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BRUNN–MINKOWSKI AND ISOPERIMETRIC INEQUALITY IN THE HEISENBERG GROUP

Roberto Monti

Universität Bern, Mathematisches Institut
Sidlerstrasse 5, CH-3012 Bern, Schweiz; roberto.monti@math-stat.unibe.ch

Abstract. We prove that the natural generalization of the Brunn–Minkowski inequality in the Heisenberg group does not hold, because it would imply the isoperimetric property for Carnot–Carathéodory balls, the property they do not have.

1. Introduction

The Brunn–Minkowski inequality in the Euclidean space states that if A and B are nonempty subsets of $\mathbf{R}^n$, $n \geq 1$, then

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{1.1}$$

where $A+B = \{x+y : x \in A,\ y \in B\}$ is the Minkowski sum of A and B and $|\cdot|$ denotes Lebesgue outer measure in $\mathbf{R}^n$ (see, for instance, [F, 3.2.41]). In this paper we show that the natural generalization of this inequality to the geometric setting of the Heisenberg group does not hold.

The Heisenberg group is a Lie group which can be identified with $\mathbf{R}^3$ endowed with the non commutative group law

$$x \cdot y = \bigl(x_1 + y_1, x_2 + y_2, x_3 + y_3 + 2(x_2 y_1 - x_1 y_2)\bigr), \tag{1.2}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in \mathbf{R}^3$. For any $x \in \mathbf{R}^3$ the map $\tau_x \colon \mathbf{R}^3 \to \mathbf{R}^3$ defined by $\tau_x(y) = x \cdot y$ is a *left translation.* The maps $\delta_\lambda \colon \mathbf{R}^3 \to \mathbf{R}^3$, $\lambda > 0$, defined by $\delta_\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$ form a group of automorphisms of $(\mathbf{R}^3, \cdot\,)$ called *dilations.*

Let $E \subset \mathbf{R}^3$ be a Lebesgue measurable set and denote by $|E|$ its Lebesgue measure. It is not difficult to check that

$$|\tau_x(E)| = |E| \quad \text{for all } x \in \mathbf{R}^3 \text{ and } \quad |\delta_\lambda(E)| = \lambda^4 |E| \quad \text{for all } \lambda > 0. \tag{1.3}$$

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The first property states that Lebesgue measure is the Haar measure of the group. The second one shows that Lebesgue measure is homogeneous of degree 4 with respect to dilations.

Define the "Minkowski sum" of two sets $A, B \subset \mathbf{R}^3$ as $A \cdot B = \{x \cdot y : x \in A,\ y \in B\}$. The natural generalization of the Brunn–Minkowski inequality (1.1) to the geometric setting of the Heisenberg group should be

$$|A \cdot B|^{1/4} \geq |A|^{1/4} + |B|^{1/4} \quad \text{for all bounded open sets } A, B \subset \mathbf{R}^3. \tag{1.4}$$

We prove by an indirect argument that this inequality is false. It is known that in Euclidean spaces inequality (1.1) implies the isoperimetric property of balls. Our argument relies upon the fact that balls in the Heisenberg group are not isoperimetric sets and it essentially shows that, in this setting, the isoperimetric inequality with sharp constant (a problem that is still open) can not be obtained through an inequality of Brunn–Minkowski type. This is the main interest of the result. By "balls" we mean the Carnot–Carathéodory balls associated with the canonical Heisenberg left invariant vector fields (see Section 2), and the isoperimetric problem we are referring to is formulated by means of a surface measure, the Heisenberg perimeter of a set, which is defined through the same vector fields (see Definition 2.2 and (3.13)). Since such balls do not solve the isoperimetric problem (this was already proved in [M] constructing a counterexample) we are able to get a contradiction and the argument will also show how to find sets for which (1.4) fails.

In the next section we introduce a metric structure in the Heisenberg group and we recall some definitions of surface measure. In the third section we study the relation between isoperimetric inequality and inequality (1.4) and we prove that this latter does not hold.

