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FINITE REPRESENTABILITY OF THE YANG OPERATOR

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Abstract. We study the finite representability of the operator $T^{\mathrm{co}}: X^{**}/X \longrightarrow Y^{**}/Y$ in $T: X \longrightarrow Y$ and its consequences in operator semigroup and operator ideal theory. The results obtained involve a delicate study of the second conjugate T^{**}.

1. Introduction

For every operator $T: X \longrightarrow Y$ acting between Banach spaces, we can associate the operator $T^{\mathrm{co}}: X^{**}/X \longrightarrow Y^{**}/Y$, defined by $T^{\mathrm{co}}(x^{**}+X) := T^{**}(x^{**})+Y$. The operator T^{co}, introduced by Yang [18], has been successfully applied in the study of several operator semigroups related to the class of tauberian operators. Here, the notion of *semigroup* is considered in the sense of [1], where it has been introduced as a natural counterpart to the notion of ideal of operators. Let us list some fields where T^{co} has been applied:

(a) exact sequences in Banach spaces and generalized Fredholm operators [18];
(b) asymmetry between tauberian operators and cotauberian operators [2];
(c) weak Calkin algebras [8];
(d) strongly tauberian operators [14];

application (c), developed by González, Saksman and Tylli, is remarkable because it exhibits the rich interplay between operator ideals and operator semigroups. We also recall that tauberian operators play an important role in Banach space theory (see for instance [15]).

The purpose of our paper is the study of the finite representability of T^{co} in T and its consequences in ultrapower-stable semigroups and ideals. Those classes of operators are interesting because they can be defined locally, in terms of the action of their operators on finite-dimensional subspaces. Examples of ultrapower-stable classes of operators are the ideal of uniformly convexifying operators [10], and the semigroup of supertauberian operators, which was introduced by Tacon [17] in order to obtain a class of tauberian operators stable under duality.

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Section 3 starts recalling the definitions of operator finite representability that we need: local supportability (that generalizes Bellenot finite representability), which presents good applications in semigroup theory, and local representability (that generalizes Heinrich finite representability), which is applicable in ideal theory. Both finite representabilities are independent [12]. The last section contains the main results. Indeed, in Theorems 3.7 and 3.8 we prove that, given any operator T, the Yang operator T^{co} is both locally supportable and locally representable in T. As a consequence, we prove in Proposition 3.9 that if $\mathscr{S}$ is an ultrapower-stable semigroup which is either left-stable and injective, or right-stable and surjective, then T^{co} belongs to $\mathscr{S}$ provided $T \in \mathscr{S}$. An analogous result is obtained for ultrapower-stable, regular ideals. Those results are achieved after giving in Theorem 3.4 a very strong result about finite representability of T^{**} in T. Actually, the fact that T^{**} is Heinrich and Bellenot finitely representable in T is already known, but we obtain some additional properties which are essential in the proof of Theorems 3.7 and 3.8.

In the following, capital letters $X, Y, \dots$ stand for Banach spaces; B_X is the closed unit ball of X, and S_X is the set of all norm-one elements of X. The successive conjugate spaces of X are denoted $X^*, X^{**}, X^{*(3)}, X^{*(4)}, \dots$. The action of $f \in X^*$ on $x \in X$ is denoted $\langle f, x\rangle$; the linear subspace generated by a subset A of X is denoted by $\mathrm{span}(A)$, its norm closure is denoted by $\bar{A}$, and its norm interior, by $\mathrm{int}\, A$. Given a bounded linear map (*operator* in short) $T\colon X \longrightarrow Y$, its kernel and range are respectively denoted by $N(T)$ and $T(X)$. The class of all operators from X into Y is represented by $\mathscr{L}(X,Y)$. If E is a closed subspace of X, I_E represents the natural embeddings of E into both X and X^{**}. We say that $P \in \mathscr{L}(X,Y)$ is a projection when Y is a closed subspace of X, $P|_Y = I_Y$ and $P^2 = P$.

Given an ultrafilter $\mathfrak{U}$ on I, the ultrapower X following $\mathfrak{U}$ is the quotient space $l_\infty(I,X)/N$, where $N := \big\{(x_i)_{i\in I} \in l_\infty(I,X) : \lim_{\mathfrak{U}} \|x_i\| = 0\big\}$; $[x_i]_i$ (or $[x_i]$ if there is no confusion) denotes the element of $X_{\mathfrak{U}}$ whose representative is $(x_i)_{i\in I}$; its norm is $\lim_{i\to\mathfrak{U}} \|x_i\|$ (or $\lim_{\mathfrak{U}} \|x_i\|$ for short). Given an operator $T \in \mathscr{L}(X,Y)$, we denote by $T_{\mathfrak{U}} \in \mathscr{L}(X_{\mathfrak{U}}, Y_{\mathfrak{U}})$ the operator that maps every $[x_i]$ onto $[T(x_i)]$. More details about ultraproducts may be found in [9]. An ultrafilter $\mathfrak{U}$ on I is said to be *countably incomplete* if there is a countable partition of I, $\{I_n\}_{n=1}^\infty$, disjoint with $\mathfrak{U}$; that is, $I_n \notin \mathfrak{U}$ for all n. All ultrafilters throughout this paper are countably incomplete.

Given $d \geq 1$, an operator $T \in \mathscr{L}(X,Y)$ is said to be a d*-injection* if $d^{-1} \leq \|T(x)\| \leq d$ for all $x \in S_X$. An operator $T \in \mathscr{L}(X,Y)$ is said to be a *metric injection* (or *isometric embedding*) if T is a 1-injection; T is said to be a *metric surjection* if T^* is a metric injection.

