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GEOMETRIC STRUCTURE OF TUBES AND BANDS OF ZERO MEAN CURVATURE IN MINKOWSKI SPACE

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Abstract. Spacelike and timelike tubes and bands of zero mean curvature in Minkowski space are investigated in a neighborhood of finite or infinite singularities. We also study the correlation between the branching of surfaces and their exterior amounts, and questions of the smooth pasting of spacelike and timelike tubes and bands. We give an asymptotic representation of the surfaces in the neighborhood of the singular point.

1. Introduction

We investigate the structure of tubes and bands of zero mean curvature in Minkowski space in a neighborhood of their singularity. An important property of the case we study is the existence of isolated singularities of cone type [20], [11] and [18]. This property is specific for surfaces in Lorentz spaces.

Geometric and topological aspects of the structure of Lorentzian manifolds having singularities were investigated in [15] and [2].

We also consider another problem. That is, we study manifolds with singularities embedded in Minkowski space. Firstly, we are interested in questions connected with the exterior structure of manifolds in a neighborhood of their singular points. We also consider some questions connected with processes of transition from spacelike to timelike manifolds at their common singular point.

We give some results in the following directions: to describe the exterior structure of spacelike bands with infinite number of branches at the infinity of $\mathbf{R}_1^{n+1}$; to obtain an asymptotic decomposition of zero mean curvature tubes and bands in the neighborhood of singular points; to investigate possibilities of the smooth pasting of spacelike tubes and bands with timelike ones at the singular point.

It is possible that Shiffman [33], Nitsche [31], and Osserman and Shiffer [32] were the first to investigate tubes of zero mean curvature. The minimal surfaces of a tubular type of arbitrary codimension in $\mathbf{R}^{n+1}$ were defined in [22] and minimal bands were introduced in [24]. The idea of investigating bands was borrowed from the theory of relative strings (for example, see [3] and [7]), where tubes and bands of zero mean curvature in Minkowski space $\mathbf{R}_1^{n+1}$ (but with timelike and

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not Riemannian structure, induced by the scalar product of $\mathbf{R}_1^{n+1}$) are important objects of research.

From the geometric viewpoint, relativistic strings and membranes are surfaces of zero mean curvature in Minkowski space-time. The surfaces of tubular type correspond to some closed strings there. The open strings are interpreted as bands of a special form. An approach to the string theory from the viewpoint of their geometric structure is promising, since even the simplest extrinsic properties of a surface in the framework of that or other models can be translated into the language of physical phenomena. Thus, the estimate of the extension of a tube or a band along the time-axis corresponds to the estimate of the lifetime of the string; the projection $e_0^T(m)$ of the time vector e_0 onto the tangent plane $T(m)$ of the surface $\mathscr{M}$ at the point $m \in M$ corresponds to the local time on the string; branch points of surfaces correspond to the beginning of change in the type of a physical process, decay of particles, and so on [27].

The ideas of string theory lie on the basis of a Nilsen conjecture [30], which states that metrics of zero mean curvature in Minkowski space are only physically significant among all metrics which are solutions of the Einstein equation. The fact that these surfaces have isolated singular points in $\mathbf{R}_1^{n+1}$ ensures possibilities for modeling some special aspects of the 'big bang' [15] by the tubes and bands of zero mean curvature surfaces. By analogy with the 'big bang', the problem of pasting is an attempt to answer the question: what could exist before the 'big bang' of a universe?

Now in spite of the many papers devoted to relativistic strings and their generalizations, there is no mathematical theory of strings. We regard the construction of this theory as a superproblem and disregard important questions of strings quantification. We restrict ourselves to a more narrow set of questions: describing the geometric structure of the string, namely, investigating the structure of spacelike and timelike tubes and bands in Minkowski space.

Among the papers devoted to the structure of zero mean curvature tubes and bands in a neighborhood of a singular point, we distinguish the pioneering paper [20], where it is shown that the set of the tangent rays to any maximal surface in a neighborhood of an essentially singular point coincides with upper or lower sheets C^+ or C^- of the light cone (also, see [11]).

In our papers [18] and [21] this result was sharpened. We gave quantitative characteristics of this property in terms of interior and exterior girth functions of tubes.