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2. Balls and surface measures in the Heisenberg group

The Lie algebra of the Heisenberg group is generated by the vector fields in $\mathbf{R}^3$

$$X_1 = \partial_1 + 2x_2\partial_3 \quad \text{and} \quad X_2 = \partial_2 - 2x_1\partial_3. \tag{2.5}$$

Indeed, these vector fields are left invariant with respect to the group law (1.2) and they satisfy the following maximal rank condition: X_1, X_2 and their commutator $[X_1, X_2] = -4\partial_3$ are linearly independent.

A Lipschitz curve $\gamma\colon [0,1] \to \mathbf{R}^3$ is *admissible* if there exist $h_1, h_2 \in L^\infty(0,1)$ such that

$$\dot\gamma(t) = h_1(t)X_1\big(\gamma(t)\big) + h_2(t)X_2\big(\gamma(t)\big) \qquad \text{for a.e. } t \in [0,1], \tag{2.6}$$

and its length is by definition

$$\mathrm{len}(\gamma) = \int_0^1 \sqrt{h_1(t)^2 + h_2(t)^2}\, ds.$$

A metric structure can be introduced in the Heisenberg group minimizing the length of admissible curves connecting points. Precisely, define the distance d: $\mathbf{R}^3 \times \mathbf{R}^3 \to [0, +\infty)$ letting for any pair of points $x, y \in \mathbf{R}^3$

$$d(x,y) = \inf\bigl\{\mathrm{len}(\gamma) : \gamma\colon [0,1] \to \mathbf{R}^3 \text{ is admissible and } \gamma(0) = x,\ \gamma(1) = y\bigr\}.$$

Since the vector fields (2.5) satisfy the maximal rank condition, by Chow theorem admissible curves connecting points do always exist and therefore $d(x,y) < +\infty$ for all $x, y \in \mathbf{R}^3$. The function d is a metric on $\mathbf{R}^3$, usually called *Carnot–Carathéodory metric*, which is left invariant and homogeneous of degree 1 with respect to dilations, i.e.

$$d(z \cdot x, z \cdot y) = d(x,y) \quad \text{and} \quad d\bigl(\delta_\lambda(x), \delta_\lambda(y)\bigr) = \lambda d(x,y)$$

for all $x, y, z \in \mathbf{R}^3$ and $\lambda > 0$. We denote by $B(x,r) = \{y \in \mathbf{R}^3 : d(x,y) < r\}$ the *Carnot–Carathéodory ball* centered at x with radius $r > 0$. The shape of this ball can be calculated explicitly (see [Br], [B] and, for instance, [M]). Precisely, the boundary of $B(0,r)$, $r > 0$, can be parameterized by the following functions

$$\begin{cases} x_1(\theta,\varphi,r) = \dfrac{\cos\theta(1-\cos\varphi r) + \sin\theta\sin\varphi r}{\varphi}, \\ x_2(\theta,\varphi,r) = \dfrac{-\sin\theta(1-\cos\varphi r) + \cos\theta\sin\varphi r}{\varphi}, \\ x_3(\theta,\varphi,r) = 2\dfrac{(\varphi r - \sin\varphi r)}{\varphi^2}, \end{cases} \tag{2.7}$$

where $\theta \in [0, 2\pi)$ and $-2\pi \le \varphi r \le 2\pi$.

Now we introduce some surface measures in the Heisenberg group: Minkowski content, Heisenberg perimeter and 3-dimensional spherical Hausdorff measure.

We begin with Minkowski content. If E is a subset of $\mathbf{R}^3$ define the distance function $\mathrm{dist}(x;E) = \inf_{y\in E} d(x,y)$ and for $\varepsilon > 0$ let

$$E_\varepsilon = \{x \in \mathbf{R}^3 : \mathrm{dist}(x;E) < \varepsilon\} = \bigcup_{x\in E} B(x,\varepsilon) = E \cdot B(0,\varepsilon)$$

be the ε-neighborhood of E.