2. Background

In this paper we deal with two types of operator finite representability: *lo-*

cal supportability (Definition 2.1(b)) introduced by the authors in [12], and *local representability* (Definition 2.2(b)), introduced by Pietsch [13]. These types of finite representability respectively generalize Bellenot finite representability (Definition 2.1(a) and [5]) and Heinrich finite representability (Definition 2.2(a) and [10]). It is important to bear in mind that Definitions 2.1 and 2.2 are mutually independent, namely: Bellenot finite representability does not imply local representability, and Heinrich finite representability does not imply local supportability [12]. We note that the notions of mere local supportability and local representability extend the definition of crude finite representability for Banach spaces, introduced by James.

Definition 2.1. Let $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(W,Z)$ be a pair of operators;

(a) T is said to be *Bellenot finitely representable in* S if for every finite dimensional subspace E of X and every $\varepsilon > 0$ there is a $(1+\varepsilon)$-injection $L \in \mathscr{L}(E,W)$ satisfying $\big|\|Tx\| - \|SLx\|\big| \leq \varepsilon\|x\|$ for all $x \in E$; equivalently, there are $(1+\varepsilon)$-injections $U \in \mathscr{L}(E,W)$, $V \in \mathscr{L}\big(T(E),Z\big)$ so that $\|SU - VT|_E\| \leq \varepsilon$;

(b) given $d \geq 1$, T is said to be *locally d-supportable in* S if for every $\varepsilon > 0$ and every finite-dimensional subspace E of X there is a $(d+\varepsilon)$-injection $U \in \mathscr{L}(E,W)$ and an operator $V \in \mathscr{L}(T(E),Z)$ satisfying $\|V\| \leq d+\varepsilon$ and $\|SU - VT|_E\| \leq \varepsilon$.

Definition 2.2. Let $T \in \mathscr{L}(X,Y)$ and $S \in \mathscr{L}(W,Z)$ be a pair of operators;

(a) T is said to be *Heinrich finitely representable in* $S \in \mathscr{L}(W,Z)$ if for every $\varepsilon > 0$, every finite-dimensional subspace E of X and every finite-codimensional subspace F of Y there is a finite-dimensional subspace E_1 of W, a finite-codimensional subspace F_1 of Z and a pair of surjective $(1+\varepsilon)$-injections $U \in \mathscr{L}(E,E_1)$, $V \in \mathscr{L}(Z/F_1, Y/F)$ so that $\|VQ_{F_1}SU - Q_F T|_E\| \leq \varepsilon$, where Q_F and Q_{F_1} are the natural quotient maps;

(b) given $c > 0$, T is said to be *locally c-representable in* S if for every $\varepsilon > 0$ and every pair of operators $A \in \mathscr{L}(E,X)$, $B \in \mathscr{L}(Y,F)$ with E and F finite-dimensional spaces there is a pair of operators $A_1 \in \mathscr{L}(E,W)$, $B_1 \in \mathscr{L}(Z,F)$ satisfying $\|A_1\| \cdot \|B_1\| \leq (c+\varepsilon)\|A\| \cdot \|B\|$ and $BTA = B_1 S A_1$.

When we do not need to specify parameters d or c in the above definitions, we will just speak of local supportability or local representability.

3. Finite representability of the operators T^{**} and T^{co} in T

There are several proofs and versions of the fact that, for every operator T, T^{**} is Bellenot finitely representable in T. The first is given by Bellenot, but Basallote and Díaz claim that it contains a gap, so they provide us with a second demonstration [3]. Behrends [4, Corollary 5.4] gives another proof which yields some additional exact conditions. Nevertheless, his proof is only valid when T is tauberian, which is an important lack of generality from the point of view of our paper. In our Theorem 3.4, we improve the proofs mentioned above by showing that T^{**} is Bellenot finitely representable in T for every operator T and obtaining additional properties. In fact, we obtain the exact conditions (a) and (b), which are essential in order to prove that T^{co} is locally supportable and locally representable in T, and also, we get the conditions (d) and (e), which play an important role in part (b) of Proposition 3.9, where we study surjective, right-stable semigroups. We start with a chain of lemmata.

Lemma 3.1. *Let E be a finite-dimensional space with $\dim E = n$ and let $0 < \varepsilon < n^{-1}$. Then every ε-net in S_E contains a basis whose coordinate functionals are norm bounded by $(1 - n\varepsilon)^{-1}$.*

Proof. Let $\mathscr{E}$ be an ε-net of S_E. By Auerbach's lemma, there is a biorthogonal system $(u_i, h_i)_{i=1}^n$ in $S_E \times S_{E^*}$. For every u_i, we choose $e_i \in \mathscr{E}$ such that $\|u_i - e_i\| \le \varepsilon$. We define the operator $L \in \mathscr{L}(E,E)$ by $L(e) := \sum_{i=1}^n \langle h_i, e\rangle e_i$. Note that $L(u_i) = e_i$ and $\|I_E - L\| \le n\varepsilon$, so L is an isomorphism and $\{e_i\}_{i=1}^n$ is a basis of E. Moreover, given $e = \sum_{i=1}^n \lambda_i e_i \in S_E$ and writing $u := L^{-1}(e) = \sum_{i=1}^n \lambda_i u_i$, we get $\|e - u\| \le n\varepsilon\|u\|$; hence, for every i, $|\lambda_i| = |\langle h_i, u\rangle| \le \|u\| \le (1 - n\varepsilon)^{-1}$; thus, the coordinate functionals associated to $\{e_i\}_{i=1}^n$ are norm bounded by $(1 - n\varepsilon)^{-1}$. □