We do not know any results relevant to asymptotic decompositions of zero mean curvature surfaces in a neighborhood of a singularity or to pasting problems.

For the structure of zero mean curvature surfaces in Minkowski space, see also [8], [13], [4], [5], [10], [12], and [25].

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2. Main results

Let $\mathbf{R}_1^{n+1}$ be a Minkowski space, that is, an $(n+1)$-dimensional real pseudo-Euclidean space with a metric of signature $(1, n)$. Let $x = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n$, $t \in \mathbf{R}^1$ and $\chi = (t, x) \in \mathbf{R}_1^{n+1}$. For an arbitrary pair of vectors $\chi' = (t', x')$ and $\chi'' = (t'', x'')$ of $\mathbf{R}_1^{n+1}$, we denote their scalar product by

$$\langle \chi', \chi'' \rangle = -t't'' + \sum_{i=1}^{n} x_i' x_i''. \tag{2.1}$$

The scalar square of a vector $\chi \in \mathbf{R}_1^{n+1}$ is

$$|\chi|^2 = -t^2 + \sum_{i=1}^{n} x_i^2.$$

A nonzero vector $\chi \in \mathbf{R}_1^{n+1}$ is called *spacelike, lightlike*, or *timelike* if $|\chi|^2 > 0$, $|\chi|^2 = 0$, or $|\chi|^2 < 0$, respectively. The totality $C = C(\chi_0)$ of the lightlike vectors with origin at a point $\chi_0 \in \mathbf{R}_1^{n+1}$ forms the light cone. We shall denote upper and lower sheets of the light cone by $C^+ = C^+(\chi_0)$ and $C^- = C^-(\chi_0)$.

Let M be a two-dimensional connected, orientable noncompact manifold of C^2 with a piecewise smooth boundary ∂M (possibly empty). Consider the surface $\mathscr{M} = (M, u)$ given by a C^2-immersion $\chi = u(m)\colon M \to \mathbf{R}_1^{n+1}$.

The surface $\mathscr{M} = (M, u)$ is said to be *spacelike* if its tangent vectors are spacelike. If the surface $\mathscr{M}$ is spacelike, then the scalar product (2.1) induces a Riemannian metric on $\mathscr{M}$, and the standard connection $\overline{\nabla}$ in $\mathbf{R}_1^{n+1}$ induces a Riemannian connection ∇ on $\mathscr{M}$. In addition, the Riemannian metric on M and the connection ∇ are coordinated [6, Addition A]. By Δ we will denote the Laplacian in this metric.

The surface $\mathscr{M} = (M, u)$ is said to be *timelike* if for each point $m \in M$ the tangent plane $T_{u(m)}$ contains both spacelike and timelike vectors.

Let $\{e_i\}_{i=0}^n$ be an orthonormal basis in $\mathbf{R}_1^{n+1}$ for which

$$\langle e_i, e_j \rangle = 0 \quad \text{for } i \neq j,\ |e_0|^2 = -1 \text{ and } |e_i|^2 = 1 \quad \text{for } i = 1, 2, \dots, n.$$

Therefore, $\chi = te_0 + \sum_{i=1}^n x_i e_i$.

We denote a hyperplane of constant time by

$$\Pi(\tau) = \{\chi \in \mathbf{R}_1^{n+1} : \langle \chi + \tau e_0, e_0 \rangle = 0\}.$$

A surface $\mathscr{M} = (M, u)$ in $\mathbf{R}_1^{n+1}$ is called a *band* with projection (α, β), $-\infty \le \alpha < \beta \le +\infty$, with the time axis $0t$ if it satisfies the following properties:

(a) for any $\tau_1, \tau_2 \in (\alpha, \beta)$ the set

$$M(\tau_1, \tau_2) = \{m \in M : \tau_1 < t(m) < \tau_2\}, \qquad t(m) = -\langle u(m), e_0 \rangle,$$

is precompact;

(b) for any $\tau \in (\alpha, \beta)$ the intersection $\Sigma(\tau) = u(M) \cap \Pi(\tau)$ is not empty;

(c) there exists $\tau \in (\alpha, \beta)$ such that at least one of the connected components of $u^{-1}(\Sigma(\tau))$ contains points of ∂M;

(d) any vector ν of the unit normal to $u(\partial M)$ on $u(M)$ satisfies $\langle \nu, e_0 \rangle = 0$;

(e) for any point $m \in \partial M$ at which the boundary ∂M does not have any tangent plane, the contingency $\operatorname{contg}_{u(m)} u(M)$ does not contain lightlike rays.

