C}^{\mathbf{N}} \text{ (or } \mathbf{R}^{\mathbf{N}});$$
$$\text{for all } n \in \mathbf{N}: \ \big(a_n(i)x(i)\big)_i \text{ converges to } 0 \big\},$$

the last space endowed with the topology induced by $\lambda_\infty(A)$. The spaces $\lambda_p(A)$ are called (Köthe) *echelon spaces* of order p, $1 \le p \le \infty$ or $p = 0$; they are Fréchet spaces with the sequence of norms $p_n = q_n^p$, $n = 1, 2, \ldots$. If A consists of a single function $a = \big(a(i)\big)_i$, we sometimes write $l_p(a)$ instead of $\lambda_p(A)$, $1 \le p \le \infty$, and $c_0(a)$ instead of $\lambda_0(A)$. The Banach space $l_p(a)$ is a diagonal transform (via a) of the space $l_p = l_p(1)$, $1 \le p \le \infty$. With this notation, $\lambda_p(A) = \operatorname{proj}_n l_p(a_n)$, i.e. $\lambda_p(A)$ is the countable projective limit of the Banach spaces $\big(l_p(a_n)\big)_{n \in \mathbf{N}}$.

For a Köthe matrix $A = (a_n)_n$, denote by $V = (v_n)_n$ the associated decreasing sequence of functions $v_n = 1/a_n$, and put

$$k_p(V) = \operatorname*{ind}_n l_p(v_n), \ 1 \le p \le \infty, \qquad \text{and} \qquad k_0(V) = \operatorname*{ind}_n c_0(v_n).$$

That is, $k_p(V)$ is the increasing union of the Banach spaces $l_p(v_n)$, respectively $c_0(v_n)$, endowed with the strongest locally convex topology under which the injection of each of these Banach spaces is continuous. The spaces $k_p(V)$ are called *co-echelon spaces* of order p. They are (LB)-spaces, and they are complete if $p \ne 0$. For a given decreasing sequence $V = (v_n)_n$ of strictly positive functions on $\mathbf{N}$ or for the corresponding Köthe matrix $A = (a_n)_n$, we associate as in [2] the system

$$\lambda_\infty(A)_+ = \Big\{ \bar{v} = \big(\bar{v}(i)\big)_i \in \mathbf{R}_+^{\mathbf{N}}; \text{ for all } n \in \mathbf{N}: \ \sup_i \frac{\bar{v}(i)}{v_n(i)} = \sup_i a_n(i)\bar{v}(i) < \infty \Big\},$$

which will be denoted by $\overline{V} = \overline{V}(V)$. The system $\overline{V}$ can be used to characterize the bounded subsets of $\lambda_p(A)$, as follows, [2]: *Let A be a Köthe matrix on $\mathbf{N}$. A subset B of $\lambda_p(A)$, $1 \le p \le \infty$, is bounded if and only if there exists a strictly positive weight $\bar{v} \in \overline{V}$ so that*

$$B \subset \bar{v}B(l_p) = \big\{ y \in \mathbf{C}^{\mathbf{N}} \ (\textit{or } \mathbf{R}^{\mathbf{N}}); \ y(i) = \bar{v}(i)z(i), \ i \in \mathbf{N}, \ \textit{for some } z \in B(l_p) \big\},$$

where $B(l_p)$ denotes the closed unit ball of the Banach space l_p. This result will be used several times in the proofs below.

The duality of the echelon and co-echelon spaces is as follows: *For $1 < p < \infty$ or $p = 0$, if $(1/p) + (1/q) = 1$ (where we take $q = 1$ for $p = 0$), then $\big(\lambda_p(A)\big)'_b = k_q(V)$ and $\big(k_p(V)\big)'_b = \lambda_q(A)$.* See [2] for the remaining cases.

Every bounded subset of the co-echelon space $k_p(V)$, $1 < p < \infty$, is contained in the unit ball of one of the steps $l_p(v_n)$. Moreover, the inductive limit topology of $k_p(V)$, $1 < p < \infty$, can be described using the system $\overline{V}$ of associated weights as follows, [2]:

$$k_p(V) = \operatorname*{proj}_{\bar v \in \overline{V}} l_p(\bar v).$$

2. Main results

We first treat the case of Köthe echelon spaces of order one which is easy.