Definition 2.2. Let $E \subset \mathbf{R}^3$ be a bounded open set. If the limit exists, define the *Minkowski content* of the boundary ∂E of E

$$\mathfrak{M}(\partial E) = \lim_{\varepsilon \downarrow 0} \frac{|E_\varepsilon \setminus E|}{\varepsilon}. \tag{2.8}$$

Minkowski content in Carnot–Carathéodory spaces is studied in [MSC]. Now we recall the definition of Heisenberg perimeter of a set. This is a special case of a more general definition of perimeter related to vector fields introduced in [GN], which has been generalized to metric spaces in [A].

Definition 2.2. Let $E \subset \mathbf{R}^3$ be a bounded Lebesgue measurable set. The *Heisenberg perimeter* of E is

$$P(E) = \sup\left\{ \int_E \big(X_1\varphi_1(x) + X_2\varphi_2(x)\big)\,dx : \varphi_1, \varphi_2 \in C_0^1(\mathbf{R}^3),\ \varphi_1^2 + \varphi_2^2 \le 1 \right\},$$

where X_1 and X_2 are the differential operators (2.5). If $P(E) < +\infty$ the set E is said to have *finite Heisenberg perimeter*.

It is not difficult to prove that Heisenberg perimeter is invariant under translations and 3-homogeneous with respect to dilations. Precisely, if $E \subset \mathbf{R}^3$ is a set of finite Heisenberg perimeter then

$$P\big(\tau_x(E)\big) = P(E) \quad \text{and} \quad P\big(\delta_\lambda(E)\big) = \lambda^3 P(E) \tag{2.9}$$

for all $x \in \mathbf{R}^3$ and for all $\lambda > 0$. See, for instance, Lemma 4.5 in [MSC].

The Heisenberg perimeter of a bounded open set E with Lipschitz boundary has a useful integral representation. The Euclidean outward unit normal to ∂E is defined at $\mathscr{H}^2$-almost every point $x \in \partial E$ (we denote by $\mathscr{H}^2$ the Euclidean 2-dimensional Hausdorff measure in $\mathbf{R}^3$). Denote this normal by $\nu(x) = \big(\nu_1(x), \nu_2(x), \nu_3(x)\big) \in \mathbf{R}^3$. Using the divergence theorem it can be shown that

$$P(E) = \int_{\partial E} \sqrt{\big(\nu_1(x) + 2x_2\nu_3(x)\big)^2 + \big(\nu_2(x) - 2x_1\nu_3(x)\big)^2}\, d\mathscr{H}^2(x). \tag{2.10}$$

Finally, it is interesting to consider the 3-dimensional spherical Hausdorff measure defined by means of the Carnot–Carathéodory metric. The metric space $(\mathbf{R}^3, d)$ has metric dimension 4 (metric dimension and homogeneous dimension are equal, see [Mi]) and therefore the correct surface dimension seems to be 3 (see also [G]).

Definition 2.3. The 3-dimensional *spherical Hausdorff measure* of a set $K \subset (\mathbf{R}^3, d)$ is

$$\mathscr{S}^3(K) = \sup_{\delta > 0} \inf\left\{ \gamma \sum_{j=1}^{+\infty} \big(\mathrm{diam}(B_j)\big)^3 : K \subset \bigcup_{j=1}^{+\infty} B_j,\ \mathrm{diam}(B_j) \le \delta \right\}, \tag{2.11}$$

where $\gamma > 0$ is a suitable normalization constant, B_j are Carnot–Carathéodory balls and $\mathrm{diam}(B_j) = \sup_{x,y \in B_j} d(x,y)$ is the diameter of B_j.

According to general theorems on Hausdorff measures, $\mathscr{S}^3$ is a Borel measure in $\mathbf{R}^3$. The following theorem describes the known relations between perimeter, Minkowski content and $\mathscr{S}^3$, and it shows that perimeter is the correct way to define the measure of the boundary of a set in the Heisenberg setting.

