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CONTINUOUS DEPENDENCE ON OBSTACLES IN DOUBLE GLOBAL OBSTACLE PROBLEMS

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Abstract. Let $A\colon H_0^1(\Omega)\to H^{-1}(\Omega)$ be Lipschitz and monotone operator; let $\langle\,\cdot\,,\cdot\,\rangle$ stand for the dual bracket between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. For given functions φ, ψ we define (under suitable assumptions) the admissible set $K_\varphi^\psi := \{v\in H_0^1(\Omega) : \varphi\le v\le\psi \text{ a.e. in } \Omega\}$.

Next for sequences (φ_n), (ψ_n), converging to φ, ψ, respectively, we consider the sequence of admissible sets $(K_{\varphi_n}^{\psi_n})$ defined as $K_{\varphi_n}^{\psi_n} := \{v\in H_0^1(\Omega) : \varphi_n\le v\le\psi_n \text{ a.e. in } \Omega\}$. Then for $f\in H^{-1}(\Omega)$ we discuss the following obstacle problems.

(P_n): Find $u_n\in K_{\varphi_n}^{\psi_n} : \langle Au_n, v_n-u_n\rangle \ge \langle f, v_n-u_n\rangle$ for all $v_n\in K_{\varphi_n}^{\psi_n}$
and

(P): Find $u\in K_\varphi^\psi : \langle Au, v-u\rangle\ge\langle f, v-u\rangle$ for all $v\in K_\varphi^\psi$.

The above problems represent the so-called double global obstacle problem. See [2], [3] for existence and regularity results.

The purpose of this paper is to study convergence (in certain sense) of the solutions u_n of (P_n) providing the sequences of impediments converge to their limits. Such problems are known as “varying obstacle problems”.

We extend here the results given in [6] where the author has considered the global obstacle problem (see [2], [3] for definitions, existence and regularity results).

1. Introduction

In this paper we study continuity properties corresponding to double global obstacle problems. We consider two sequences of impediments (φ_n) and (ψ_n) converging to φ and ψ, respectively. Let u_n for $n\in\mathbf{N}$ denote the solution of the double global obstacle problem with the admissible set $K_{\varphi_n}^{\psi_n} = \{v\in H_0^1(\Omega) : \varphi_n\le v\le\psi_n \text{ a.e. in } \Omega\}$. We analyze the convergence of the sequence (u_n) to the solution u of the obstacle problem with the admissible set $K_\varphi^\psi = \{v\in H_0^1(\Omega) : \varphi\le v\le\psi \text{ a.e. in } \Omega\}$. Certain assumptions are imposed on the functions φ_n and ψ_n.

We present two results. In the first one we assume that the sequence (φ_n) converges to φ from below while (ψ_n) converges to ψ from above. We prove that the solutions u_n approach u. We also give an example illustrating that if one of the assumptions is not satisfied then the convergence of impediments does not imply convergence of the solutions to the solution of the limit problem.

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In the second one we assume that both (φ_n) and (ψ_n) converge to φ and ψ, respectively, from above. Moreover, the functions φ_n are convex while ψ_n are concave. Then the solutions u_n approach the solution of the limit problem.

The proofs are based on properties of $H_0^1(\Omega)$ functions, Sobolev's embedding theorem and Minty's lemma.

2. Notation and basic definitions

Throughout this paper we shall use the following notation and assumptions if not stated otherwise:

$\Omega \subset \mathbf{R}^n$ bounded domain with smooth boundary $\partial\Omega$;

$H^1(\Omega)$ is the Sobolev space $W^{1,2}(\Omega)$;

$H_0^1(\Omega)$ represents H^1 closure of $C_0^\infty(\Omega)$;

$(\cdot\,,\cdot)$ stands for the inner product in $L^2(\Omega)$;

$Q(\Omega)$ is the vector space of equivalent classes of quasi-continuous functions;

$L_c^2(\Omega) := \big\{\varphi \in Q(\Omega) : \widetilde{v} \geq |\varphi|$ quasi-everywhere in Ω for $v \in H_0^1(\Omega)$, where $\widetilde{v}$ is a quasi-continuous representative of $v\big\}$.

The definitions of a quasi-continuous function, properties which hold quasi-everywhere and $Q(\Omega)$ can be found in [5].

