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A CHARACTERIZATION OF CALORIC MORPHISMS BETWEEN MANIFOLDS

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Abstract. In this paper, we give a characterization of mappings which preserve caloric functions between semi-riemannian manifolds.

1. Introduction

The Appell transformation plays important roles in the study of the heat equation—especially in the study of positive solutions of the heat equation, because it preserves the solution of the heat equation, and also its positivity. In this note, we shall give a characterization of such transformations, called caloric morphisms between manifolds. We treat not only riemannian manifolds but also semi-riemannian manifolds.

H. Leitwiler [4] and the author [7] studied the characterization of caloric morphism on Euclidean domains. For riemannian manifolds, the characterization can be obtained in almost parallel form. But it does not go similarly for semi-riemannian manifolds. The biggest difference is that f^0 can depend on the space variables, although f^0 depends only on the time variable for riemannian manifolds.

We organize this paper as follows: In Section 2, we define the caloric morphism, and state the main theorem and related results. In Section 3, we shall give some lemmas, and prove the main theorem in Section 4. Examples are given in the final Section 5.

2. Notation and results

In this paper we always consider manifolds to be connected and infinitely differentiable. Let (M, g) be a semi-riemannian manifold, that is, M is a manifold endowed with non-degenerate and symmetric metric g which is not necessarily positive definite. If g is positive definite, then (M, g) is called a riemannian

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manifold. For details on semi-riemannian manifolds, we refer to [6] and for our purpose see also [1].

When M is the euclidean space with a translation invariant metric g, we call (M,g) a semi-euclidean space. In [5], the authors determined caloric morphisms between semi-euclidean spaces of same dimension, which is a generalization of the result by H. Leutwiler [4].

We denote by Δ_g the Laplace–Beltrami operator on (M,g) which is given in local coordinates $(x^i)_{i=1}^m$ by

$$\Delta_g u = \sum_{i,j=1}^m \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}\, g^{ij} \frac{\partial u}{\partial x^j}\right),$$

where

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \qquad g = \det(g_{ij})$$

and (g^{ij}) denotes the inverse matrix of (g_{ij}).

A C^2 function $u(t,x)$ defined on an open set in $\mathbf{R}\times M$ is said to be caloric if u satisfies the heat equation

$$H_g u := \frac{\partial u}{\partial t} - \Delta_g u = 0.$$

Definition 1. Let M and N be semi-riemannian manifolds and D be a domain in $\mathbf{R}\times M$. A pair (f,φ) of a C^2 mapping $f\colon D\to \mathbf{R}\times N$ and a C^2 function $\varphi>0$ on D is said to be a caloric morphism if:

(1) $f(D)$ is a domain in $\mathbf{R}\times N$,

(2) for any caloric function u defined on a domain $E\subset \mathbf{R}\times N$, the function

$$\varphi(t,x)\cdot(u\circ f)(t,x)$$

is caloric on $f^{-1}(E)$.

Evidently, the composition of two caloric morphisms is also a caloric morphism. To be precise, let M, N and L be semi-riemannian manifolds and D, E be domains in $\mathbf{R}\times M$, $\mathbf{R}\times N$, respectively. If $(f,\varphi)\colon E\to \mathbf{R}\times L$ and $(g,\psi)\colon D\to \mathbf{R}\times N$ are caloric morphisms such that $g(D)\subset E$, then we can make a caloric morphism $(F,\Phi)\colon D\to\mathbf{R}\times L$ from (f,φ) and (g,ψ) by the composition $(F,\Phi)=\big(f\circ g,(\varphi\circ g)\psi\big)$.

Now we shall state our main theorems, first in the case that M is a riemannian manifold and next in the general case. The author gave the characterization theorem of caloric morphisms on Euclidean spaces in [7] (cf. [4]). It can be generalized to the riemannian case in a very natural form as follows.

Theorem 2.1. *Let (M, g) be a riemannian manifold and (N, h) be a semi-riemannian manifold. For a C^2 mapping f on a domain $D \subset \mathbf{R} \times M$ to $\mathbf{R} \times N$ such that $f(D)$ is a domain and for a C^2 function $\varphi > 0$ on D, the following four statements are equivalent:*

(1) *(f, φ) is a caloric morphism;*

(2) *We write $f = (f^0, f^1, \ldots, f^n)$ for a local coordinate $(y^1, \ldots, y^n)$ of N. Then $f^0, f^1, \ldots, f^n$ and φ satisfy the following equations* (E-1)–(E-4):