Proposition 1. *Let L: $\lambda_1(A^3) \to \lambda_1(A^4)$ and R: $\lambda_1(A^1) \to \lambda_1(A^2)$ be non-zero operators on Köthe echelon spaces of order 1. The wedge operator $R \wedge L$: $L_b\big(\lambda_1(A^2), \lambda_1(A^3)\big) \to L_b\big(\lambda_1(A^1), \lambda_1(A^4)\big)$ is reflexive if and only if L and R are Montel.*

Proof. First suppose that $R \wedge L$ is reflexive. By [4, 2.1], L and R^t are reflexive. By [5, 1.2], R is also reflexive. Since every weakly compact subset of a Köthe echelon space of order one is compact [18, Chapter 2, Section 2.2(2), p. 214], L and R are Montel operators. Now, if L and R are Montel, then L and R^t are Montel, see [6, 2.3]. The extension due to Apiola [1] of Vala's classical result implies that $R \wedge L$ is a Montel operator, hence reflexive. ▫

The following well-known result will be used several times. It can be seen e.g. in [8, Proposition 3, p. 123]: *Let E and F be locally convex spaces. Suppose that $f \in L(E,F)$ and $g \in L(F,E)$ satisfy that $g \circ f$ coincides with the identity on E. Then f is a topological isomorphism from E onto $f(E) \subset F$ and $f(E)$ is a complemented subspace of F.* The following result about compact operators on classical Banach sequence spaces is proved for example in the proofs of [15, Propositions 3.2 and 3.3]. We state it in the form which will be needed later.

Lemma 2. (a) *Let $1 < p < \infty$. If $f \in L(l_p, l_p)$ is not compact, then there exist a complemented subspace X_0 of l_p isomorphic to l_p and a complemented subspace Y_0 of l_p such that $f \mid X_0$ is an isomorphism from X_0 onto Y_0.*

(b) *If $f \in L(c_0, c_0)$ is not compact, then there exist a subspace X_0 of c_0 isomorphic to c_0 and a subspace Y_0 of c_0 such that $f \mid X_0$ is an isomorphism from X_0 onto Y_0. In particular, f is not weakly compact.*

Proposition 3. *Let $1 < p < \infty$ and let $A = (a_n)_n$ and $B = (b_n)_n$ be Köthe matrices. If $f \in L\big(\lambda_p(A), \lambda_p(B)\big)$ is not Montel, there is a complemented* *subspace $X_0 \subset \lambda_p(A)$ isomorphic to l_p and there is a complemented subspace Y_0 of $\lambda_p(B)$ such that $f \mid X_0$ is a isomorphism from X_0 onto Y_0.*

Proof. Since $f\colon \lambda_p(A) \to \lambda_p(B)$ is not a Montel operator, there is a bounded subset M of $\lambda_p(A)$ whose image $f(M)$ is not relatively compact in $\lambda_p(B)$. By [2, Proposition 2.5], there is $\bar{v} \in \overline{V}(A)$ such that $\bar{v}(i) > 0$ for each $i \in \mathbf{N}$ and

$$M \subset M_{\bar{v}} := \left\{ x = \big(x(i)\big) \in \lambda_p(A) \;\Big|\; \sum_{i=1}^{\infty} \left(\frac{|x(i)|}{\bar{v}(i)} \right)^p \leq 1 \right\}.$$

Therefore $M \subset M_{\bar{v}}$ and $M_{\bar{v}}$ can be seen as the unit ball of the Banach sequence space $l_p(1/\bar{v})$. We denote by $j\colon l_p(1/\bar{v}) \to \lambda_p(A)$ the canonical inclusion. The set $fj(M_{\bar{v}})$ is not relatively compact in $\lambda_p(B)$. As $\lambda_p(B) = \operatorname{proj}_n l_p(b_n)$, there is $n \in \mathbf{N}$ such that $\pi_n fj\colon l_p(1/\bar{v}) \to l_p(b_n)$ is not compact. Here $\pi_n\colon \lambda_p(B) \to l_p(b_n)$ is the canonical injection with dense range.