It is well known (see [4]) that the sequence of solutions (u_n) converges strongly to u in $H_0^1(\Omega)$ provided $(K_{\varphi_n}^{\psi_n})$ is a sequence of closed, convex subsets of $H_0^1(\Omega)$ which converges to K_φ^ψ in the sense of Mosco.

The capacity problems allowed to introduce necessary and sufficient conditions for convergence in the sense of Mosco of sequences of the convex admissible sets; see [1]. Referring to the result mentioned we recall that (u_n) converges strongly to u in $H_0^1(\Omega)$ provided (φ_n) and (ψ_n) converge to φ, ψ in $L_c^2(\Omega)$.

Similarly as in [6] we define the class of obstacles converging from above and from below. More precisely:

Definition 1. We say that the sequence (γ_n) converges to γ from below (above) a.e. in Ω provided:

(i) $\gamma_n \underset{n\to\infty}{\longrightarrow} \gamma$ a.e. in Ω,

(ii) for all $n \in \mathbf{N}$ $\gamma_n \leq \gamma$ $(\gamma_n \geq \gamma)$ a.e. in Ω.

Definition 2. We say that the sequence (γ_n) converges to γ from below (above) in $L^2(\Omega)$ provided:

(i) $\gamma_n \underset{n\to\infty}{\longrightarrow} \gamma$ in $L^2(\Omega)$,

(ii) for all $n \in \mathbf{N}$ $\gamma_n \leq \gamma$ $(\gamma_n \geq \gamma)$ a.e. in Ω.

Throughout this paper we agree that we are given an operator $A\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ which is Lipschitz and monotone i.e. there exist $\alpha, \gamma > 0$ such that for all $u, v \in H_0^1(\Omega)$

$$\|Au - Av\| \leq \gamma \|u - v\|,$$
$$\langle Au - Av, u - v\rangle \geq \alpha \|u - v\|^2;$$

$f \in H^{-1}(\Omega)$ and functions $\varphi_n, \varphi, \psi_n, \psi \in L_c^2(\Omega)$, $n \in \mathbf{N}$.

We consider double global obstacle problems P_n with impediments φ_n, ψ_n given by:

P_n: Find $u_n \in K_{\varphi_n}^{\psi_n} := \left\{v \in H_0^1(\Omega) : \varphi_n \leq v \leq \psi_n \text{ a.e. in } \Omega\right\}$ such that

$$\langle Au_n, v_n - u_n\rangle \geq \langle f, v_n - u_n\rangle \quad \text{for all } v_n \in K_{\varphi_n}^{\psi_n}$$

and the double global obstacle problem P with impediments φ, ψ defined as:

P: Find $u \in K_{\varphi}^{\psi} := \left\{v \in H_0^1(\Omega) : \varphi \leq v \leq \psi \text{ a.e. in } \Omega\right\}$ such that

$$\langle Au, v - u\rangle \geq \langle f, v - u\rangle \quad \text{for all } v \in K_{\varphi}^{\psi}.$$

3. Continuity results

Our first result consists in showing that the solution of the double global obstacle problem with impediments φ and ψ can be obtained as the limit (in the $H_0^1(\Omega)$ space) of the solutions u_n with impediments φ_n, ψ_n converging to φ, ψ from below and above, respectively. Observe that monotonicity of the sequences (φ_n) and (ψ_n) is not required.

The proof is based on properties of $H_0^1(\Omega)$ functions taking advantage of Sobolev's embedding theorem and Minty's lemma.

Theorem 1. *If we assume that the sequences (φ_n) and (ψ_n) converge a.e. in Ω to φ and ψ from below and above, respectively, and $\varphi_n \leq \psi_n$ a.e. in Ω, $\varphi_{n|\partial\Omega} \leq 0$, $\psi_{n|\partial\Omega} \geq 0$ then the solutions u_n of the double global obstacle problem P_n converge strongly in $H_0^1(\Omega)$ to the solution u of the double global obstacle problem P.*

Here we present an example which illustrates that convergence from above of the impediments does not necessarily imply convergence of solutions to the limit solution.