Theorem 2.4. *The following relations hold*:

(i) $P(E) = \mathscr{S}^3(\partial E)$ *for all bounded open sets* $E \subset \mathbf{R}^3$ *of class* C^1;

(ii) $P(E) = \mathfrak{M}(\partial E)$ *for all bounded open sets* $E \subset \mathbf{R}^3$ *of class* C^2.

Proof. The identity $P(E) = \mathscr{S}^3(\partial E)$ for open sets with boundary of class C^1, which holds with a suitable choice of the constant γ in (2.11), is proved in [FSSC], Corollary 7.7. The identity $P(E) = \mathfrak{M}(\partial E)$ for sets with boundary of class C^2 is proved in [MSC], Theorem 5.1, in the general setting of Carnot–Carathéodory spaces. We refer to these papers for the proofs. □

The boundary of a Carnot–Carathéodory ball is not of class C^2 because, when its center is translated to the origin of $\mathbf{R}^3$, it has two Lipschitz points on the x_3-axis. Anyway, the identity between Heisenberg perimeter and Minkowski content still holds for balls as the following proposition shows.

Proposition 2.5. *Let* B *be a Carnot–Carathéodory ball. Then* $P(B) = \mathfrak{M}(\partial B)$.

Proof. Without loss of generality we can assume that B is the ball centered at the origin with radius 1.

Let $\varepsilon > 0$, denote by B_ε the ε-neighborhood of B and note that $B_\varepsilon = B(0, 1+\varepsilon)$. Consider the distance function $x \mapsto \varrho(x) = \operatorname{dist}(x; \partial B)$. Let X_1 and X_2 be the differential operators in (2.5), denote by $X = (X_1, X_2)$ the Heisenberg gradient and write $|X\varrho(x)| = \sqrt{\big(X_1\varrho(x)\big)^2 + \big(X_2\varrho(x)\big)^2}$. By the coarea formula in Carnot–Carathéodory spaces (see, for instance, [MSC], Theorem 4.2 or [GN], Theorem 5.2)

$$\int_{B_\varepsilon \setminus B} |X\varrho(x)|\, dx = \int_0^\varepsilon P\big(B(0, 1+t)\big)\, dt. \tag{2.12}$$

The function $x \mapsto d(x, 0)$ is of class C^1 in the open set $\mathbf{R}^3 \setminus \{x_1 = x_2 = 0\}$ and it satisfies the Eikonal equation $|X d(x, 0)| = 1$ (see [M], Theorem 3.8). Moreover, if $x \in \mathbf{R}^3$ is such that $d(x, 0) > 1$ we have $\varrho(x) = d(x, 0) - 1$, and therefore ϱ is of class C^1 and satisfies $|X\varrho| = 1$ in the open set $\{x \in \mathbf{R}^3 : d(x, 0) > 1 \text{ and } x_1^2 + x_2^2 \neq 0\}$. Using the scaling property (2.9) of perimeter we have $P\big(B(0, 1+t)\big) = (1+t)^3 P(B)$, and then from (2.12) we find

$$|B_\varepsilon \setminus B| = \int_{B_\varepsilon \setminus B} dx = \int_{B_\varepsilon \setminus B} |X\varrho(x)|\, dx = P(B) \int_0^\varepsilon (1+t)^3\, dt.$$

Finally, we easily calculate

$$\mathfrak{M}(\partial B) = \lim_{\varepsilon \downarrow 0} \frac{|B_\varepsilon \setminus B|}{\varepsilon} = P(B) \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon (1+t)^3 \, dt = P(B). \square$$

3. Brunn–Minkowski and isoperimetric inequality

In this section we show that the Brunn–Minkowski inequality (1.4) does not hold. The argument will be the following: inequality (1.4) implies that Carnot–Carathéodory balls are optimal isoperimetric sets, at least within the class of sets with sufficiently regular boundary (Proposition 3.5). But such balls are not isoperimetric sets (Proposition 3.4) and thus (1.4) cannot hold.