We apply Lemma 2(a) to obtain a subspace X_0 of $l_p(1/\bar{v})$ which is complemented, isomorphic to l_p, and a subspace Y_0 of $l_p(b_n)$ which is complemented and such that the operator $(\pi_n fj) \mid X_0$ is an isomorphism from X_0 onto Y_0. The proof is now completed in several steps:

(1): $j\colon X_0 \to \lambda_p(A)$ is a topological isomorphism into.

Clearly j is injective and continuous. Suppose that $(x_k)_k \subset X_0$ satisfies that $j(x_k) \to 0$ in $\lambda_p(A)$. Then $\pi_n fj(x_k) \to 0$ in $l_p(b_n)$. Since $(\pi_n fj) \mid X_0$ is an isomorphism, $x_k \to 0$ in X_0.

We denote by $P_2\colon l_p(b_n) \to l_p(b_n)$ the continuous projection onto Y_0. We have $P_2(y) = y$ for each $y \in Y_0$. The operator $(\pi_n fj)^{-1} P_2(\pi_n f)\colon \lambda_p(A) \to X_0$ is well-defined, linear and continuous. Moreover, if $x \in X_0$, we have

$$(\pi_n fj)^{-1} P_2(\pi_n f) j(x) = (\pi_n fj)^{-1} (\pi_n fj)(x) = x,$$

since $(\pi_n fj)(x_0) \in Y_0$. We apply [8, Proposition 3, p. 123] mentioned above to obtain that

(2): X_0 is isomorphic to $j(X_0)$, hence to l_p, and $j(X_0)$ is a complemented subspace of $\lambda_p(A)$.

(3): If we consider $fj(X_0)$ as a subspace of $\lambda_p(B)$, then $\pi_n\colon fj(X_0) \to Y_0$ is a topological isomorphism.

Indeed, π_n is linear and continuous, and it is injective on $fj(X_0)$ because $\pi_n fj$ is an isomorphism on X_0. Moreover, π_n is surjective: given $y \in Y_0$, $x = (\pi_n fj)^{-1} y \in X_0$ and $\pi_n\big(fj(x)\big) = y$. Suppose now that $(x_k)_k \subset X_0$ satisfies $\pi_n\big(fj(x_k)\big) \to 0$ in Y_0. Then $x_k \to 0$ in X_0, thus $fj(x_k) \to 0$ in $\lambda_p(B)$.

Part (3) implies that $\pi_n^{-1}(Y_0)$ is a Banach subspace of $\lambda_p(B)$ which is isomorphic to X_0, hence to l_p.

(4): $\pi_n^{-1}(Y_0)$ is a complemented subspace of $\lambda_p(B)$.

Consider $\pi_n^{-1}\colon Y_0 \to \lambda_p(B)$, which is linear and continuous by (3), and $P_2\pi_n\colon \lambda_p(B) \to Y_0$ which is also linear and continuous. As $(P_2\pi_n)\pi_n^{-1}(y) = P_2 y = y$ for each $y \in Y_0$, the conclusion follows again from [8, Proposition 3, p. 123]. □

Proposition 4. *Let $1 < p < \infty$ and let $V = (v_n)_n$ and $W = (w_n)_n$ be decreasing sequences of strictly positive weights on $\mathbf{N}$. If $f \in L\bigl(k_p(V), k_p(W)\bigr)$ is not Montel, there is a complemented subspace $X_0 \subset k_p(A)$ isomorphic to l_p and there is a complemented subspace Y_0 of $k_p(B)$ such that $f \mid X_0$ is a topological isomorphism from X_0 onto Y_0.*

Proof. This can be proved along the lines of the proof of Proposition 3. Since $f\colon k_p(V) \to k_p(W)$ is not a Montel operator, there is a bounded subset M of $k_p(V)$ whose image $f(M)$ is not relatively compact in $k_p(W)$. There is n such that, if we denote by $j_n\colon l_p(v_n) \to k_p(V)$ the canonical inclusion, the image by fj_n of the closed unit ball of $l_p(v_n)$ is not relatively compact in $k_p(W)$. By the description of the topology of $k_p(W)$ mentioned in the introduction, there is a strictly positive element $\overline{w} \in \overline{W}$ such that $\pi f j_n\colon l_p(v_n) \to l_p(\overline{w})$ is not compact, where $\pi\colon k_p(W) \to l_p(\overline{w})$ is the canonical inclusion with dense range. The proof now continues as before, applying Lemma 2(a) to $\pi f j_n\colon l_p(v_n) \to l_p(\overline{w})$. □