Example. Let $\Omega = (-1; 1)$, $A = -d^2/dx^2$ and $f \equiv 0$ in Ω. We take

$$\varphi_n = \begin{cases} 1 & \text{in } \left[-\dfrac{1}{n+1}; \dfrac{1}{n+1}\right], \\ 0 & \text{otherwise.} \end{cases}$$

We put

$$\psi_n = \begin{cases} \dfrac{1}{2} & \text{in } \left(-1; -\dfrac{2}{n+2}\right] \cup \left[\dfrac{2}{n+2}; 1\right), \\ 2 & \text{otherwise.} \end{cases}$$

Note that $\varphi = 0$ and $\psi = \frac{1}{2}$ a.e. in Ω. Moreover, for all $n \in \mathbf{N}$

$$\big(\psi_n \geq \varphi_n \text{ a.e. in } \Omega,\ \varphi_{n|\partial\Omega} \leq 0,\ \psi_{n|\partial\Omega} \geq 0\big).$$

Next we have for all $n \in \mathbf{N}$, $(\varphi_n \geq \varphi, \psi_n \geq \psi)$.

We consider the sequence (P_n) of the double global obstacle problems: Find $u_n \in K_{\varphi_n}^{\psi_n} := \big\{v \in H_0^1((-1;1)) : \varphi_n \leq v \leq \psi_n \text{ a.e. in } (-1;1)\big\}$ such that:

$$\int_{-1}^{1} u_n' \cdot (u_n' - v') \geq 0 \quad \text{for all } v \in K_{\varphi_n}^{\psi_n}.$$

We know that $H_0^1\big((-1;1)\big)$ is embedded in $C_0^{1,1-1/2}\big((-1;1)\big)$ from the Sobolev theorem, so after an easy computation we arrive at the solutions of P_n given by

$$u_n(x) = \begin{cases} \dfrac{n+2}{2n}x + \dfrac{n+2}{2n}, & x \in \left(-1, \dfrac{-2}{n+2}\right], \\ \dfrac{(n+2)(n+1)}{2n}x + \dfrac{3n+2}{2n}, & x \in \left[\dfrac{-2}{n+2}, \dfrac{-1}{n+1}\right], \\ 1, & x \in \left[\dfrac{-1}{n+1}, \dfrac{1}{n+1}\right], \\ \dfrac{-(n+2)(n+1)}{2n}x + \dfrac{3n+2}{2n}, & x \in \left[\dfrac{1}{n+1}, \dfrac{2}{n+2}\right], \\ \dfrac{-(n+2)}{2n}x + \dfrac{n+2}{2n}, & x \in \left[\dfrac{2}{n+2}, 1\right). \end{cases}$$

We notice that $u_n \to u^*$ a.e. in Ω where

$$u^* = \begin{cases} \dfrac{x}{2} + \dfrac{1}{2}, & x \in [-1;0], \\ -\dfrac{x}{2} + \dfrac{1}{2}, & x \in [0;1]. \end{cases}$$

Now the limit obstacle problem P is:

Find $u \in K_\varphi^\psi = \big\{v \in H_0^1\big((-1;1)\big) : 0 \leq v \leq \frac{1}{2} \text{ a.e. in } (-1;1)\big\}$ such that $\int_{-1}^{1} u' \cdot (u' - v') \geq 0$ for all $v \in K_\varphi^\psi$. It is also very easy to see that $u \equiv 0$. Obviously (u_n) does not converge to u in any sense.

Now we concentrate on the case of the double obstacle problems with impediments converging from above. In order to obtain convergence of solutions we introduce some additional assumptions imposed on the functions φ_n and ψ_n. This result is obtained due to a certain approximation procedure applied to a $H_0^1(\Omega)$-function and the theory of singular perturbation problems.

Theorem 2. *Let the following be satisfied*:

$$\begin{aligned} \varphi_n, \psi_n, \varphi, \psi \in H_0^1(\Omega), \\ -\Delta\varphi_n \leq 0 \text{ in } H^{-1}(\Omega), \\ -\Delta\psi_n \geq 0 \end{aligned} \tag{1}$$

in $H^{-1}(\Omega)$, $n \in \mathbf{N}$. *If* $(\varphi_n), (\psi_n)$ *approach* φ, ψ, *respectively, from above in* $L^2(\Omega)$ *then*

$$u_n \underset{n\to\infty}{\longrightarrow} u$$

strongly in $H_0^1(\Omega)$.

Remark. We have also managed to derive a similar result to the one given in [6] for the global double obstacle problem when the impediments φ_n, ψ_n are concave and convex, respectively.