Sometimes it is necessary to use the following property:

(c$'$) each connected component of the set $u^{-1}(\Sigma(\tau))$ contains points of ∂M.

This condition is stronger than (c). A surface $\mathscr{M} = (M, u)$ in $\mathbf{R}_1^{n+1}$ is called a *strict band* if it satisfies (a), (b), (c$'$), (d) and (e).

The (finite or infinite) quantity $\beta - \alpha$ is said to be *the time existence* (*length*) of the band.

The surface $\mathscr{M} = (M, u)$ in $\mathbf{R}_1^{n+1}$ is said to be a *surface of tubular type* with a projection (α, β) if M is a manifold without a boundary and (M, u) has properties (a) and (b).

A tube or a band is called the *tube or band in large* if its projection is $(-\infty, +\infty)$.

Some examples of tubes and bands of zero mean curvature $\mathbf{R}_1^{n+1}$ can be found in [18], [19] and [27].

Let $\mathscr{M} = (M, u)$ be a surface and $C(\chi_0)$ be a light cone. If for some $m \in M$

$$l^2(m, \chi_0) = -\big(u_0(m) - t_0\big)^2 + \sum_{i=1}^{n} (u_i(m) - x_{0\,i})^2 < 0,$$

then the point $u(m)$ lies inside $C(\chi_0)$.

If for some $m \in M$ the magnitude $l^2(m, \chi_0) > 0$, then $u(m)$ lies outside $C(\chi_0)$.

Suppose that for some χ_0

$$\limsup_{t(m) \to \alpha} l^2(m, \chi_0) \le 0, \tag{2.2}$$

and that the set $M^+ = \{m \in M : l^2(m, \chi_0) > 0\}$ is not empty. For arbitrary $p, q > 0$, we consider a *counting function* $N_u(t) = N_u(t; p, q)$ of the (p, q)-connected components of M^+ which are defined by (3.34). Roughly speaking, the number $N_u(t)$ denotes the number of the (p, q)-components of $M^+ \cap \Sigma(t)$.

Next we denote a flow of time through $\Sigma(t)$ by

$$\mu(t) = \int_{\Sigma(t)} |\nabla t(m)|.$$

If $\mathscr{M}$ is a tube or band of zero mean curvature, then μ is independent of t [27].

2.3. Theorem. *Let $\mathscr{M}$ be a two-dimensional tube or band of zero mean curvature with the projection (α, ∞) and the condition* (2.2).

If the set M^+ is not empty, then for any p, $q > 0$ and arbitrary $\tau > \alpha$, it is true that

$$N_u(\tau) \exp\left\{\frac{1}{\mu N_u(\tau)} \int_\alpha^\tau N_u^2(t)\, dt\right\} \le 4\mu \max_{m \in \Sigma(\tau+1)} l^4(m, \chi_0).$$

Fix an arbitrary vector $e \in \mathbf{R}_1^{n+1}$ and consider the function $h(m) = \langle u(m), e\rangle$. Suppose that

$$\limsup_{t(m) \to \alpha} h(m) \le 0, \tag{2.4}$$

and that the set $M^+ = \{m \in M : h(m) > 0\}$ is not empty.

The following theorem strengthens a corresponding theorem in [27], which treats the case that M^+ has a finite number of connected components.

2.5. Theorem. *Let $\mathscr{M}$ be a band or tube of zero mean curvature with a projection (α, ∞) and with* (2.4).

For any p, $q > 0$ and an arbitrary $\tau > \alpha$, it is true that

$$N_u(\tau) \exp\left\{\frac{1}{\mu N_u(\tau)} \int_\alpha^\tau N_u^2(t)\, dt\right\} \le 4\mu \max_{m \in \Sigma(\tau+1)} h^2(m). \tag{2.6}$$

Both Theorem 2.3 and 2.5 are geometric corollaries of the more general Theorem 3.39 for arbitrary solutions of (3.18) on $\mathscr{M}$.