Proposition 5. *An operator $f\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive if and only if it is Montel.*

Proof. We proceed again as in the proof of Proposition 3. If the operator $f\colon \lambda_0(A) \to \lambda_0(B)$ is not Montel, we find $\bar v \in \overline{V}(A)$ and $n \in \mathbf{N}$ such that $\pi_n f j\colon c_0(1/\bar v) \to c_0(b_n)$ is not compact. Here $j\colon c_0(1/\bar v) \to \lambda_0(a)$ and $\pi_n\colon \lambda_0(B) \to c_0(b_n)$ are the canonical inclusions. By Lemma 2(b), there are a closed subspace X_0 of $c_0(1/\bar v)$ isomorphic to c_0 and a closed subspace Y_0 of $c_0(b_n)$ such that $(\pi_n f j) \mid X_0$ is an isomorphism from X_0 onto Y_0. As in the part (1) of the proof of Proposition 3 we conclude that $j\colon X_0 \to \lambda_0(A)$ is a topological isomorphism into, and as in the proof of part (3), $\pi_n\colon fj(X_0) \subset \lambda_0(B) \to Y_0$ is also a topological isomorphism into. Therefore, the restriction of f to $j(X_0) \subset \lambda_0(A)$ is an isomorphism into $\lambda_0(B)$ and $j(X_0)$ is isomorphic to c_0. This implies that f is not reflexive. □

If A is a Köthe matrix and we apply Proposition 5 to the identity id: $\lambda_0(A) \to \lambda_0(A)$, we conclude the following result of Valdivia [18, Chapter 2, Section 4, 2(1)]: *a Köthe echelon space $\lambda_0(A)$ is reflexive if and only if it is a Montel space.*

Proposition 6. *Let $L\colon \lambda_0(A^3) \to \lambda_0(A^4)$ and $R\colon \lambda_0(A^1) \to \lambda_0(A^2)$ be non-zero operators on Köthe echelon spaces of order 0. The wedge operator $R \wedge L\colon L_b\bigl(\lambda_0(A^2), \lambda_0(A^3)\bigr) \to L_b\bigl(\lambda_0(A^1), \lambda_0(A^4)\bigr)$ is reflexive if and only if L and R are Montel.*

Proof. First suppose that $R \wedge L$ is reflexive. By [4, 2.1], L and R^t are reflexive. By [5, 1.2], R is also reflexive. Proposition 5 implies now that L and R are Montel operators. Conversely, if L and R are Montel, then L and R^t are Montel, see [6, 2.3]. The extension due to Apiola [1] of Vala's classical result implies that $R \wedge L$ is a Montel operator, hence reflexive. □

Theorem 7. *Let $1 < p < \infty$. Let $L\colon \lambda_p(A^3) \to \lambda_p(A^4)$ and $R\colon \lambda_p(A^1) \to \lambda_p(A^2)$ be non-zero operators between Köthe echelon spaces of order p. The wedge operator $R \wedge L\colon L_b\big(\lambda_p(A^2), \lambda_p(A^3)\big) \to L_b\big(\lambda_p(A^1), \lambda_p(A^4)\big)$ is reflexive if and only if L or R is a Montel operator.*