Lemma 3.2. *Let $E \subset X^{**}$ be a finite-dimensional subspace with $\dim E = n$, $\{e_i\}_{i=1}^p$ an ε-net in S_E with $0 < \varepsilon < (2n)^{-1}$ and $\mathscr{V}$ a weak* neighborhood of $0 \in X^{**}$. If $(L_\alpha)_\alpha$ is a net of operators from E into X^{**} such that $\|L_\alpha(e_i)\| \le (1 - n\varepsilon)^{-1}$ and $w^*\text{-}\lim_\alpha L_\alpha(e_i) = e_i$ for all $1 \le i \le p$, then there is an α_0 such that L_α is a $(1 - 2n\varepsilon)^{-1}$-injection and $L_\alpha(e) \in e + \mathscr{V}$ for all $e \in S_E$ and all $\alpha \ge \alpha_0$.*

Proof. By Lemma 3.1, we may assume that $\{e_i\}_{i=1}^n$ is a basis of E whose coordinate functionals are norm bounded by $(1 - n\varepsilon)^{-1}$. Consequently, $\|L_\alpha\| \le (1 - n\varepsilon)^{-1} n$ for all α. Since $w^*\text{-}\lim_\alpha L_\alpha(e_i) = e_i$ and $(1 - n\varepsilon)^{-1} < 2$, we can select β satisfying $\|L_\alpha e_i\| \ge 1 - n\varepsilon\big(2 - (1-n\varepsilon)^{-1}\big)$ for all $1 \le i \le p$ and $\alpha \ge \beta$, so L_α is a $(1 - 2n\varepsilon)^{-1}$-injection. Indeed, given $e \in S_E$, by choosing e_i such that $\|e - e_i\| \le \varepsilon$, we obtain

$$\begin{aligned}
\|L_\alpha(e)\| &\le \|L_\alpha(e_i)\| + \|L_\alpha(e-e_i)\| \\
&\le (1 - n\varepsilon)^{-1} + n\varepsilon(1 - n\varepsilon)^{-1} \le (1 - 2n\varepsilon)^{-1} \quad \text{and} \\
\|L_\alpha(e)\| &\ge \|L_\alpha(e_i)\| - \|L_\alpha(e-e_i)\| \\
&\ge 1 - n\varepsilon\big(2-(1-n\varepsilon)^{-1}\big) - n\varepsilon(1-n\varepsilon)^{-1} = 1-2n\varepsilon,
\end{aligned}$$

which proves that L_α is a $(1 - 2n\varepsilon)^{-1}$-injection. Now, with no loss of generality, we can assume that $\mathscr{V}$ is absolutely convex. By choosing $\alpha_0 \ge \beta$ and such that $L_\alpha(e_i) \in e_i + n^{-1}(1 - n\varepsilon)\mathscr{V}$ for all $1 \le i \le n$ and $\alpha \ge \alpha_0$, the proof is complete. □

Lemma 3.3 ([14, Lemma 23]). *Let $T \in \mathscr{L}(X,Y)$, $y \in Y$, $x^{**} \in X^{**}$ and $\eta > 0$ such that $\|x^{**}\| < 1$ and $\|T^{**}(x^{**}) + y\| < \eta$. Then x^{**} belongs to the $\sigma(X^{**}, X^{*})$-closure of $\{x \in X : \|x\| < 1,\ \|T(x) + y\| < \eta\}$.*

The following result includes the classical principle of local reflexivity for Banach spaces.

Theorem 3.4. *Let $T \in \mathscr{L}(X,Y)$ be an operator, E a finite-dimensional subspace of X^{**} and F a finite-dimensional subspace of Y^{**} satisfying $F \cap T^{**}(E) = \{0\}$. Let $0 < \varepsilon < 1$ and a pair of weak* neighborhoods $\mathscr{U}$ of $0 \in X^{**}$ and $\mathscr{V}$ of $0 \in Y^{**}$ be given. Then there is a pair of $(1-\varepsilon)^{-1}$-injections $U \in \mathscr{L}(E,X)$ and $V \in \mathscr{L}\big(T^{**}(E) \oplus F, Y\big)$ satisfying the following statements*:

(a) $U|_{E\cap X} = I_{E\cap X}$,

(b) $V|_{(T^{**}E\oplus F)\cap Y} = I_{(T^{**}E\oplus F)\cap Y}$,

(c) $\|TU - VT^{**}|_E\| < \varepsilon$,

(d) $U(e) \in e + \mathscr{U}$ *for all* $e \in S_E$,

(e) $V(f) \in f + \mathscr{V}$ *for all* $f \in S_{T^{**}(E)\oplus F}$.

In particular, T^{**} is Bellenot finitely representable in T.

Proof. Without loss of generality, we may assume that $\|T\| = 1$. Let $\{x_i^1\}_{i=1}^p \cup \{x_i^2\}_{i=1}^q \cup \{x_i^3\}_{i=1}^t$ be a basis of E taken in $\operatorname{int} B_E$ satisfying

$$
\begin{aligned}
&\{x_i^1\}_{i=1}^p \text{ is a basis in } E \cap X,\\
&\{x_i^1\}_{i=r+1}^p \text{ spans } N(T|_{E\cap X}),\\
&\{x_i^1\}_{i=1}^p \cup \{x_i^2\}_{i=1}^q \text{ is a basis in } (T^{**}|_E)^{-1}Y,\\
&\{x_i^1\}_{i=r+1}^p \cup \{x_i^2\}_{i=s+1}^q \text{ spans } N(T^{**}|_E).
\end{aligned}
$$

We write $y_i^k := T^{**}x_i^k$ and also take a basis $\{y_i^4\}_{i=1}^u \cup \{y_i^5\}_{i=1}^v$ in $\operatorname{int} B_F$ such that $\{y_i^4\}_{i=1}^u$ spans $F \cap Y$. Let $(h_i)_{i=1}^q$ be the coordinate functionals associated with $(x_i^2)_{i=1}^q$ and let $H = (1-n\varepsilon)^{-1}\sum_{i=1}^q \|h_i\|$.