  (E-1) $H_g \varphi = 0$,

  (E-2) $H_g f^\alpha = 2g(\nabla_g \log \varphi, \nabla_g f^\alpha) - \sum_{\beta,\gamma=1}^n g(\nabla_g f^\beta, \nabla_g f^\gamma) \cdot {}^h\Gamma^\alpha_{\beta\gamma} \circ f$ *for* $\alpha = 1, \ldots, n$,

  (E-3) $\nabla_g f^0 = 0$,

  (E-4) $g(\nabla_g f^\alpha, \nabla_g f^\beta) = (h^{\alpha\beta} \circ f) \cdot (df^0/dt)$ *for* $\alpha, \beta = 1, \ldots, n$,

where ∇_g denotes the gradient operator of (M, g) and ${}^h\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbol of (N, h);

(3) *There exists a continuous function λ on D, depending only on t, such that*

$$H_g(\varphi \cdot u \circ f)(t, x) = \lambda(t) \cdot \varphi(t, x) \cdot H_h u \circ f(t, x)$$

for any C^2 function u on $\mathbf{R} \times N$;

(4) *There exists a continuous function λ on D such that*

$$H_g(\varphi \cdot u \circ f)(t, x) = \lambda(t, x) \cdot \varphi(t, x) \cdot H_h u \circ f(t, x)$$

for any C^2 function u on $\mathbf{R} \times N$.

In the case of semi-riemannian manifolds, we have the following characterization.

Theorem 2.2. *Let (M, g) and (N, h) be semi-riemannian manifolds. For a C^2 mapping f on a domain $D \subset \mathbf{R} \times M$ to a domain $f(D)$ in $\mathbf{R} \times N$ and for a C^2 function $\varphi > 0$ on D, the following three statements are equivalent:*

(1) *(f, φ) is a caloric morphism;*

(2) *$f = (f^0, f^1, \ldots, f^n)$ and φ satisfy the following equations* (E-5)–(E-8) *for a local coordinate of N:*

  (E-5) $H_g \varphi = 0$,

  (E-6) $H_g f^\alpha = 2g(\nabla_g \log \varphi, \nabla_g f^\alpha) - \sum_{\beta,\gamma=1}^n g(\nabla_g f^\beta, \nabla_g f^\gamma) \cdot {}^h\Gamma^\alpha_{\beta\gamma} \circ f$ *for* $\alpha = 1, \ldots, n$,

  (E-7) $g(\nabla_g f^0, \nabla_g f^\alpha) = 0$ *for* $\alpha = 0, 1, \ldots, n$,

  (E-8) $g(\nabla_g f^\alpha, \nabla_g f^\beta) = (h^{\alpha\beta} \circ f) \cdot \lambda$ *for* $\alpha, \beta = 1, \ldots, n$,

where $\lambda = H_g f^0 - 2g(\nabla_g \log \varphi, \nabla_g f^0)$;

(3) *There exists a continuous function λ on D such that*

$$H_g(\varphi \cdot u \circ f)(t, x) = \lambda(t, x) \cdot \varphi(t, x) \cdot H_h u \circ f(t, x)$$

for any C^2 function u on $\mathbf{R} \times N$.

Remark 1. If $\dim M = \dim N$ or (M,g) is riemannian, then it follows from (E-7) that f^0 depends only on t and then $\lambda = df^0/dt$, which shows that (E-7) and (E-8) are equivalent to (E-3) and (E-4), respectively. Thus Theorem 2.1 follows from Theorem 2.2. Moreover, since $f(D)$ is a domain in our definition, λ does not vanish. In the case that M is not riemannian, it happens that f^0 depends both on t and on x (see Example 4). Also, it happens that λ changes its sign in the case that M is not riemannian (see Example 5).

Corollary 1. *Let (M,g), (N,h) be semi-riemannian manifolds and let the signature of the metric g (respectively h) be (p,q) (respectively (r,s)). If there exists a caloric morphism from $D \subset \mathbf{R} \times M$ to $\mathbf{R} \times N$ such that $\lambda(t,x) \neq 0$ at some point $(t,x) \in D$, then*

$$p \geqq r,\ q \geqq s \qquad \text{or} \qquad p \geqq s,\ q \geqq r$$

holds. Especially, if M and N have the same dimension, then

$$p = r,\ q = s \qquad \text{or} \qquad p = s,\ q = r.$$

Corollary 2. *Let (M,g), (N,h) be semi-riemannian manifolds of same dimensions. If (f,φ) is a caloric morphism from $D \subset \mathbf{R} \times M$ to $\mathbf{R} \times N$ such that f has an inverse mapping f^{-1} on a open set $E \subset f(D)$, then $\left(f^{-1}, 1/\varphi \circ f^{-1}\right)$ is a caloric morphism from E into $\mathbf{R} \times M$.*