Proof. As all the Köthe echelon spaces here are reflexive, the operators L and R are reflexive. Therefore, it is enough to show that the condition is necessary by [4, 2.11] and [6, 2.3]. Suppose now that neither R nor L is a Montel operator. We apply Proposition 3 to find a complemented subspace X_0 of $\lambda_p(A^1)$ isomorphic to l_p such that the restriction $R_1\colon X_0 \to R(X_0)$ is a topological isomorphism and $R(X_0)$ is complemented in $\lambda_p(A^2)$. We denote by $j_1\colon X_0 \to \lambda_p(A^1)$ the inclusion and by $P_1\colon \lambda_p(A^2) \to R(X_0)$ the continuous projection. Analogously, Proposition 3 provides us with a complemented subspace Y_0 of $\lambda_p(A^3)$ isomorphic to l_p such that the restriction $L_2\colon Y_0 \to L(Y_0)$ is a topological isomorphism and $L(Y_0)$ is complemented in $\lambda_p(A^4)$. We denote by $j_2\colon Y_0 \to \lambda_p(A^3)$ the injection and by $P_2\colon \lambda_p(A^4) \to L(Y_0)$ the continuous projection.

Since R_1 and L_2 are isomorphisms, it is easy to see that the wedge operator

$$R_1 \wedge L_2\colon L_b\big(R(X_0), Y_0\big) \to L_b\big(X_0, L(Y_0)\big)$$

is an isomorphism, too. Moreover,

$$L_b\big(R(X_0), Y_0\big) \simeq L_b\big(X_0, L(Y_0)\big) \simeq L_b(l_p, l_p)$$

is not a reflexive space, because it contains a copy of l_∞.

Suppose that the wedge operator

$$R \wedge L\colon L_b\big(\lambda_p(A^2), \lambda_p(A^3)\big) \to L_b\big(\lambda_p(A^1), \lambda_p(A^4)\big)$$

is reflexive. Then the operator

$$(j_1 \wedge P_2) \circ (R \wedge L) \circ (P_1 \circ j_2)\colon L_b\big(R(X_0), Y_0\big) \to L_b\big(X_0, L(Y_0)\big)$$

is also reflexive. It is easy to see that

$$(j_1 \wedge P_2) \circ (R \wedge L) \circ (P_1 \circ j_2) = R_1 \circ L_2,$$

which is an isomorphism between non-reflexive spaces; a contradiction. □

If we use Proposition 4 instead of Proposition 3, the proof above shows the following characterization in the case of co-echelon spaces.

Theorem 8. *Let $1 < p < \infty$. Let $L\colon k_p(V^3) \to k_p(V^4)$ and $R\colon k_p(V^1) \to k_p(V^2)$ be non-zero operators between Köthe co-echelon spaces of order p. The wedge operator $R \wedge L\colon L_b\big(k_p(V^2), k_p(V^3)\big) \to L_b\big(k_p(V^1), k_p(V^4)\big)$ is reflexive if and only if L or R is a Montel operator.*

As a consequence of [4, 2.14, 2.15], if $1 < p < \infty$, $L\colon \lambda_p(A^3) \to \lambda_p(A^4)$ and $R\colon \lambda_p(A^1) \to \lambda_p(A^2)$ satisfy that L and R^t are weakly compact and L or R^t is compact, the wedge operator $R \wedge L\colon L_b\big(\lambda_p(A^2), \lambda_p(A^3)\big) \to L_b\big(\lambda_p(A^1), \lambda_p(A^4)\big)$ is weakly compact. It is a natural question whether a result similar to Theorem 7 would hold for weakly compact operators on Köthe echelon spaces of order $1 < p < \infty$. The answer is negative as the following example shows. The authors thank P. Domański for his suggestions concerning the example.

Example 9. Let $1 < p < \infty$. Grothendieck gave an example of a Köthe echelon space $\lambda_p(A)$ which is a Fréchet Montel space with a quotient isomorphic to the Banach space l_p. See e.g. [10, 33.6(2)]. We denote by $Q\colon \lambda_p(A) \to l_p$ the quotient map. The operator Q is weakly compact but not compact.

We set $\lambda_p(A^1) = \lambda_p(A)$, $\lambda_p(A^2) = l_p$, $\lambda_p(A^3) = \lambda_p(A)$, $\lambda_p(A^4) = l_p$, $R\colon \lambda_p(A^1) \to \lambda_p(A^2)$, $R = Q$ and $L\colon \lambda_p(A^3) \to \lambda_p(A^4)$, $L = Q$.