4. Proof of Theorem 1

The existence and uniqueness of the solutions of the problems P_n and P follow directly from the Lions–Stampacchia theorem ([3], [5]). First we show adapting the ideas from [6] that if there exists $v_0 \in \bigcap_{n\geq 1} K_{\varphi_n}^{\psi_n}$ then the following estimate holds:

$$\|u_n - v_0\| \leq \frac{1}{\alpha}(\|f\| + \|Av\|).$$

Indeed we put $v = v_0$ at every problem P_n to get

$$\langle Au_n, v_0 - u_n\rangle \geq \langle f, v_0 - u_n\rangle. \tag{2}$$

Using the fact that A is monotone and Lipschitz, $f \in H^{-1}(\Omega)$ we have

$$\begin{aligned} \alpha\|u_n - v_0\|^2 &\leq \langle Au_n - Av_0, u_n - v_0\rangle \\ &= \langle Au_n, u_n - v_0\rangle - \langle Av_0, u_n - v_0\rangle \\ &\leq \langle f, u_n - v_0\rangle - \langle Av_0, u_n - v_0\rangle \\ &\leq \|f\|\,\|u_n - v_0\| + M\|v_0\|\,\|u_n - v_0\|. \end{aligned} \tag{3}$$

Whence

$$\|u_n - v_0\| \leq \frac{1}{\alpha}(\|f\| + M\|v_0\|).$$

Since v_0 is a priori known we get that (u_n) is bounded which implies that there exists a subsequence

$$\begin{aligned} &u_n \to u^* \text{ weakly in } H_0^1(\Omega);\\ &u_n \to u^* \text{ strongly in } L^2(\Omega);\\ &u_n \to u^* \text{ a.e. in } \Omega \end{aligned} \tag{4}$$

for some $u^* \in H_0^1(\Omega)$. Since $\varphi_n \le u_n \le \psi_n$ a.e. in Ω if we let $n \to \infty$ we obtain that $\varphi \le u^* \le \psi$ a.e. in Ω and this implies that $u^* \in K_\varphi^\psi$.

Now let us remark that for any $v \in K_\varphi^\psi$ we get: $\varphi_n \le \varphi \le v \le \psi \le \psi_n$ a.e. in Ω which gives that

$$K_\varphi^\psi \subset K_{\varphi_n}^{\psi_n} \quad \text{for all } n \in \mathbf{N}. \tag{5}$$

Let us note that $\varphi^+ - \psi^- \in K_\varphi^\psi$ (see [1], [3], [5] for definitions and details). Therefore there exists

$$v_0 \in \bigcap_{n\ge 1} K_{\varphi_n}^{\psi_n}$$

and

$$u_n \to u^* \quad \text{weakly in } H_0^1(\Omega).$$

From Minty's lemma ([3]) we can identify the problem P_n with: $u_n \in K_{\varphi_n}^{\psi_n}$ such that

$$\langle Av, v - u_n\rangle \ge \langle f, v - u_n\rangle \quad \text{for all } v \in K_{\varphi_n}^{\psi_n}.$$

Having (5), we replace $K_{\varphi_n}^{\psi_n}$ by K_φ^ψ and get:

$$\langle Av, v - u_n\rangle \ge \langle f, v - u_n\rangle \quad \text{for all } v \in K_\varphi^\psi.$$

Now we let $n \to \infty$ and we arrive at the following:

$$u^* \in K_\varphi^\psi : \langle Av, v - u^*\rangle \ge \langle f, v - u^*\rangle \quad \text{for all } v \in K_\varphi^\psi.$$

Applying Minty's lemma to the last problem we get:

$$u^* \in K_\varphi^\psi : \langle Au^*, v - u^*\rangle \ge \langle f, v - u^*\rangle \quad \text{for all } v \in K_\varphi^\psi.$$

Uniqueness of the solution of the problem P allows us to state that $u^* = u$. So we have shown that $u_n \to u$ weakly in $H_0^1(\Omega)$.

In order to show strong convergence we observe that

$$u \in \bigcap_{n\ge 1} K_{\varphi_n}^{\psi_n}$$

so we can put $v_0 = u$ in (3). We arrive at the following estimate:

$$\alpha \cdot \|u_n - u\|^2 \le \langle f - Au, u_n - u\rangle.$$

If we let $n \to \infty$ we obtain that $u_n \to u$ strongly in $H_0^1(\Omega)$ which finishes the proof.