The isoperimetric problem in the Heisenberg group is the following minimum problem

$$\min\big\{P(E) : E \subset \mathbf{R}^3 \text{ is a bounded open set such that } |E| = 1\big\}. \tag{3.13}$$

Among all bounded open sets with given Lebesgue measure find the one that has minimum Heisenberg perimeter. If this problem has a solution in the class of regular sets then by Theorem 2.4 we also get a solution in this class for the problem of minimizing Minkowski content or spherical Hausdorff measure $\mathscr{S}^3$.

If F is a solution of (3.13) and we let $c = P(F)^{-4/3}$ then by the scaling properties of Lebesgue measure (1.3) and perimeter (2.9) the following isoperimetric inequality immediately follows

$$|E| \le cP(E)^{4/3} \quad \text{for all bounded open sets } E \subset \mathbf{R}^3. \tag{3.14}$$

Sets satisfying equality with the sharp constant c will be called *isoperimetric sets.* If F is an isoperimetric set, then $\tau_x(F)$ and $\delta_\lambda(F)$ are also isoperimetric for all $x \in \mathbf{R}^3$ and $\lambda > 0$.

The isoperimetric inequality (3.14) was proved by Pansu in [P] for regular sets, without sharp constant and with $\mathscr{S}^3(\partial E)$ replacing $P(E)$. Afterwards, many other generalizations have been established (see, for instance, [FGW] and [GN]) but always without sharp constants.

Problem (3.13) has a solution. This result has been recently proved by Leonardi and Rigot. Let us first give a definition.

Definition 3.1. An open set $F \subset \mathbf{R}^3$ satisfies *Condition* (B) if there exists $\beta \in (0,1)$ such that for any ball B centered at $x \in \partial F$ and with radius $0 < r \le r_0$ there exist two balls B_1 and B_2 with radius βr such that $B_1 \subset F \cap B$ and $B_2 \subset B \setminus \overline{F}$.

Theorem 3.2. *There exists a bounded open set $F \subset \mathbf{R}^3$ solving problem* (3.13)*. Moreover, F satisfies Condition* (B).

The existence statement and the regularity statement are respectively Theorem 2.5 and Theorem 2.11 in [LR]. Unfortunately, the regularity results proved in this paper do not suffice to characterize and compute the solution.

However, consider the group G of all orthogonal transformations (matrices) $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ of the form

$$T = \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix},$$

where $A \in O(2)$ is a 2×2 orthogonal matrix. It can be checked that if $E \subset \mathbf{R}^3$ is a set with finite Heisenberg perimeter then $P\big(T(E)\big) = P(E)$ for all $T \in G$. This suggests that sets solving problem (3.13) and having barycenter at the origin should satisfy $T(E) = E$ for all $T \in G$.

Definition 3.3. We say that an open set $E \subset \mathbf{R}^3$ belongs to the class $\mathscr{A}$ if $E = \big\{x \in \mathbf{R}^3 : |x_3| < \varphi\big(\sqrt{x_1^2 + x_2^2}\big)\big\}$ for some non negative function $\varphi \in C\big([0,\varrho]\big) \cap C^2\big([0,\varrho)\big)$, $\varrho > 0$, with $\varphi(\varrho) = 0$ and $\varphi'(\varrho) = -\infty$.

No proof that isoperimetric sets necessarily are in the class $\mathscr{A}$ is known. However, Carnot–Carathéodory balls centered at the origin belong to this class. This fact, which can be seen from the parametric equations written in (2.7), makes possible the following argument.