Pick $0 < \delta < 2^{-1}(p+q+t+u+v)^{-1}\varepsilon$ and δ-nets $(e_j)_{j=1}^n$ in S_E and $(f_j)_{j=1}^m$ in $S_{T^{**}(E)\oplus F}$, and write $e_j = \sum_{k,i} \lambda_{ki}^j x_i^k$ and $f_j = \sum_{k,i} \mu_{ki}^j y_i^k$ for the appropriate k and i in each case.

Define $S\colon l_\infty^q(X) \oplus_\infty l_\infty^t(X) \oplus_\infty l_\infty^v(Y) \longrightarrow l_\infty^n(X) \oplus_\infty l_\infty^m(Y) \oplus_\infty l_\infty^q(Y)$ by $S = (S_1, S_2, S_3)$, where

$$
\begin{aligned}
S_1\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) &= \left(\sum_{i=1}^q \lambda_{2i}^j a_i + \sum_{i=1}^t \lambda_{3i}^j b_i\right)_{j=1}^n \in l_\infty^n(X),\\
S_2\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) &= \left(\sum_{i=1}^t \mu_{3i}^j T b_i + \sum_{i=1}^v \mu_{5i}^j c_i\right)_{j=1}^m \in l_\infty^m(Y),\\
S_3\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) &= (\varepsilon^{-1} H\, T a_i)_{i=1}^q \in l_\infty^q(Y),
\end{aligned}
$$

with $a_i \in X$, $b_i \in X$ and $c_i \in Y$ for all i. Consider the element

$$z = \left(\left(\sum_{i=1}^{p} \lambda_{1i}^{j} x_i^1 \right)_{j=1}^{n}, \left(\sum_{i=1}^{r} \mu_{1i}^{j} y_i^1 + \sum_{i=1}^{s} \mu_{2i}^{j} y_i^2 + \sum_{i=1}^{u} \mu_{4i}^{j} y_i^4 \right)_{j=1}^{m}, (-\varepsilon^{-1} H y_i^2)_{i=1}^{q} \right).$$

Then $S\big((a_i)_{i=1}^q, (b_i)_{i=1}^t, (c_i)_{i=1}^v\big) + z$ is equal to

$$\begin{aligned} &\Bigg(\left(\sum_{i=1}^{p} \lambda_{1i}^{j} x_i^1 + \sum_{i=1}^{q} \lambda_{2i}^{j} a_i + \sum_{i=1}^{t} \lambda_{3i}^{j} b_i \right)_{j=1}^{n}, \\ &\qquad \left(\sum_{i=1}^{r} \mu_{1i}^{j} y_i^1 + \sum_{i=1}^{s} \mu_{2i}^{j} y_i^2 + \sum_{i=1}^{t} \mu_{3i}^{j} T b_i + \sum_{i=1}^{u} \mu_{4i}^{j} y_i^4 + \sum_{i=1}^{v} \mu_{5i}^{j} c_i \right)_{j=1}^{m}, \\ &\qquad \big(\varepsilon^{-1} H (T(a_i) - y_i^2) \big)_{i=1}^{q} \Bigg). \end{aligned}$$

Besides, $S^{**}\big((x_i^2)_{i=1}^q, (x_i^3)_{i=1}^t, (y_i^5)_{i=1}^v\big) + z = \big((e_j)_{j=1}^n, (f_j)_{j=1}^m, (0)_{i=1}^q\big)$ is norm-one, so Lemma 3.3 provides us with a net $\big((a_i^\alpha)_{i=1}^q, (b_i^\alpha)_{i=1}^t, (c_i^\alpha)_{i=1}^v\big)$ in the unit ball of $l_\infty^q(X) \oplus_\infty l_\infty^t(X) \oplus_\infty l_\infty^v(Y)$ which is weak* converging to $\big((x_i^2)_{i=1}^q, (x_i^3)_{i=1}^t, (y_i^5)_{i=1}^v\big)$ and such that $\big\|S\big((a_i^\alpha)_{i=1}^q, (b_i^\alpha)_{i=1}^t, (c_i^\alpha)_{i=1}^v\big) + z\big\| < (1-n\delta)^{-1}$; in particular, $\|Ta_i^\alpha - y_i^2\| \le \varepsilon H^{-1}(1-n\varepsilon)^{-1}$ for all $1 \le i \le q$.

Now we define $U_\alpha \in \mathscr{L}(E, X)$ and $V_\alpha \in \mathscr{L}(T^{**}(E) \oplus F, Y)$ by

$$\begin{aligned} U_\alpha(x_i^1) &:= x_i^1 && \text{for all } i \in \{1, \dots, p\}, \\ U_\alpha(x_i^2) &:= a_i^\alpha && \text{for all } i \in \{1, \dots, q\}, \\ U_\alpha(x_i^3) &:= b_i^\alpha && \text{for all } i \in \{1, \dots, t\}, \\ V_\alpha(y_i^1) &:= y_i^1 && \text{for all } i \in \{1, \dots, r\}, \\ V_\alpha(y_i^2) &:= y_i^2 && \text{for all } i \in \{1, \dots, s\}, \\ V_\alpha(y_i^3) &:= Tb_i^\alpha && \text{for all } i \in \{1, \dots, t\}, \\ V_\alpha(y_i^4) &:= y_i^4 && \text{for all } i \in \{1, \dots, u\}, \\ V_\alpha(y_i^5) &:= c_i^\alpha && \text{for all } i \in \{1, \dots, v\}. \end{aligned}$$

Note that $U_\alpha|_{E\cap X} = I_{E\cap X}$ and $V_\alpha|_{(T^{**}(E)\oplus F)\cap Y} = I_{(T^{**}(E)\oplus F)\cap Y}$ for all α, so conditions (a) and (b) hold. Besides, for all $e \in S_E$, $\|(TU_\alpha - V_\alpha T^{**})(e)\| \le \sum_{i=1}^q |\langle h_i, e\rangle| \|Ta_i^\alpha - y_i^2\| \le \varepsilon$ so $\|TU_\alpha - V_\alpha T^{**}|_E\| \le \varepsilon$, and part (c) is done.