Proof. Let v be any C^2 function on $\mathbf{R} \times M$. Then $u = (1/\varphi \circ f^{-1}) \cdot v \circ f^{-1}$ is a C^2 function on E. By (3), there exists a continuous function λ on D such that $H_g(\varphi \cdot u \circ f) = \lambda \cdot \varphi \cdot (H_h u) \circ f$. Since $\dim M = \dim N$, λ does not vanish (see Remark 1). Hence we have

$$\frac{1}{\lambda \circ f^{-1}} \frac{1}{\varphi \circ f^{-1}} (H_g v) \circ f^{-1} = H_h\left(\frac{1}{\varphi \circ f^{-1}} \cdot v \circ f^{-1}\right).$$

Therefore $\left(f^{-1}, 1/\varphi \circ f^{-1}\right)$ is a caloric morphism from E into D by (3). □

As an application of Theorem 2.2, we have the following propositions, which enable us to construct new caloric morphisms. These are proved by a calculation similar to the one in [7, Proposition 5].

Proposition 2.1 (direct product). *Let I be an open interval of $\mathbf{R}$ and M_j be a semi-riemannian manifold $(j = 1,2)$. For two caloric morphisms of form $\left(\left(f_0(t), f_j(t,x_j)\right), \varphi_j(t,x_j)\right)$ from $I \times M_j$ to $\mathbf{R} \times N_j$, we consider a map (f_0, f_1, f_2) from $I \times M_1 \times M_2$ to $\mathbf{R} \times N_1 \times N_2$:*

$$(t, x_1, x_2) \mapsto \left(f_0(t), f_1(t,x_1), f_2(t,x_2)\right)$$

and a function $\varphi_1\varphi_2$ on $I \times\ M_1 \times M_2$:

$$(t, x_1, x_2) \mapsto \varphi_1(t,x_1)\varphi_2(t,x_2).$$

Then a pair $\left((f_0, f_1, f_2), \varphi_1\varphi_2\right)$ is a caloric morphism.

Proposition 2.2 (direct sum). *Let E be a semi-Euclidean space, I an open interval and M_j a semi-riemannian manifold $(j = 1, 2)$. For two caloric morphisms (f_j, φ_j) from $I \times M_j$ to $\mathbf{R} \times E$, we put*

$$f(t, x_1, x_2) = f_1(t, x_1) + f_2(t, x_2),$$
$$\varphi(t, x_1, x_2) = \varphi_1(t, x_1)\varphi_2(t, x_2)$$

for $(t, x_1, x_2) \in I \times M_1 \times M_2$. Then (f, φ) is a caloric morphism from $I \times M_1 \times M_2$ to $\mathbf{R} \times E$.

Remark 2. Let g be a harmonic morphism with constant dilatation from a domain $\Omega \subset M$ to N (see [1] for the definition of harmonic morphism). Putting

$$f(t, x) = \big(t, g(x)\big), \qquad \varphi(t, x) = 1,$$

we obtain a caloric morphism (f, φ).

Finally, we make a remark on the relation between harmonic maps and caloric morphisms. Let (M, g) and (N, h) be semi-riemannian manifolds. For a mapping $f \colon M \to N$, the tension field $\tau(f)$ is defined as

$$\tau^\alpha(f) = \Delta_g f^\alpha + \sum_{\gamma,\eta} g(\nabla_g f^\gamma, \nabla_g f^\eta) \cdot ({}^h\Gamma^\alpha_{\gamma\eta} \circ f), \qquad \alpha = 1, \ldots, n,$$

in the local coordinates. A harmonic map is the solution of

$$\tau(f) = 0. \tag{1}$$

Then (E-2) or (E-6) can be written as

$$\frac{\partial f^\alpha}{\partial t} = \tau^\alpha(f) + 2g(\nabla_g \log\varphi, \nabla_g f^\alpha), \qquad \alpha = 1, \ldots, n.$$

Now we introduce a new field $\tau_\varphi(f)$ by

$$\tau^\alpha_\varphi(f) := \tau^\alpha(f) + 2g(\nabla_g \log\varphi, \nabla_g f^\alpha), \qquad \alpha = 1, \ldots, n.$$