Let $G \subset \mathbf{R}^2$ be a domain in the plane and let $(0, 0) \in G$. We consider a solution $f \in C^2$ of the equation

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1 - f_x^2 - f_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1 - f_x^2 - f_y^2}}\right) = 0. \tag{2.7}$$

This equation describes the spacelike zero mean curvature surfaces in Minkowski space-time.

We assume that f is defined on G and has an isolated singularity at the origin $(0,0)$. Ecker [11] has shown that

$$|f(x,y)-f(0,0)| \sim \sqrt{x^2+y^2} \qquad \text{as } (x,y)\to(0,0).$$

In [18] and [27], we sharpened this asymptotic. That is, we proved that

$$\bar{\nu} = 6 \limsup_{x,y\to 0} \frac{\sqrt{x^2+y^2} - |f(x,y)-f(0,0)|}{(x^2+y^2)^{3/2}} < \infty, \tag{2.8}$$

and, moreover, $\bar{\nu} \geq \mu^{-2}$.

Let

$$\kappa^*(x,y) = \kappa^*(\varrho e^{i\psi}) = \frac{K(x,y)}{\sinh^4 \alpha(x,y)} = \frac{f_{xy}^2 - f_{xx}f_{yy}}{(f_x^2+f_y^2)^2}$$

be the curvature expression in the coordinates $x+iy = \varrho e^{i\psi}$.

2.9. Theorem. *Let $f(x,y)$ be a solution of (2.7) with a singularity at $(0,0)$. Then,*

(1) *there exist 2π-periodic real analytic functions $c_k(\psi)$ defined on $[0,2\pi]$ such that the following decomposition holds*:

$$f(\varrho e^{i\psi}) = \varrho + \sum_{k=1}^{\infty} c_k(\psi)\varrho^{2k+1}; \tag{2.10}$$

(2) *there exists a limit*

$$\lim_{x,y\to 0} \kappa^*(x,y) = \lim_{\varrho\to 0} \kappa^*(\varrho e^{i\psi}) = -6c_1(\psi) = \kappa^*(\psi); \tag{2.11}$$

(3) *the following equalities are true*:

$$6 \lim_{\varrho\to 0} \frac{\varrho - f(\varrho e^{i\psi})}{\varrho^3} = \kappa^*(\psi) \tag{2.12}$$

and

$$\int_0^{2\pi} \frac{d\psi}{\sqrt{\kappa^*(\psi)}} = \mu. \tag{2.13}$$

Further, we suppose that $f(x,y)$ is a solution of timelike zero mean curvature surfaces equation in the Minkowski space-time

$$\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{f_x^2+f_y^2-1}}\right) + \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{f_x^2+f_y^2-1}}\right) = 0. \tag{2.14}$$

We assume f is defined on a domain $G \subset \mathbf{R}^2$ and has an isolated singularity at $(0,0)$.

The following statement is similar to a known theorem for spacelike zero mean curvature surfaces [11], which says that the totality of the tangent rays in the singular point forms an upper or lower sheet of the light cone.

2.15. Theorem. *Let $\mathscr{M}$ be a two-dimensional timelike tube of zero mean curvature with a singularity at $\{0\} \in \mathbf{R}_1^3$. The tangent rays to $\mathscr{M}$ at this point are lightlike.*

Let $f_1(x,y)$ and $f_2(x,y)$ be solutions of (2.7) and (2.14), respectively, having isolated singularities at $(0,0)$ such that

$$f_1(0,0) = f_2(0,0) = 0, \qquad f_1(x,y) < 0, \qquad f_2(x,y) > 0.$$

We will say that solutions f_1 and f_2 are C^k*-pasted* if

$$\delta(\varrho e^{i\psi}) \in C^k, \quad \text{where } \delta(\varrho e^{i\psi}) = f_1(\varrho e^{i\psi}) + f_2(\varrho e^{i\psi}).$$

The following statement asserts the possibility of the smooth gluing of solutions.