In order to apply Lemma 3.2 to both U_α and V_α, note that $\dim E \le (p+q+t+u+v)$ and $\dim\big(T^{**}(E) \oplus F\big) \le (p+q+t+u+v)$; moreover, $\|U_\alpha(e_j)\| \le (1-n\delta)^{-1}$ and $w^*\text{-}\lim_\alpha U_\alpha(e_j) = e_j$ for all $1 \le j \le n$; analogously, we have $\|V_\alpha(f_j)\| \le (1-n\delta)^{-1}$ and $w^*\text{-}\lim_\alpha V_\alpha(f_j) = f_j$ for all $1 \le j \le m$, so it is possible to choose α such that U_α and V_α are $(1-\varepsilon)^{-1}$-injections, and such that conditions (d) and (e) hold, concluding the proof. ▫

Theorem 3.5. *For every operator $T \in \mathscr{L}(X,Y)$ there exists an ultrafilter $\mathfrak{U}$ and there are metric injections $U \in \mathscr{L}(X^{**}, X_{\mathfrak{U}})$ and $V \in \mathscr{L}(Y^{**}, Y_{\mathfrak{U}})$, and metric surjections $P \in \mathscr{L}(X_{\mathfrak{U}}, X^{**})$ and $Q \in \mathscr{L}(Y_{\mathfrak{U}}, Y^{**})$ such that*

(a) $T_{\mathfrak{U}} \circ U = V \circ T^{**}$,

(b) $T^{**} \circ P = Q \circ T_{\mathfrak{U}}$,

(c) $T^{**} = Q \circ T_{\mathfrak{U}} \circ U$.

Moreover, $U(x) = [x]$ and $P([x]) = x$ for all $x \in X$, and analogously, $V(y) = [y]$ and $Q([y]) = y$ for all $y \in Y$.

Proof. Let J be the set of all tuples $j \equiv (E_j, F_j, \varepsilon_j, \mathscr{U}_j, \mathscr{V}_j)$ where E_j and F_j are finite-dimensional subspaces of X^{**} and Y^{**}, respectively, $\varepsilon_j \in (0,1)$, $\mathscr{U}_j$ is a weak* neighborhood of $0 \in X^{**}$, and $\mathscr{V}_j$ is a weak* neighborhood of $0 \in Y^{**}$. We define an order $\preceq$ in J by $i \preceq j$ if $E_i \subset E_j$, $F_i \subset F_j$, $\varepsilon_i \geq \varepsilon_j$, $\mathscr{U}_i \supset \mathscr{U}_j$ and $\mathscr{V}_i \supset \mathscr{V}_j$. Let $\mathfrak{U}$ be an ultrafilter refining the order filter on J.

For every $j \in J$, Theorem 3.4 yields a pair of $(1+\varepsilon_j)$-injections $U_j \in \mathscr{L}(E_j, X)$ and $V_j \in \mathscr{L}\big(T^{**}(E_j) + F_j, Y\big)$ such that

$$
\begin{aligned}
&U_j(e) = e \text{ for all } e \in E_j \cap X,\\
&V_j(f) = f \text{ for all } f \in (T^{**}(E_j) + F_j) \cap Y,\\
&\|(TU_j - V_jT^{**})(e)\| < \varepsilon \text{ for all } e \in S_{E_j},\\
&U_j(e) \in e + \mathscr{U}_j \text{ for all } e \in S_{E_j},\\
&V_j(f) \in f + \mathscr{V}_j \text{ for all } f \in S_{T^{**}(E_j)+F_j}.
\end{aligned}
$$

The operators U, V, P and Q are defined as follows:

$$
\begin{aligned}
&U(x^{**}) = [x_j] \text{ where } x_j := U_j(x^{**}) \text{ if } x^{**} \in E_j, \text{ and } x_j := 0 \text{ otherwise;}\\
&V(y^{**}) = [y_j] \text{ where } y_j := V_j(y^{**}) \text{ if } y^{**} \in T^{**}(E_j) + F_j, \text{ and } y_j := 0 \text{ otherwise;}\\
&P([x_j]) = w^*\text{-} \lim_{j \to \mathfrak{U}} x_j \in X^{**};\\
&Q([y_j]) = w^*\text{-} \lim_{j \to \mathfrak{U}} y_j \in Y^{**}.
\end{aligned}
$$

Fix $x^{**} \in S_{X^{**}}$ and $\delta > 0$. Let us write $U(x^{**}) = [x_j]$ as in the definition of U. Take $j_0 \in J$ such that $x^{**} \in E_{j_0}$ and $\varepsilon_{j_0} < \delta$. It follows that $(1+\delta)^{-1} \leq \|U_j(x^{**})\| \leq 1 + \delta$ for all $j_0 \preceq j$, so $\lim_{j \to \mathfrak{U}} \|x_j\| = 1$, which proves that U is a metric injection. Analogously we prove that V is also a metric injection. The fact that P is a metric surjection follows from $P(B_{X_{\mathfrak{U}}}) = B_{X^{**}}$. The same applies for Q.