Then (E-2) or (E-6) can be simplified as

$$\frac{\partial f}{\partial t} = \tau_\varphi(f).$$

The equation

$$\tau_\varphi(f) = 0$$

is the Euler–Lagrange equation of the weighted energy functional

$$E_{\Omega,\varphi}(f) = \int_\Omega e(f)\varphi^2\, d\mu_g,$$

while (1) is the Euler–Lagrange equation of the energy functional

$$E_\Omega(f) = \int_\Omega e(f)\, d\mu_g,$$

where

$$e(f) = \frac{1}{2}|df|^2 := \frac{1}{2}\sum_{\alpha,\beta} g(\nabla_g f^\alpha, \nabla_g f^\beta)\cdot(h_{\alpha\beta}\circ f), \qquad d\mu_g(x) = \sqrt{|g|}\, dx$$

and Ω is a relatively compact subdomain of M.

3. Preliminaries

First, we quote a theorem by L. Hörmander from [2], in order to construct local solutions of the heat equation with prescribed derivatives at a given point.

Let P be a second order differential operator of C^∞ coefficients on $\mathbf{R}^n$ of the form

$$P = \sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{k=1}^n b^k(x)\frac{\partial}{\partial x^k} + c(x),$$

where the matrix $\big(a^{ij}(x)\big)$ is symmetric and non-degenerate for all x.

Theorem A. *Let $m \geqq 2$ be an integer. If $u \in C^{m+1}(\mathbf{R}^n)$ and $Pu(x) = O(|x|^{m-1})$ as $x \to 0$, then for any $s > m$ there exists $U \in C^s(\mathbf{R}^n)$ such that $U(x) - u(x) = O(|x|^{m+1})$ as $x \to 0$ and $PU = 0$ on a neighborhood of 0.*

In the proof, he uses the following theorem, which is also useful for our purpose.

Theorem B ([2, Theorem 7.1]). *For any positive integer s, there exists a bounded linear operator G_s from the Sobolev space $H^s(\mathbf{R}^n)$ to $H^{s+1}(\mathbf{R}^n)$ such that for every $f \in H^s(\mathbf{R}^n)$, $PG_s f = f$ on a neighborhood of the origin.*

Next we prepare lemmas for the proof of the main theorem. Combining the Sobolev imbedding theorem with Theorem B, we have the following lemma.

Lemma 3.1. *For any integers $s \geqq 2$ and $s' > \frac{1}{2}n + s - 1$ and any $C^{s'}$ function f defined on a neighborhood of the origin in $\mathbf{R}^n$, there exists a C^s function u such that $Pu = f$ on a neighborhood of the origin.*

By Theorem A, we have the following lemma.

Lemma 3.2. *Let* $\left((\eta_{ij})_{i,j=1}^n, (\eta_k)_{k=1}^n, \eta\right) \in \mathbf{R}^{n^2+n+1}$ *satisfy* $\eta_{ij} = \eta_{ji}$ *and*

$$\sum_{i,j=1}^n a^{ij}(0)\eta_{ij} + \sum_{k=1}^n b^k(0)\eta_k + c(0)\eta = 0.$$

Then for any $s > 2$ *there exists a* C^s *function* U *such that*

$$\frac{\partial^2 U}{\partial x^i \partial x^j}(0) = \eta_{ij}, \quad \frac{\partial U}{\partial x^k}(0) = \eta_k, \quad U(0) = \eta, \quad \text{for } i, j, k = 1, \ldots, n,$$

and $PU = 0$ *on a neighborhood of* 0.

Proof. Put $m = 2$ and apply Theorem A to the function

$$u(x) = \frac{1}{2}\sum_{i,j=1}^n \eta_{ij}x^i x^j + \sum_{k=1}^n \eta_k x^k + \eta. \ \square$$

Using these lemmas, we have the following existence theorem for the solution of the heat equation with prescribed derivatives.

Lemma 3.3. *For any* $\left((\eta_{ij})_{i,j=0}^n, (\eta_k)_{k=0}^n, \eta\right) \in \mathbf{R}^{(n+1)^2+(n+1)+1}$ *such that* $\eta_{ij} = \eta_{ji}$ *and*

$$\eta_0 = \sum_{i,j=1}^n a^{ij}(0)\eta_{ij} + \sum_{k=1}^n b^k(0)\eta_k + c(0)\eta,$$

there exists a C^2 *function* u *on a neighborhood of the origin of* $\mathbf{R} \times \mathbf{R}^n$ *such that*

$$\text{(2)} \quad \frac{\partial^2 u}{\partial x^i \partial x^j}(0,0) = \eta_{ij}, \quad \frac{\partial u}{\partial x^k}(0,0) = \eta_k, \quad u(0,0) = \eta, \quad \text{for } i, j, k = 0, 1, \ldots, n,$$

and $(\partial u/\partial t) - Pu = 0$. *Here we use a convention* $(\partial/\partial x^0) = (\partial/\partial t)$.