2.16. Theorem. *For an arbitrary solution $f_1(x,y)$ of (2.7) with a singularity at $(0,0)$, there exists a solution $f_2(x,y)$ of (2.14) with a singularity at $(0,0)$ such that $f_1(x,y)$ and $f_2(x,y)$ are C^2-pasted.*

3. A structure at infinity

Let $\mathscr{M} = (M,u)$, $\dim M = 2$, be a spacelike tube or band of zero mean curvature in $\mathbf{R}_1^{n+1}$ with projection (α,β) onto the time axis $0t$. Below we will use the notation and terminology of [27, Section 3]. That is to say, we will need a concept of the ends $\xi_M(\alpha)$, $\xi_M(\beta)$ of the surface $\mathscr{M}$ which are determined by analogy with the prime ends of a planar domain (for example, see [34]).

Let $f(m)$, $f(m) \not\equiv 0$, be an arbitrary function of $C^0(\overline{M}) \cap C^2(M)$ such that

$$f|_{\partial M} = 0. \tag{3.17}$$

We suppose

$$f\Delta f \geq 0 \quad \text{everywhere on } M. \tag{3.18}$$

The differential inequality (3.18) is not traditional. We give some simple properties of functions f satisfying (3.17) and (3.18).

At first, solutions of (3.17) and (3.18) do not have a strict maximum in the domain.

3.19. Lemma. *Let f be a solution of (3.18) satisfying (3.17). Any connected component $\mathscr{O}$ of the set $\{m \in M : f(m) > 0\}$ does not have a compact closure $\overline{\mathscr{O}}$.*

Proof. We suppose that the closure $\bar{\mathscr{O}}$ is compact. By Gauss' formula, we have

$$\int_{\partial\mathscr{O}} f\langle\nabla f, \nu\rangle = \int_{\mathscr{O}} |\nabla f|^2 + \int_{\mathscr{O}} f\Delta f,$$

where ν is a unit outward normal to $\partial\mathscr{O}$.

Since $f|_{\partial\mathscr{O}} = 0$, the contour integral vanishes. Therefore, from (3.18) it follows that

$$\int_{\mathscr{O}} |\nabla f|^2 = 0.$$

Hence $f \equiv \text{const}$ on $\mathscr{O}$ contradicts the definition of $\mathscr{O}$.

These arguments are strict only if the boundary $\partial\mathscr{O}$ is rectifiable. In the general case, the function f is extended by zero outside $\bar{\mathscr{O}}$. Further, the function obtained can be approximated by C^2-smooth, compactly-supported functions on M.

We denote

$$\operatorname{osc}\{f, \Sigma(t)\} = \sup_{x,y\in\Sigma(t)} |f(x) - f(y)|.$$

3.20. Lemma. *If*

$$\liminf_{t\to\alpha} \operatorname{osc}\{f, \Sigma(t)\} = 0, \tag{3.21}$$

then

$$\lim_{m\to\xi_M(\alpha)} f(m) = 0, \tag{3.22}$$

and for an arbitrary connected component $\mathscr{O}$, there exists $t_0 \in (\alpha, \beta)$ such that

$$\mathscr{O} \cap \Sigma(t) \neq \emptyset \qquad \textit{for all } t > t_0. \tag{3.23}$$

Proof. The proof follows from the weak maximum-minimum principle for solutions of differential inequalities (3.18). The maximum principle is proved by Lemma 3.19. In order to prove the minimum principle, it is sufficient to note that both f and $-f$ satisfy (3.18).

We fix an arbitrary connected component $\mathscr{O}$ and suppose that the relation (3.21) holds. We denote by $\tau(\mathscr{O})$ the smallest among values $t_0 \geq \alpha$ for which (3.23) is true. For each fixed $t > \tau(\mathscr{O})$, let $\gamma_1(t), \gamma_2(t), \dots$ be all connected components of $\mathscr{O} \cap \Sigma(t)$ having properties: $\overline{\gamma_i(t)} \cap \partial M \neq \emptyset$, $i = 1, 2, \dots$.

We denote

$$\mu(t, \mathscr{O}) = \sup_i \int_{\gamma_i(t)} |\nabla t(m)|.$$

For an arbitrary $t > \tau(\mathscr{O})$, we put

$$J(t, \mathscr{O}) = \int_{\mathscr{O} \cap M(\alpha, t)} |\nabla f|^2,$$

where $M(t_1, t_2) = \{m \in M : t_1 < t(m) < t_2\}$.