Since R and L are bounded (i.e. they map a 0-neighbourhood into a bounded set), it is easy to see that the wedge operator

$$R \wedge L\colon L_b\big(l_p, \lambda_p(A)\big) \to L_b\big(\lambda_p(A), l_p\big)$$

is also bounded. As $\lambda_p(A)$ is a Montel Köthe echelon space, the space $L_b\big(\lambda_p(A), l_p\big)$ is a reflexive (DF)-space by [3]. Therefore the wedge operator $R \wedge L$ is weakly compact. However, neither R nor L is compact.

An example in which all the spaces E_1, E_2, E_3, E_4 coincide can be easily obtained as follows: define the spaces $E_1 := \lambda_p(A) \times l_p$, $i = 1, 2, 3, 4$, and the operators

$$R\colon \lambda_p(A) \times l_p \to \lambda_p(A) \times l_p, R(x, y) = \big(0, Q(x)\big),$$
$$L\colon \lambda_p(A) \times l_p \to \lambda_p(A) \times l_p, L(x, y) = \big(0, Q(x)\big).$$

In this case $T\colon \lambda_p(A) \times l_p \to \lambda_p(A) \times l_p$ has the form

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

and

$$(R \wedge L)(T)(x, y) = \big(0, QT_{12}Q(x)\big)$$

for each (x, y).

References

[1] Apiola, H.: On the tensor product and Hom (f, g) of compact operators in locally convex topological vector spaces. - Ann. Acad. Sci. Fenn. Ser. A I Math. 544, 1973.

[2] Bierstedt, K. D., R. Meise, and W. H. Summers: Köthe sets and Köthe sequence spaces. - In: Functional Analysis, Holomorphy and Approximation Theory, edited by J. H. Barroso, North-Holland Math. Studies 71, 1982, 27–91.

[3] Bonet, J., and J. C. Díaz: The problem of topologies of Grothendieck and the class of Fréchet T-spaces. - Math. Nachr. 150, 1991, 109–118.

[4] Bonet, J., and M. Friz: Weakly compact composition operators on locally convex spaces. - Math. Nachr. 245, 2002, 26–44.

[5] Díaz, J., and P. Domański: Reflexive operators with domain in Köthe spaces. - Manuscripta Math. 97, 1998, 189–204.

[6] Dierolf, S., and P. Domański: Factorization of Montel operators. - Studia Math. 107, 1993, 15–32.

[7] Geue, A. S.: Precompact and collectively semi-precompact sets of semi-precompact continuous linear operators. - Pacific J. Math. 52, 1974, 377–401.

[8] Horváth, J.: Topological Vector Spaces and Distributions. - Addison-Wesley, Massachussets, 1969.

[9] Jarchow, H.: Locally Convex Spaces. - Math. Leitfäden, B.G. Teubner, Stuttgart, 1981.

[10] Köthe, G.: Topological Vector Spaces I and II. - Springer-Verlag, Berlin–Heidelberg–New York, 1969 and 1979.

[11] Lindström, M., and G. Schlüchtermann: Composition of operator ideals. - Math. Scand. 84:2, 1999, 284–296.

[12] Meise, R., and D. Vogt: Introduction to Functional Analysis. - Clarendon Press, Oxford, 1997.

[13] Pérez Carreras, P., and J. Bonet: Barreled Locally Convex Spaces. - North-Holland Math. Studies 131, Amsterdam, 1987.

[14] Racher, G.: On the tensor product of weakly compact operators. - Math. Ann. 294, 1992, 267–275.

[15] Saksman, E., and H.-O. Tylli: Weak compactness of multiplication operators on spaces of bounded linear operators. - Math. Scand. 70, 1992, 91–111.

[16] Saksman, E., and H.-O. Tylli: Weak essential spectra of multiplication operators on spaces of bounded linear operators. - Math. Ann. 299, 1994, 299–309.

[17] Vala, K.: On compact sets of compact operators. - Ann. Acad. Sci. Fenn. Ser. A I Math. 351, 1964.

[18] Valdivia, M.: Topics in Locally Convex Spaces. - North-Holland Math. Studies 67, Amsterdam, 1982.

[19] Wrobel, V.: Some generalizations of Schauder's theorem in locally convex spaces. - Pacific J. Math. 61, 1975, 579–586.

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