Proposition 3.4. *If the isoperimetric problem* (3.13) *has a solution in the class $\mathscr{A}$, then it is a dilation of the set $F = \big\{x \in \mathbf{R}^3 : |x_3| < \varphi\big(\sqrt{x_1^2 + x_2^2}\big)\big\}$ where $\varphi(t) = \arccos t + t\sqrt{1-t^2}$, $t \in [0,1)$. Thus, Carnot–Carathéodory balls are not isoperimetric sets.*

Proof. Let E be a set in the class $\mathscr{A}$ with $E = \big\{x \in \mathbf{R}^3 : |x_3| < \varphi\big(\sqrt{x_1^2 + x_2^2}\big)\big\}$ and write $f\colon D \to [0,+\infty)$, $f(x_1,x_2) = \varphi\big(\sqrt{x_1^2 + x_2^2}\big)$, $D = \big\{(x_1,x_2) \in \mathbf{R}^2 : \sqrt{x_1^2 + x_2^2} < \varrho\big\}$, $\varrho > 0$.

Denote by $\nu(x) = \big(\nu_1(x), \nu_2(x), \nu_3(x)\big)$ the Euclidean outward unit normal to ∂E at $x \in \partial E$. Since ∂E is symmetric with respect to the x_1x_2-plane, the representation formula (2.10) and the Area formula yield

$$\begin{aligned} P(E) &= \int_{\partial E} \sqrt{\big(\nu_1(x) + 2x_2\nu_3(x)\big)^2 + \big(\nu_2(x) - 2x_1\nu_3(x)\big)^2}\, d\mathscr{H}^2(x) \\ &= 2\int_D \sqrt{(\nu_1 + 2x_2\nu_3)^2 + (\nu_2 - 2x_1\nu_3)^2}\, \sqrt{1 + |\nabla f(x_1,x_2)|^2}\, dx_1\, dx_2, \end{aligned}$$

where in the last integral we have written $\nu = \nu\big(x_1, x_2, f(x_1,x_2)\big)$ and $\nabla f = (\partial_1 f, \partial_2 f)$. Since

$$\nu\big(x_1, x_2, f(x_1,x_2)\big) = \frac{\big(-\nabla f(x_1,x_2), 1\big)}{\sqrt{1 + |\nabla f(x_1,x_2)|^2}},$$

we finally get

$$P(E) = 2\int_D \sqrt{\big(\partial_1 f(x_1,x_2) - 2x_2\big)^2 + \big(\partial_2 f(x_1,x_2) + 2x_1\big)^2}\, dx_1\, dx_2$$
$$= 2\int_D \sqrt{|\nabla f(x_1,x_2)|^2 + 4\big(x_1\partial_2 f(x_1,x_2) - x_2\partial_1 f(x_1,x_2)\big) + 4(x_1^2+x_2^2)}\, dx_1\, dx_2.$$

In order to simplify computations it is useful to introduce the function $\psi(t) = \varphi\big(\sqrt{t}\,\big)$, in such a way that $f(x_1,x_2) = \psi(x_1^2+x_2^2)$. Since $\partial_1 f = 2x_1\psi'$ and $\partial_2 f = 2x_2\psi'$, then $x_1\partial_2 f - x_2\partial_1 f \equiv 0$ and using polar coordinates we find

$$P(E) = 4\int_D \sqrt{(x_1^2+x_2^2)\big(\psi'(x_1^2+x_2^2)^2+1\big)}\, dx_1\, dx_2 = 8\pi\int_0^{\varrho} r^2\sqrt{1+\psi'(r^2)^2}\, dr$$
$$= 4\pi\int_0^{\varrho^2} \sqrt{r}\,\sqrt{1+\psi'(r)^2}\, dr.$$

In the same way we obtain

$$|E| = 2\int_D f(x_1,x_2)\, dx_1\, dx_2 = 2\pi\int_0^{\varrho^2} \psi(r)\, dr.$$

If E solves problem (3.13) then the function ψ minimizes the functional

$$J(\psi) = 4\pi\int_0^{\sigma} \sqrt{r}\,\sqrt{1+\psi'(r)^2}\, dr$$

among non negative functions satisfying

$$\psi \in C([0,\sigma]) \cap C^2(0,\sigma), \quad \psi(\sigma) = 0, \quad \psi'(\sigma) = -\infty, \quad 2\pi\int_0^{\sigma} \psi(r)\, dr = 1,\ \sigma > 0.$$