The following lemma is basic in the present paragraph.

3.24. Lemma. *If a function $f(m)$ satisfies (3.17), (3.18) and (3.21), then for almost every $t > \tau(\mathscr{O})$ we have*

$$J(t, \mathscr{O}) \leq \frac{\mu(t, \mathscr{O})}{2\pi} J'(t, \mathscr{O}). \tag{3.25}$$

Proof. Using the Stokes formula and (3.17), we can write

$$\int_{\mathscr{O} \cap \Sigma(t)} f\langle \nabla f, \nu\rangle = \int_{\partial(\mathscr{O} \cap M(\alpha, t))} f\langle \nabla f, \nu\rangle = J(t, \mathscr{O}) + \int_{\mathscr{O} \cap M(\alpha, t)} f\, \Delta f,$$

where ν is an inward normal to the boundary of $\mathscr{O} \cap M(\alpha, t)$ on M.

From (3.18) we obtain

$$J(t, \mathscr{O}) \leq \int_{\mathscr{O} \cap \Sigma(t)} f\, \langle \nabla f, \nu\rangle. \tag{3.26}$$

As

$$\nu = \frac{\nabla t}{|\nabla t|}(m) \qquad \text{for all } m \in \Sigma(t),$$

the condition (3.17) and Cauchy's inequality give

$$\begin{aligned} \int_{\mathscr{O} \cap \Sigma(t)} f\, \langle \nabla f, \nu\rangle &= \int_{\mathscr{O} \cap \Sigma(t)} f\, \langle \nabla f, \nabla t\rangle \frac{1}{|\nabla t|} = \sum_i \int_{\gamma_i(t)} f\, \langle \nabla f, \nabla t\rangle \frac{1}{|\nabla t|} \\ &\leq \sum_i \Big(\int_{\gamma_i(t)} f^2\, |\nabla t| \Big)^{1/2} \Big(\int_{\gamma_i(t)} \Big\langle \nabla f, \frac{\nabla t}{|\nabla t|} \Big\rangle^2 \frac{1}{|\nabla t|} \Big)^{1/2}. \end{aligned} \tag{3.27}$$

The following arguments are close to arguments from [27, Lemma 5.1]. Let $\gamma = \gamma_i(t)$ be an arbitrary connected component of $\mathscr{O} \cap \Sigma(t)$ and $m(s)$: $[0, \text{length}\,(\gamma)] \to \gamma$ be its natural parameterization. We put

$$v(s) = \int_0^s \big|\nabla t\big(m(s)\big)\big|\, ds, \qquad \tilde{v} = v\big(\text{length}\,(\gamma)\big).$$

Since the arc $\gamma = \gamma_i(t)$ is open and $\gamma_i \cap \partial M \neq \emptyset$, we get from (3.17)

$$f|_{s=0} = f|_{s=\text{length}\,(\gamma)} = 0.$$

By Wirtinger's inequality,

$$\int_\gamma f^2|\nabla t| = \int_0^{\tilde v} f^2\,dv(s) \le \left(\frac{\tilde v}{\pi}\right)^2 \int_0^{\tilde v} \left(\frac{df}{dv}\right)^2 dv = \left(\frac{\tilde v}{\pi}\right)^2 \int_\gamma \left(\frac{df}{ds}\right)^2 \frac{ds}{|\nabla t|}.$$

Now we have

$$\tilde v = \int_\gamma |\nabla t| \le \sup_i \int_{\gamma_i} |\nabla t| \le \mu(t, \mathscr{O}),$$

and, therefore,

$$\int_\gamma f^2|\nabla t| \le \frac{\mu^2(t,\mathscr{O})}{\pi^2} \int_\gamma \left(\frac{df}{ds}\right)^2 \frac{1}{|\nabla t(m)|}.$$