To prove (a), we take $x^{**} \in S_{X^{**}}$ and $\delta > 0$. Select $j_0 \in J$ such that $\varepsilon_j \leq \delta$ and $x^{**} \in E_j$. Thus $\{j \in J : \|(TU_j - V_jT^{**})(x^{**})\| \leq \delta\} \supset \{j \in J : j_0 \preceq j\} \in \mathfrak{U}$ which shows that $T_{\mathfrak{U}}U - VT^{**} = 0$. For statement (b), take $[x_j] \in X_{\mathfrak{U}}$. Then

$$T^{**}P([x_j]) = T^{**}\Big(w^*\text{-} \lim_{j \to \mathfrak{U}} x_j\Big) = w^*\text{-} \lim_{j \to \mathfrak{U}} T(x_j) = QT_{\mathfrak{U}}([x_j]).$$

Part (c) is achieved by using similar arguments. The facts that $U(x) = [x]$ and $P([x]) = x$ for all $x \in X$, and $V(y) = [y]$ and $P([y]) = y$ for all $y \in Y$ are trivial. □

It is proved in [10] that for every operator T, T^{**} is Heinrich finitely representable in T. Part (b) in Theorem 3.5 leads to an alternative proof of the same fact. In order to show that T^{co} is locally supportable by T, we first establish the following lemma based upon an argument in [11]. Note that we need Theorem 3.4 to get statement (b).

Lemma 3.6. *Let X be a Banach space, $R\colon X^{**} \longrightarrow X^{**}/X$ the associated quotient operator, M a finite-dimensional subspace of X^{**}/X and $0 < \varepsilon < 1$. Write $Z := R^{-1}M$, take any projection $Q\colon Z \longrightarrow X$ and denote its kernel by G. Then we have*

(a) *there is a finite-dimensional subspace F of X such that for each $g \in (I_Z - Q)B_Z$ there is $e \in F$ satisfying $\|g - e\| \le 1 + \varepsilon$;*

(b) *let $L\colon F \oplus G \longrightarrow X$ be a $(1+\varepsilon)$-injection satisfying $L|_F = I_F$ and define $P := Q + L(I_Z - Q)$; then $P\colon Z \longrightarrow X$ is a projection with $\|P\| \le 3 + 4\varepsilon$;*

(c) *the operator $U := R|_{N(P)}$ is a norm-one isomorphism onto M, $\|U^{-1}\| \le 1 + \|P\|$ and $U^{-1}(g + X) = g - Lg$ for all $g \in G$.*

Proof. (a) Since $(I_Z - Q)(B_Z)$ is compact, we can choose a finite set $\{z_i\}_{i=1}^n$ in B_Z so that for $g_i := (I_Z - Q)z_i$, the family $\{g_i\}_{i=1}^n$ is an ε-net of $(I_Z - Q)(B_Z)$. Let $x_i := Qz_i$ for all $1 \le i \le n$ and prove that $F := \mathrm{span}\{x_i\}_{i=1}^n$ is the wanted subspace. Indeed, given $z \in B_Z$, we write $g := (I_Z - Q)z$. Take g_i so $\|g - g_i\| \le \varepsilon$. Thus

$$\|g + x_i\| \le \|g - g_i\| + \|g_i + x_i\| \le \varepsilon + 1.$$

(b) It is straightforward that $P^2 = P$, so P is a projection. To evaluate $\|P\|$, take $z \in B_Z$ and write $g := (I_Z - Q)z$. By part (a) there is $h \in F$ satisfying $\|g - h\| \le 1 + \varepsilon$. Thus

$$\|L(g) - h\| = \|L(g - h)\| \le \|L\|\|g - h\| \le (1 + \varepsilon)^2$$

and

$$\|Q(z) + h\| \le \|Q(z) + g\| + \|g - h\| = \|z\| + \|g - h\| \le 2 + \varepsilon.$$

It follows that

$$\|Pz\| = \|Qz + Lg\| \le \|Qz + h\| + \|L(g) - h\| \le 3 + 4\varepsilon.$$

(c) For every $z \in N(P)$, we have

$$\|U(z)\| = \inf_{x \in X} \|z + x\| \ge \|I_Z - P\|^{-1} \inf_{x \in X} \|(I_Z - P)(z + x)\| = \|I_Z - P\|^{-1}\|z\|.$$

It follows that U is an isomorphism onto its image and $\|U^{-1}\| \le 1 + \|P\|$. Moreover, for every $g \in G$, we have that $g - L(g) \in N(P)$, so $U\big(N(P)\big) = M$ and $U^{-1}(g + X) = g - L(g)$. □

Theorem 3.7. *The Yang operator $T^{co} \in \mathscr{L}(X^{**}/X, Y^{**}/Y)$ is locally supportable in T for every $T \in \mathscr{L}(X,Y)$.*

Proof. Let M_0 be a finite-dimensional subspace of X^{**}/X and $0 < \varepsilon < \frac{1}{2}$. We denote by $R_0 \in \mathscr{L}(X^{**}, X^{**}/X)$ and $R_1 \in \mathscr{L}(Y^{**}, Y^{**}/Y)$ the respective quotient operators.

Let $M_1 := T^{co}(M_0)$, $Z_0 := R_0^{-1}(M_0)$ and $Z_1 := R_1^{-1}(M_1)$. We choose a finite-dimensional subspace G_0 of X^{**} such that $Z_0 = X \oplus G_0$, and denote $K_0 := \|R_0|_{G_0}^{-1}\|$. We decompose $T^{**}(G_0)$ as $T^{**}(G_0) = H_1 \oplus G_1$, where $H_1 \subset Y$ and $G_1 \cap Y = \{0\}$. Obviously $Z_1 = Y \oplus G_1$.