5. Proof of Theorem 2

From the maximum principle we find that $\varphi_n \le 0$, $\psi_n \ge 0$ a.e. in Ω. Therefore $0 \in \bigcap_{n\in\mathbf{N}} K_{\varphi_n}^{\psi_n}$ and using the results of Theorem 1 we can estimate $\|u_n\| \le C$ (where C does not depend on n).

Next for $v \in K_\varphi^\psi$ we define its approximation v_n as:

$$v_n \in H_0^1(\Omega) : -\frac{1}{n}\Delta v_n + v_n = (v \vee \varphi_n) - (v \wedge \psi_n) + v, \tag{6}$$

where $v \vee \varphi_n = \max(v, \varphi_n)$ and $v \wedge \psi_n = \min(v, \psi_n)$. By the theory of singular perturbation problems ([5]) (v_n) converges to v in $H_0^1(\Omega)$ and $v_n \to \max(v,\varphi) - \min(v,\psi) + v = v$ strongly in $H_0^1(\Omega)$.

Now we show that $v_n \in K_{\varphi_n}^{\psi_n}$.

First we multiply both sides of (6) by $(\varphi_n - v_n)^+$ and we integrate:

$$\begin{aligned}\int_\Omega \Big(\frac{1}{n}\Delta v_n - v_n\Big)(\varphi_n - v_n)^+ &= -\frac{1}{n}\int_\Omega \nabla v_n \nabla(\varphi_n - v_n)^+ - \int_\Omega v_n(\varphi_n - v_n)\\ &= -\int_\Omega (v \vee \varphi_n - v \wedge \psi_n + v)(\varphi_n - v_n)^+.\end{aligned}$$

Next we add $\int_\Omega \big(-(1/n)\Delta\varphi_n + \varphi_n\big)(\varphi_n - v_n)^+$ to both sides and obtain

$$\begin{aligned}&-\frac{1}{n}\int_\Omega \nabla v_n \nabla(\varphi_n - v_n)^+ - \int_\Omega v_n(\varphi_n - v_n)^+ + \int_\Omega (-\frac{1}{n}\Delta\varphi_n + \varphi_n)(\varphi_n - v_n)^+\\ &= -\frac{1}{n}\int_\Omega \nabla v_n \nabla(\varphi_n - v_n)^+ - \frac{1}{n}\int_\Omega \Delta\varphi_n(\varphi_n - v_n)^+ + \int_\Omega (\varphi_n - v_n)(\varphi_n - v_n)^+\\ &= -\frac{1}{n}\int_\Omega \nabla v_n \nabla(\varphi_n - v_n)^+ + \frac{1}{n}\int_\Omega \nabla\varphi_n \nabla(\varphi_n - v_n)^+ + \int_\Omega (\varphi_n - v_n)(\varphi_n - v_n)^+\\ &= \int_\Omega \frac{1}{n}\nabla(\varphi_n - v_n)\nabla(\varphi_n - v_n)^+ + \int_\Omega (\varphi_n - v_n)(\varphi_n - v_n)^+\\ &= \int_\Omega \frac{1}{n}|\nabla(\varphi_n - v_n)^+|^2 + \int_\Omega |(\varphi_n - v_n)^+|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_\Omega \Big(-\frac{1}{n}\Delta\varphi_n + \varphi_n - v \vee \varphi_n + v \wedge \psi_n - v\Big)(\varphi_n - v_n)^+\\ &\qquad = \int_\Omega \Big(-\frac{1}{n}\Delta\varphi_n - (v - \varphi_n)^+ - (v - \psi_n)^+\Big)(\varphi_n - v_n)^+.\end{aligned}$$

The last integral is nonpositive, since $\varphi_n-(v\vee\varphi_n)=-(v-\varphi_n)^+$ and $(v\wedge\psi_n)-v=-(v-\psi_n)^+$. This gives us that

$$\int_\Omega \frac{1}{n}|\nabla(\varphi_n-v_n)^+|^2+\int_\Omega |(\varphi_n-v_n)^+|^2\le 0$$

which means that $\varphi_n-v_n\le 0$ a.e. in Ω; so $v_n\ge\varphi_n$ a.e. in Ω.