Proof. We shall construct a solution of form

$$\text{(3)} \qquad u(t, x) = u_0(x) + u_1(x)t + \tfrac{1}{2}u_2(x)t^2.$$

Substitute (3) into the heat equation $(\partial u/\partial t) - Pu = 0$ and compare the coefficients as the function of t. Then we obtain the equations

$$Pu_2 = 0, \qquad Pu_1 = u_2, \qquad Pu_0 = u_1.$$

From the initial value condition (2), u_0, u_1, u_2 must satisfy

$$\begin{aligned}
u_2(0) &= \frac{\partial^2 u}{\partial t^2}(0,0) = \eta_{00}, \\
u_1(0) &= \frac{\partial u}{\partial t}(0,0) = \eta_0, \\
\frac{\partial u_1}{\partial x^j}(0) &= \frac{\partial^2 u}{\partial t \partial x^j}(0,0) = \eta_{0j} \quad \text{for } j = 1, \dots, n, \\
u_0(0) &= u(0,0) = \eta, \\
\frac{\partial u_0}{\partial x^k}(0) &= \frac{\partial u}{\partial x^k}(0,0) = \eta_k \quad \text{for } k = 1, \dots, n, \\
\frac{\partial^2 u_0}{\partial x^i \partial x^j}(0) &= \frac{\partial^2 u}{\partial x^i \partial x^j}(0,0) = \eta_{ij} \quad \text{for } i, j = 1, \dots, n.
\end{aligned}$$

First we shall solve the equation $Pu_2 = 0$ under the condition $u_2(0) = \eta_{00}$. Put $\tilde{\eta} = \eta_{00}$ and $\tilde{\eta}_k = 0$ for $k = 1, \dots, n$. Since the matrix $(a^{ij}(0))_{i,j=1}^n$ is non-degenerate, there exists a symmetric matrix $(\tilde{\eta}_{ij})_{i,j=1}^n$ satisfying that

$$\sum_{i,j=1}^n a^{ij}(0)\tilde{\eta}_{ij} + \sum_{k=1}^n b^k(0)\tilde{\eta}_k + c(0)\tilde{\eta} = 0. \tag{4}$$

Then we can solve the equation $Pu_2 = 0$ by Lemma 3.2.

Next we shall solve the equation $Pu_1 = u_2$ under the conditions $u_1(0) = \eta_0$ and $(\partial u_1/\partial x^j)(0) = \eta_{0j}$ for $j = 1, \dots, n$. We can take w_1 with $Pw_1 = u_2$ by Lemma 3.1. Putting $\tilde{\eta} = \eta_0 - w_1(0)$ and $\tilde{\eta}_k = \eta_{0k} - (\partial w_1/\partial x^k)(0)$ for $k = 1, \dots, n$, we can take $\tilde{\eta}_{ij}$ for $i, j = 1, \dots, n$ with (4). Then we can solve the equation $Pv_1 = 0$ under the conditions $v_1(0) = \eta_0 - w_1(0)$ and $(\partial v_1/\partial x^k)(0) = \eta_{0k} - (\partial W_1/\partial x^k)(0)$ for $k = 1, \dots, n$ by Lemma 3.2. Then $u_1 = v_1 + w_1$ is a desired solution.

Finally, we shall solve the equation $Pu_0 = u_1$ under the conditions $u_0(0) = \eta$, $(\partial u_0/\partial x^k)(0) = \eta_k$ and $(\partial^2 u_0/\partial x^i \partial x^j)(0) = \eta_{ij}$ for $i, j, k = 1, \dots, n$. We take w_0 with $Pw_0 = u_1$ by Lemma 3.1 and solve the equation $Pv_0 = 0$ under the conditions $v_0(0) = \eta - w_0(0)$, $(\partial v_0/\partial x^k)(0) = \eta_k - (\partial w_0/\partial x^k)(0)$ and $(\partial^2 v_0/\partial x^i \partial x^j)(0) = \eta_{ij} - (\partial^2 w_0/\partial x^i \partial x^j)(0)$ for $i, j, k = 1, \dots, n$, because the condition (4) holds for $\tilde{\eta} = \eta - w_0(0)$, $\tilde{\eta}_k = \eta_k - (\partial w_0/\partial x^k)(0)$ and $\tilde{\eta}_{ij} = \eta_{ij} - (\partial^2 w_0/\partial x^i \partial x^j)(0)$. Putting $u_0 = v_0 + w_0$, we have the lemma. □

4. Proof of Theorem 2.2

In this section we shall prove our main theorem. Before the proof, we prepare a small lemma from linear algebra.