Take the projections $Q_0 \in \mathscr{L}(Z_0, X)$, $Q_1 \in \mathscr{L}(Z_1, Y)$ whose respective kernels are G_0 and G_1. By Lemma 3.6(a), there are finite-dimensional subspaces $F_0 \subset X$, $F_1 \subset Y$ such that for every $z_0 \in B_{Z_0}$ and $z_1 \in B_{Z_1}$ there are $e_0 \in F_0$ and $e_1 \in F_1$ satisfying $\|(I_{Z_0} - Q_0)(z_0) - e_0\| \leq \frac{3}{2}$ and $\|(I_{Z_1} - Q_1)(z_1) - e_1\| \leq \frac{3}{2}$.

By Theorem 3.4 there are $\frac{3}{2}$-injections $L_0\colon F_0 \oplus G_0 \longrightarrow X$ and $L_1\colon (H_1 + F_1) \oplus G_1 \longrightarrow Y$ satisfying $\|TL_0 - L_1 T^{**}|_{F_0 \oplus G_0}\| \leq \varepsilon K_0^{-1}$. Lemma 3.6(b) enables us to say that the operators $P_0 := Q_0 + L_0(I_{Z_0} - Q_0)$ and $P_1 := Q_1 + L_1(I_{Z_1} - Q_1)$ are projections with norm equal or smaller than 5, so Lemma 3.6(c) shows that the operators $U_0 := R_0|_{N(P_0)}$ and $U_1 := R_1|_{N(P_1)}$ are 6-injections. It only remains (cf. Lemma 3.6(c)) to prove that $\|T^{**}U_0^{-1} - U_1^{-1}T^{co}|_{M_0}\| \leq \varepsilon$. For, take $g \in G_0$. Note that $T^{**}U_0^{-1}(g + X) = T^{**}(g) - TL_0(g)$ and $U_1^{-1}T^{co}(g + X) = T^{**}(g) - L_1 T^{**}(g)$, so

$$\|(T^{**}U_0^{-1} - U_1^{-1}T^{co})(g + X)\| \leq \varepsilon K_0^{-1}\|g\| \leq \varepsilon \|g + X\|.$$

Thus, T^{co} is locally supportable in T^{**}, and so, in T. □

Theorem 3.8. *For every $T \in \mathscr{L}(X,Y)$, the Yang operator T^{co} is locally representable in T.*

Proof. Let E and F be a pair of finite-dimensional spaces, $A \in \mathscr{L}(E, X^{**}/X)$ and $B \in \mathscr{L}(Y^{**}/Y, F)$ a pair of operators, and $0 < \varepsilon < 1$.

We denote $R_X \in \mathscr{L}(X^{**}, X^{**}/X)$ and $R_Y \in \mathscr{L}(Y^{**}, Y^{**}/Y)$ the natural quotient operators. Let $Z := R_X^{-1}\big(A(E)\big)$, take a projection $Q \in \mathscr{L}(Z, X)$ and let $G := N(Q)$.

By Lemma 3.6(a) there is a finite-dimensional subspace F of X such that for every $z \in B_Z$ there is $e \in F$ so that $\|(I_Z - Q)(z) - e\| \leq \frac{3}{2}$. By Theorem 3.4 there is a $(1+\varepsilon)$-injection $L \in \mathscr{L}(F \oplus G, X)$ satisfying $L|_F := I_F$. Hence, Lemma 3.6(b) shows that $P := Q + L(I_Z - Q)$ is a projection with $\|P\| \leq 5$, and part (c) says that $U := R_X|_{N(P)}$ is a norm one isomorphism, its image is $R_X\big(N(P)\big) = A(E)$, $\|U^{-1}\| \leq 6$, and, moreover, $U^{-1}(g + X) = g - L(g)$ for all $g \in G$.

We define operators $A_1 := U^{-1}A$ and $B_1 := BR_Y$, and thus we get

$$B_1 T^{**} A_1 = BR_Y T^{**} U^{-1} A = BT^{co} R_X U^{-1} A = BT^{co} A.$$

Moreover,

$$\|A_1\| \cdot \|B_1\| \leq \|U^{-1}\| \cdot \|A\| \cdot \|B\| \leq 6\|A\| \cdot \|B\|,$$

and therefore T^{co} is locally 6-representable in T^{**} and so in T. ▫

A class $\mathscr{A}$ of operators is said to be *ultrapower-stable* if for every $T \in \mathscr{A}$ and every ultrafilter $\mathfrak{U}$, the operator $T_{\mathfrak{U}}$ belongs to $\mathscr{A}$ (for definitions and facts about ultrapowers of Banach spaces and operators, see [9]). Given a class $\mathscr{A}$ of operators and any pair of Banach spaces X and Y, we denote $\mathscr{A}(X,Y) := \mathscr{A} \cap \mathscr{L}(X,Y)$.

The following result is concerned with operator semigroups which are either injective and left-stable or surjective and right-stable. We recall [1, Definition 2.1] that an *operator semigroup* is a class of operators $\mathscr{S}$ satisfying the following three properties:

(1) $\mathscr{S}$ contains all bijective operators;
(2) if $S \in \mathscr{S}(X,Y)$ and $T \in \mathscr{S}(Y,Z)$ then $TS \in \mathscr{S}(X,Z)$;
(3) $S \in \mathscr{S}(U,V)$ and $T \in \mathscr{S}(X,Y)$ if and only if $S \oplus T \in \mathscr{S}(U \oplus X, V \oplus Y)$.

An operator semigroup $\mathscr{S}$ is said to be *left-stable* (respectively *right-stable*) if $S \in \mathscr{S}$ ($T \in \mathscr{S}$) whenever $TS \in \mathscr{S}$ [1, Definition 2.9]; $\mathscr{S}$ is said to be *injective* (respectively *surjective*) if it contains all upper semi-Fredholm operators (all lower semi-Fredholm operators) [1, Definition 2.13].