Now we multiply both sides of (6) by $(v_n-\psi_n)^+$ and integrate. Thus

$$\int_\Omega \frac{1}{n}\nabla v_n\nabla(v_n-\psi_n)^++\int_\Omega v_n(v_n-\psi_n)^+=\int_\Omega\big((v\vee\varphi_n)-(v\wedge\psi_n)+v\big)(v_n-\psi_n)^+.$$

Now we add

$$\int_\Omega\left(\frac{1}{n}\Delta\psi_n-\psi_n\right)(v_n-\psi_n)^+$$

to both sides of the above equality and transform, using the Stampacchia result, in order to get

$$\begin{aligned}\int_\Omega&\left(\frac{1}{n}\nabla(v_n-\psi_n)\nabla(v_n-\psi_n)^++(v_n-\psi_n)(v_n-\psi_n)^+\right)\\&=\int_\Omega\frac{1}{n}|\nabla(v_n-\psi_n)^+|^2+\int_\Omega|(v_n-\psi_n)^+|^2\\&=\int_\Omega\left(\frac{1}{n}\Delta\psi_n-\psi_n+(v\vee\varphi_n)-(v\wedge\psi_n)+v\right)(v_n-\psi_n)^+.\end{aligned}$$

Having (1) we deduce that $\varphi_n\le\psi_n$ a.e. in Ω. Moreover, since $\psi_n\ge\psi$ for all $n\in\mathbf{N}$ we note that the first factor in the last integral is nonpositive.

Therefore

$$\int_\Omega\left(\frac{1}{n}|\nabla(v_n-\psi_n)^+|^2+|(v_n-\psi_n)^+|^2\right)\le 0$$

which implies that $v_n\le\psi_n$ a.e. in Ω. This finishes the argumentation demanded for showing that $v_n\in K_{\varphi_n}^{\psi_n}$.

Applying Minty's lemma to P_n we get

$$\langle Aw,w-u_n\rangle\ge\langle f,w-u_n\rangle\quad\text{for all } w\in K_{\varphi_n}^{\psi_n}. \tag{7}$$

Taking into account the above results we deal with the approximation (v_n) of $v\in K_\varphi^\psi$. Therefore we can replace $w=v_n$ in (7) and we have:

$$\langle Av_n,v_n-u_n\rangle\ge\langle f,v_n-u_n\rangle.$$

If we let $n\to\infty$ we arrive at the following:

$$\langle Av,v-u^*\rangle\ge\langle f,v-u^*\rangle\quad\text{for all } v\in K_\varphi^\psi,$$

where u^* was introduced during the proof of Theorem 1.

Since $u_n \in K_{\varphi_n}^{\psi_n}$ we have $\varphi_n \le u_n \le \psi_n$ a.e. in Ω. Letting $n \to \infty$, we obtain, using (4), that $\varphi \le u^* \le \psi$ a.e. in Ω, which indicates that $u^* \in K_\varphi^\psi$. Using Minty's lemma again we transform the last variational inequality into $\langle Au^*, v - u^*\rangle \ge \langle f, v - u^*\rangle$ for all $v \in K_\varphi^\psi$.

Using the uniqueness of the solution of P we have $u = u^*$. This implies that $u_n \to u$ weakly in $H_0^1(\Omega)$.

To get the strong convergence we can take an approximation $(\bar u_n)$ of u. By coerciveness of A we evaluate

$$\alpha \|u_n - \bar u_n\|^2 \le \langle Au_n - A\bar u_n, u_n - \bar u_n\rangle.$$

Next, since we have $\langle Au_n, \bar u_n - u_n\rangle \ge \langle f, \bar u_n - u_n\rangle$, we estimate

$$\begin{aligned} \alpha \|u_n - \bar u_n\|^2 &\le \langle Au_n - A\bar u_n, u_n - \bar u_n\rangle \\ &= \langle Au_n, u_n - \bar u_n\rangle + \langle -A\bar u_n, u_n - \bar u_n\rangle \\ &\le \langle f, u_n - \bar u_n\rangle + \langle -A\bar u_n, u_n - \bar u_n\rangle \\ &= \langle f - A\bar u_n, u_n - \bar u_n\rangle. \end{aligned}$$

So if we let $n \to \infty$ we have that $u_n \to u$ strongly in $H_0^1(\Omega)$, which completes the proof.

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