Lemma 4.1. *Let $a, b \in \mathbf{R}^l$ and consider linear forms $\varrho(x) = (x, a)$ and $\mu(x) = (x, b)$, which are the usual inner product in $\mathbf{R}^l$. If $\varrho(x) = 0$ implies $\mu(x) = 0$ for all $x \in \mathbf{R}^l$, then there exists $\nu \in \mathbf{R}$ such that $b = \nu a$.*

Proof of Theorem 2.2. (1) $\Rightarrow$ (2): For any point $(t,P)\in D$, we put $(\tau,Q)=f(t,P)$ and take a local coordinate y of N near Q such that Q corresponds to the origin. Then since

$$\Delta_h=\sum_{\alpha,\beta=1}^{n} h^{\alpha\beta}(y)\frac{\partial^2}{\partial y^\alpha\partial y^\beta}+\sum_{\gamma=1}^{n}\left(\sum_{\alpha,\beta=1}^{n} h^{\alpha\beta}(y)\,{}^h\Gamma^{\gamma}_{\alpha\,\beta}(y)\right)\frac{\partial}{\partial y^\gamma},$$

we put $a^{\alpha\beta}(y)=h^{\alpha\beta}(y)$, $b^\gamma(y)=\sum_{\alpha,\beta=1}^{n}h^{\alpha\beta}(y)\,{}^h\Gamma^{\gamma}_{\alpha\,\beta}(y)$ and $c(y)=0$ for $\alpha,\beta,\gamma=1,\ldots,n$. Hence for every $\big((\eta_{\alpha\beta})_{\alpha,\beta=0}^{n},(\eta_\gamma)_{\gamma=0}^{n},\eta\big)\in\mathbf{R}^{(n+1)^2+(n+1)+1}$ such that

$$\eta_0=\sum_{\alpha,\beta=1}^{n}a^{\alpha\beta}(0)\eta_{\alpha\beta}+\sum_{\gamma=1}^{n}b^\gamma(0)\eta_\gamma+c(0)\eta,$$

it follows from Lemma 3.3 that there exists a caloric function u on a neighborhood of (τ,Q) such that

$$\frac{\partial^2 u}{\partial y^\alpha\partial y^\beta}(\tau,Q)=\eta_{\alpha\beta},\qquad \frac{\partial u}{\partial y^\gamma}(\tau,Q)=\eta_\gamma,\qquad u(\tau,Q)=\eta$$

for $\alpha,\beta,\gamma=0,1,\ldots,n$. Since (f,φ) is assumed to be a caloric morphism, $\varphi\cdot(u\circ f)$ must be caloric, that is,

$$\begin{aligned}
0&=H_g\big(\varphi\cdot(u\circ f)\big)\\
&=(H_g\varphi)\cdot(u\circ f)+\varphi\cdot H_g(u\circ f)-2g\big(\nabla_g\varphi,\nabla_g(u\circ f)\big)\\
&=(H_g\varphi)\cdot(u\circ f)+\sum_{\gamma=0}^{n}\big(\varphi\cdot H_gf^\gamma-2g(\nabla_g\varphi,\nabla_gf^\gamma)\big)\cdot\left(\frac{\partial u}{\partial y^\gamma}\right)\circ f\\
&\quad-\varphi\sum_{\alpha,\beta=0}^{n}g(\nabla_gf^\alpha,\nabla_gf^\beta)\cdot\left(\frac{\partial^2 u}{\partial y^\alpha\partial y^\beta}\right)\circ f.
\end{aligned}$$