Proposition 3.9. *Let $\mathscr{S}$ be an ultrapower-stable semigroup of operators and $T \in \mathscr{S}(X,Y)$. Then T^{**} and T^{co} belong to $\mathscr{S}$ if we have*

(a) *$\mathscr{S}$ is injective and left-stable*; *or*
(b) *$\mathscr{S}$ is surjective and right-stable.*

Proof. (a) First we prove that $T^{**} \in \mathscr{S}$. By Theorem 3.5(a), there exists an ultrafilter $\mathfrak{U}$ and a pair of metric injections $U \in \mathscr{L}(X^{**}, X_{\mathfrak{U}})$ and $V \in \mathscr{L}(Y^{**}, Y_{\mathfrak{U}})$ such that $V \circ T^{**} = T_{\mathfrak{U}} \circ U$. Since $\mathscr{S}$ is an injective, ultrapower-stable semigroup, we have that $T_{\mathfrak{U}} \circ U \in \mathscr{S}$. So $V \circ T^{**} \in \mathscr{S}$, and by left-stability, we get $T^{**} \in \mathscr{S}$.

In order to prove that $T^{\mathrm{co}} \in \mathscr{S}$, consider the set J of tuples $j \equiv (E_j, \varepsilon_j)$ where E_j runs over all finite-dimensional subspaces of X^{**}/X and ε_j does the same over $(0,1)$. By Theorem 3.7, for each $j \in J$ there exist two $(6+\varepsilon_j)$-injections $U_j \in \mathscr{L}(E_j, X)$ and $V_j \in \mathscr{L}\big(T^{\mathrm{co}}(E_j), Y\big)$ such that $\|TU_j - V_j T^{\mathrm{co}}|_{E_j}\| \leq \varepsilon_j$. Let $\preceq$ be an order on J defined by $i \preceq j$ if $E_i \subset E_j$ and $\varepsilon_i \geq \varepsilon_j$. Take an ultrafilter $\mathfrak{U}$ on J refining the order filter. Now we define operators $U \in \mathscr{L}(X^{**}/X, X)$ and $V \in \mathscr{L}\big(\overline{T^{\mathrm{co}}(X^{**}/X)}, Y_{\mathfrak{U}}\big)$ by $U(x^{**} + X) := [x_j]$ where $x_j := U_j(x^{**} + X)$ if $x^{**} + X \in E_j$ and $x_j := 0$ otherwise, and $V\big(T^{**}(x^{**}) + Y\big) = [y_j]$, where $y_j := V_j(T^{**}(x^{**}) + Y)$ if $T^{**}(x^{**}) + Y \in T^{\mathrm{co}}(E_j)$ and $y_j := 0$ otherwise. Let us decompose $T^{\mathrm{co}} = J\widetilde{T}$, where the operator $\widetilde{T} \in \mathscr{L}\big(X^{**}/X, \overline{T^{\mathrm{co}}(X^{**}/X)}\big)$ maps $x^{**} + X$ onto $T^{\mathrm{co}}(x^{**} + X)$, and J is the natural embedding of $\overline{T^{\mathrm{co}}(X^{**}/X)}$ into Y^{**}/Y. Computations like those in Theorem 3.5 show that U and V are isomorphisms and that $V\widetilde{T} = TU$. The same arguments as in the proof for T^{**} lead to $\widetilde{T} \in \mathscr{S}$; the injectivity of $\mathscr{S}$ yields that $T^{\mathrm{co}} \in \mathscr{S}$.

(b) By Theorem 3.5(b), there exists an ultrafilter $\mathfrak{U}$ and metric surjections $P \in \mathscr{L}(X_{\mathfrak{U}}, X^{**})$ and $Q \in \mathscr{L}(Y_{\mathfrak{U}}, Y^{**})$ such that $T^{**} \circ P = Q \circ T_{\mathfrak{U}}$. Arguing like in part (a), surjectivity and right-stability show $T^{**} \in \mathscr{S}$.

In order to prove that $T^{\mathrm{co}} \in \mathscr{S}$, let us consider the natural quotient operators $P \in \mathscr{L}(X^{**}, X^{**}/X)$ and $Q \in \mathscr{L}(Y^{**}, Y^{**}/Y)$. As $T^{**} \in \mathscr{S}$ and $Q \circ T^{**} = T^{\mathrm{co}} \circ P$, we obtain that $T^{\mathrm{co}} \in \mathscr{S}$. □

Tacon ([16] and [17]) introduces the class Ψ_+ of supertauberian operators and proves that T^{**} is supertauberian whenever T is. Since Ψ_+ is a semigroup satisfying the hypothesis of statement (a) [6], Proposition 3.9 includes Tacon's result. More examples of ultrapower-stable semigroups can be found in [7].

The last result is concerned with ultrapower-stable operator ideals. We recall that an operator ideal $\mathscr{A}$ is said to be *regular* if $T \in \mathscr{A}(X, Y)$ whenever $JT \in \mathscr{A}(X, Y^{**})$, where J stands for the canonical embedding of Y into Y^{**}.

Proposition 3.10. *Let $\mathscr{A}$ be an ultrapower-stable ideal of operators and $T \in \mathscr{A}(X, Y)$. Then $T^{**} \in \mathscr{A}$. Moreover, if $\mathscr{A}$ is regular then $T^{\mathrm{co}} \in \mathscr{A}$.*

Proof. That $T^{**} \in \mathscr{A}$ is directly derived from Theorem 3.5(c) is clear. On the other hand, we have shown in Theorem 3.8 that T^{co} is locally representable in T, so [13, 6.6] implies $T^{\mathrm{co}} \in \mathscr{A}$. □

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