By the Lagrange multiplier theorem for variational problems with integral constraint (see Proposition 1.17 in [BGH]) there exists $\lambda \neq 0$ such that the function ψ solves the Euler–Lagrange equation

$$\frac{d}{dr}\frac{\partial H(r,\psi,\psi')}{\partial z} = \frac{\partial H(r,\psi,\psi')}{\partial u}$$

where $H(r,u,z) = 4\pi\sqrt{r}\,\sqrt{1+z^2} + 2\pi\lambda u$. This gives the ordinary differential equation

$$\frac{d}{dr}\left(2\sqrt{r}\,\frac{\psi'(r)}{\sqrt{1+\psi'(r)^2}}\right) = \lambda. \tag{3.15}$$

We want to integrate equation (3.15) between 0 and r and to this aim we preliminary show that

$$\lim_{r\downarrow 0} 2\sqrt{r}\,\frac{\psi'(r)}{\sqrt{1+\psi'(r)^2}} = 0.$$

Note first of all that $2\sqrt{r}\,\psi'(r) = \varphi'\big(\sqrt{r}\big)$, and thus we have to show that $\varphi'(0)=0$. Here Condition (B) is involved. Assume by contradiction that $\alpha = \varphi'(0) < 0$ and let $\bar{x} = (0,0,\bar{x}_3)$ be the intersection point of the set E with the half line $\{(0,0,x_3) \in \mathbf{R}^3 : x_3 > 0\}$. Then, for some $r_0 > 0$ we have $E \cap B(\bar{x}, r_0) \subset K \cap B(\bar{x}, r_0)$, where $K = \big\{(x_1,x_2,x_3) \in \mathbf{R}^3 : -\alpha\sqrt{x_1^2+x_2^2} < 2(\bar{x}_3 - x_3)\big\}$ is the (downward) cone with x_3-axis, vertex at $\bar{x}$ and aperture $-2/\alpha$. Let $\operatorname{Box}(\bar{x}, r) = \big\{(x_1,x_2,x_3) \in \mathbf{R}^3 : \sqrt{x_1^2+x_2^2} \le r,\ |x_3 - \bar{x}_3| \le r^2\big\}$. It can be checked that $B(\bar{x}, r) \subset \operatorname{Box}(\bar{x}, \delta r)$ for some $\delta > 0$ (for instance with $\delta = \sqrt{2/\pi}$). Moreover

$$|K \cap \operatorname{Box}(\bar{x}, \delta r)| = \int_0^{\delta^2 r^2} \frac{4\pi t^2}{\alpha^2}\,dt = kr^6$$

for some $k > 0$. Therefore $|K \cap B(\bar{x}, r)| \le kr^6$ for all $0 < r \le r_0$, and $E \cap B(\bar{x}, r)$ cannot contain a ball B_1 with radius comparable with r because $|B(\bar{x}, r)| = |B(0,1)|r^4$. Thus Condition (B) is violated at the point $\bar{x} \in \partial E$ and this is not possible because by Theorem 1.2 solutions of the isoperimetric problem must satisfy this condition. A similar argument shows that Condition (B) is violated in $\mathbf{R}^3 \setminus \overline{E}$ if $\varphi'(0) > 0$.

Now we can integrate equation (3.15) obtaining

$$2\sqrt{r}\,\frac{\psi'(r)}{\sqrt{1+\psi'(r)^2}} = \lambda r \quad \text{and thus} \quad \psi'(r) = -\sqrt{\frac{\lambda^2 r}{4 - \lambda^2 r}}\,.$$

The condition $\psi'(\varrho^2) = -\infty$ gives $\lambda^2\varrho^2 = 4$ and using $\psi(\varrho^2) = 0$ we finally find

$$\varphi(t) = \psi(t^2) = 2\varrho^2 \int_0^{\arccos(t/\varrho)} \cos^2\theta\,d\theta = \varrho^2\left[\arccos\frac{t}{\varrho} + \frac{t}{\varrho}\sqrt{1 - \left(\frac{t}{\varrho}\right)^2}\,\right].$$

The variable ϱ is fixed by the volume constraint $|E| = 1$. □

Now we conclude the argument showing that inequality (1.4) implies that Carnot–Carathéodory balls solve problem (3.13).