Substituting (t,P), we have

$$\begin{aligned}
0=(H_g\varphi)(t,P)\cdot\eta&+\sum_{\gamma=0}^{n}\big(\varphi\cdot H_gf^\gamma-2g(\nabla_g\varphi,\nabla_gf^\gamma)\big)(t,P)\cdot\eta_\gamma\\
&-\sum_{\alpha,\beta=0}^{n}\big(\varphi\cdot g(\nabla_gf^\alpha,\nabla_gf^\beta)\big)(t,P)\cdot\eta_{\alpha\beta},
\end{aligned}$$

which implies that there exists $\nu(t,P)\in\mathbf{R}$ such that

$$\begin{aligned}
&H_g\varphi(t,P)=0,\\
&\big(\varphi\cdot H_gf^0-2g(\nabla_g\varphi,\nabla_gf^0)\big)(t,P)=\nu(t,P),\\
\text{(A)}\quad&\big(\varphi\cdot H_gf^\gamma-2g(\nabla_g\varphi,\nabla_gf^\gamma)\big)(t,P)=-\nu(t,P)\cdot b^\gamma(0)\text{ for }\gamma=1,\ldots,n,\\
&\big(\varphi\cdot g(\nabla_gf^0,\nabla_gf^\beta)\big)(t,P)=0\text{ for }\beta=0,1,\ldots,n,\\
\text{(B)}\quad&\big(\varphi\cdot g(\nabla_gf^\alpha,\nabla_gf^\beta)\big)(t,P)=\nu(t,P)\cdot a^{\alpha\beta}(0)\text{ for }\alpha,\beta=1,\ldots,n,
\end{aligned}$$

because of Lemma 4.1. Putting $\lambda(t,P)=\nu(t,P)/\varphi(t,P)$ and substituting (B) into (A), we obtain (2).

(2) $\Rightarrow$ (3): By a direct calculation, we have

$$\begin{aligned}
H_g\bigl(\varphi\cdot(u\circ f)\bigr) &= (H_g\varphi)\cdot(u\circ f)+\sum_{\gamma=0}^{n}\bigl(\varphi\cdot H_g f^{\gamma}-2g(\nabla_g\varphi,\nabla_g f^{\gamma})\bigr)\cdot\left(\frac{\partial u}{\partial y^{\gamma}}\right)\circ f\\
&\quad -\varphi\sum_{\alpha,\beta=0}^{n} g(\nabla_g f^{\alpha},\nabla_g f^{\beta})\cdot\left(\frac{\partial^2 u}{\partial y^{\alpha}\partial y^{\beta}}\right)\circ f\\
&= (\lambda\cdot\varphi)\cdot\left(\frac{\partial u}{\partial y^0}+\sum_{\gamma=1}^{n}\left(\sum_{\alpha,\beta=1}^{n} h^{\alpha\beta}{}^h\Gamma^{\gamma}_{\alpha\beta}\right)\frac{\partial u}{\partial y^{\gamma}}\right)\circ f\\
&\quad -(\lambda\cdot\varphi)\cdot\left(\sum_{\alpha,\beta=1}^{n} h^{\alpha\beta}\frac{\partial^2 u}{\partial y^{\alpha}\partial y^{\beta}}\right)\circ f\\
&= (\lambda\cdot\varphi)\cdot(H_h u)\circ f.
\end{aligned}$$

(3) $\Rightarrow$ (1) is trivial, which completes the proof. □

5. Examples

In the following Examples 1–3, we consider the case that $M=\mathbf{R}^n\setminus\{0\}$ and g has the form $g_{ij}=\varrho(|x|)\delta_{ij}$, where ϱ is a C^∞ function on $(0,\infty)$.

Example 1. Let $n=2$, $\varrho(r)=1/r^2$. Then

$$f(t,x)=\left(t+b,\frac{e^t}{|x|^2}R(t)x\right),\quad \varphi(t,r,\theta)=r^{-1/2}\exp\left(\frac{1}{2}\theta+\frac{1}{2}t\right),$$

is a caloric morphism, where

$$R(t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$$

and (r,θ) is the polar coordinate of $\mathbf{R}^2$.

Example 2. Let $n\geqq 3$, $\varrho(r)=1/r^2$. Then

$$f(t,x)=\left(t,e^t\frac{x}{|x|^2}\right),\qquad \varphi(t,x)=|x|^{-1/2}\exp\left(\frac{1}{4}t\right)$$

is a caloric morphism.

Example 3 (Appell transformation). Let $\varrho(r) = r^k$, $k \in \mathbf{R}$, $k \neq -2$. Then

$$f(t,x) = \left(\frac{ct+d}{at+b}, \frac{x}{|at+b|^{2/(k+2)}}\right),$$
$$\varphi(t,x) = \frac{1}{|at+b|^{n/2}} \exp\left(-\frac{a|x|^{k+2}}{(k+2)^2(at+b)}\right)$$

is a caloric morphism, where $a, b, c, d \in \mathbf{R}$, $bc - ad = 1$. In particular, in the case of $k = 0$, this example includes the usual Appell transformation:

$$f(t,x) = \left(-\frac{1}{t}, \frac{x}{t}\right), \qquad \varphi(t,x) = \frac{1}{(4\pi|t|)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Next we consider the non-riemannian case.

Example 4. Let $M = \mathbf{R}^3$ be a semi-Euclidean space whose metric is $g = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$ and $N = \mathbf{R}^1$ be the Euclidean space. Then

$$f(t,x) = (t + x^2 + x^3, x^1), \qquad \varphi(t,x) = 1$$

satisfy (E-5)–(E-8). In this case $\lambda(t,x) = 1$. Composing this to the Appell transformation on $\mathbf{R} \times \mathbf{R}$

$$\left(-\frac{1}{\tau}, \frac{y}{\tau}\right), \qquad \frac{1}{\sqrt{|\tau|}} \exp\left(-\frac{y^2}{4\tau}\right),$$

we have another example

$$f(t,x) = \left(-\frac{1}{t + x^2 + x^3}, \frac{x^1}{t + x^2 + x^3}\right),$$
$$\varphi(t,x) = \frac{1}{\sqrt{|t + x^2 + x^3|}} \exp\left(-\frac{(x^1)^2}{4(t + x^2 + x^3)}\right)$$

such that $\lambda(t,x) = 1/\tau^2 = 1/(t + x^2 + x^3)^2$ depends on both t and x.

Example 5. Let $M = \mathbf{R}^4$ be a semi-Euclidean space whose metric is $g = (dx^1)^2 - (dx^2)^2 + (dx^3)^2 - (dx^4)^2$ and $N = \mathbf{R}^2$ be a semi-Euclidean space whose metric is $g = (dx^1)^2 - (dx^2)^2$. Then

$$f(t,x) = \left(\frac{1}{t(t-1)}, \frac{x^1}{t} + \frac{x^4}{t-1}, \frac{x^2}{t} + \frac{x^3}{t-1}\right),$$
$$\varphi(t,x) = \frac{1}{|t(t-1)|} \exp\left(-\frac{(x^1)^2 - (x^2)^2}{4t} - \frac{(x^3)^2 - (x^4)^2}{4(t-1)}\right)$$

satisfy (E-5)–(E-8) on $(0,1) \times M$. Then $\lambda(t) = (1/t^2) - \big(1/(t-1)^2\big)$, which changes the sign at $t = \frac{1}{2}$. Note that (f, φ) is a direct sum of two Appell transfomations.

Example 6 (Reverse). Put $f(t,x)=(-t,x)$ and $\varphi(t,x)=1$. Then (f,φ) is a caloric morphism from $\mathbf{R}\times(M,g)$ onto $\mathbf{R}\times(M,-g)$. Moreover, if M is a $2n$-dimensional semi-euclidean space such that the signature of g is equal to (n,n), then there exists a $(2n\times 2n)$-matrix R such that $R^2=I$ and ${}^tRGR=-G$, where I is the identity matrix and $G=(g_{ij})$. Hence putting $f(t,x)=(-t,Rx)$ and $\varphi(t,x)=1$, we have a caloric morphism on $\mathbf{R}\times(M,g)$. We remark that $(f^0)'$ is negative in these examples.

Example 7 (Appell transformation). For integers $p,q\geqq 0$ and $k\in\mathbf{R}\setminus\{-2\}$, put

$$M=\bigl\{x\in\mathbf{R}^{p+q}:(x^1)^2+\dots+(x^p)^2-(x^{p+1})^2-\dots-(x^{p+q})^2>0\bigr\},$$
$$\langle x\rangle=\sqrt{(x^1)^2+\dots+(x^p)^2-(x^{p+1})^2-\dots-(x^{p+q})^2}$$

and

$$g_{ij}(x)=\begin{cases}\langle x\rangle^k, & i=j=1,\dots,p,\\ -\langle x\rangle^k, & i=j=p+1,\dots,p+q,\\ 0, & i\neq j.\end{cases}$$

Then

$$f(t,x)=\left(-\frac{1}{t},\frac{x}{t^{2/(k+2)}}\right),\qquad \varphi(t,x)=\frac{1}{t^{(p+q)/2}}\exp\left(-\frac{\langle x\rangle^{k+2}}{(k+2)^2t}\right)$$

is a caloric morphism from $(0,\infty)\times M$ to $(-\infty,0)\times M$. This is a generalization of Example 3.

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