Proposition 3.5. *The Brunn–Minkowski inequality* (1.4) *implies the isoperimetric property for Carnot–Carathéodory balls.*

Proof. We denote by $B = B(0,1)$ the Carnot–Carathéodory ball centered at the origin with radius 1. We show that $P(B) = 4|B|$. Indeed, letting $B_\varepsilon = B(0, 1+\varepsilon)$, $\varepsilon > 0$, by Proposition 2.5 we get

$$P(B) = \mathfrak{M}(B) = \lim_{\varepsilon\downarrow 0} \frac{|B_\varepsilon| - |B|}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \frac{(1+\varepsilon)^4 - 1}{\varepsilon}|B| = 4|B|. \tag{3.16}$$

We used once again the scaling property (1.3) of Lebesgue measure.

Now let $E \subset \mathbf{R}^3$ be a bounded open set with boundary of class C^2 and denote by

$$E_\varepsilon = \{x \in \mathbf{R}^3 : \mathrm{dist}(x; E) < \varepsilon\} = \bigcup_{x\in E} B(x,\varepsilon) = E \cdot B(0,\varepsilon)$$

its ε-neighborhood. By Theorem 2.4

$$P(E) = \mathfrak{M}(E) = \lim_{\varepsilon\downarrow 0} \frac{|E_\varepsilon| - |E|}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \frac{|E\cdot B(0,\varepsilon)| - |E|}{\varepsilon}.$$

If inequality (1.4) were true, then

$$|E\cdot B(0,\varepsilon)| \ge \big(|B(0,\varepsilon)|^{1/4} + |E|^{1/4}\big)^4 = \big(\varepsilon|B|^{1/4} + |E|^{1/4}\big)^4,$$

and thus

$$P(E) \ge \lim_{\varepsilon\downarrow 0} \frac{\big(\varepsilon|B|^{1/4} + |E|^{1/4}\big)^4 - |E|}{\varepsilon} = 4|B|^{1/4}|E|^{3/4}.$$

Finally, taking into account (3.16) we get for any bounded open set $E \subset \mathbf{R}^3$ of class C^2

$$\frac{P(E)}{|E|^{3/4}} \ge \frac{P(B)}{|B|^{3/4}}. \tag{3.17}$$

Now, let F be a solution of problem (3.13). A priori we do not know whether this set is of class C^2. However, by Theorem 7.1 in [MSC] there exists a sequence of sets $(E_n)_{n\in\mathbf{N}}$ of class C^∞ such that

$$\lim_{n\to\infty} P(E_n) = P(F) \quad \text{and} \quad \lim_{n\to\infty} |E_n| = |F|.$$

Applying (3.17) to each E_n and using the minimality of F we get

$$\frac{P(F)}{|F|^{3/4}} = \frac{P(B)}{|B|^{4/4}}.$$

Therefore Carnot–Carathéodory balls are solution of the isoperimetric problem. But this is not possible because of Proposition 3.4. □

Corollary 3.6. *There exist two open sets $A, B \subset \mathbf{R}^3$ such that*

$$|A\cdot B|^{1/4} < |A|^{1/4} + |B|^{1/4}.$$

Proof. Let $(E_n)_{n\in\mathbf{N}}$ be the sequence of sets in the proof of Proposition 3.5. There exist $n \in \mathbf{N}$ and $\varepsilon > 0$ such that $|E_n \cdot B(0,\varepsilon)|^{1/4} < |E_n|^{1/4} + |B(0,\varepsilon)|^{1/4}$. □

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