This relation is true for any open arc $\gamma = \gamma_i(t)$ such that $\gamma_i \cap \partial M \neq \emptyset$. From (3.26) and (3.27), we find that

$$\begin{aligned}
J(t,\mathscr{O}) &\le \frac{\mu(t,\mathscr{O})}{\pi} \sum_i \left(\int_{\gamma_i(t)} \left(\frac{df}{ds}\right)^2 \frac{1}{|\nabla f|}\right)^{1/2} \left(\int_{\gamma_i(t)} \left\langle \nabla f, \frac{\nabla t}{|\nabla t|}\right\rangle^2 \frac{1}{|\nabla t|}\right)^{1/2} \\
&\le \frac{\mu(t,\mathscr{O})}{2\pi} \sum_i \int_{\gamma_i(t)} \left(\left(\frac{df}{ds}\right)^2 + \left\langle \nabla f, \frac{\nabla t}{|\nabla t|}\right\rangle^2\right) \frac{1}{|\nabla t|} \\
&\le \frac{\mu(t,\mathscr{O})}{2\pi} \int_{\mathscr{O}\cap\Sigma(t)} \left(\left(\frac{df}{ds}\right)^2 + \left\langle \nabla f, \frac{\nabla t}{|\nabla t|}\right\rangle^2\right) \frac{1}{|\nabla t|}.
\end{aligned}$$

At each point $m \in \mathscr{O} \cap \Sigma(t)$, we have

$$\left(\frac{df}{ds}\right)^2 + \left\langle \nabla f, \frac{\nabla t}{|\nabla t|}\right\rangle^2 = |\nabla f|^2.$$

Therefore, from the previous inequality we obtain

$$J(t,\mathscr{O}) \le \frac{\mu(t,\mathscr{O})}{2\pi} \int_{\mathscr{O}\cap\Sigma(t)} |\nabla f|^2 \frac{1}{|\nabla t|}. \tag{3.28}$$

Now we use the following co-area formula for integration over level sets of $t = t(m)$:

$$J(t,\mathscr{O}) = \int_{\tau(\mathscr{O})}^t d\tau \int_{\mathscr{O}\cap\Sigma(\tau)} |\nabla f|^2 \frac{1}{|\nabla t|}.$$

Hence, for almost every $t \in (\tau(\mathscr{O}), \infty)$ we have

$$J'(t,\mathscr{O}) = \int_{\mathscr{O}\cap\Sigma(t)} |\nabla f|^2 \frac{1}{|\nabla t|}.$$

Combining this relation with (3.28), we get (3.25).

3.29. Definition. Let $p, q > 0$ be an arbitrary pair of numbers. We will say that a *connected component* $\mathscr{O}$ *of* f *has the type* (p,q) (*or it is a* (p,q)*-component*) if for any $t > \tau(\mathscr{O}) + p$,

$$\max_{m \in \Sigma(t) \cap \mathscr{O}} |f(m)| \geq q.$$

3.30. Lemma. *If a domain* $\mathscr{O}$ *has the type* (p,q), *then for any* $t > \tau(\mathscr{O}) + p'$ *with* $p' > p$, *it is true that*

$$\frac{(p'-p)q^2}{\mu} \exp\left\{\int_{\tau(\mathscr{O})+p'}^{t} \frac{ds}{\mu(s)}\right\} \leq J(t, \mathscr{O}), \tag{3.31}$$

where

$$\mu = \int_{\Sigma(t)} |\nabla t(m)|$$

does not depend on t *and* $\mu(s) = \mu(s, \mathscr{O})$.

Proof. From (3.25) for any $t > \tau(\mathscr{O}) + p'$, we can write

$$\int_{\tau+p'}^{t} \frac{ds}{\mu(s)} \leq \log \frac{J(t, \mathscr{O})}{J(\tau + p', \mathscr{O})}$$

with $\tau = \tau(\mathscr{O})$ and $\mu(s) = \mu(s, \mathscr{O})$. Therefore,

$$J(\tau + p', \mathscr{O}) \exp\left\{\int_{\tau+p'}^{t} \frac{ds}{\mu(s)}\right\} \leq J(t, \mathscr{O}). \tag{3.32}$$

As the connected component $\mathscr{O}$ has the type (p,q), then for any $t > \tau(\mathscr{O}) + p$, we have

$$\begin{aligned} q^2 &\leq \max_{m \in \Sigma(t) \cap \mathscr{O}} |f(m)|^2 \leq \left(\int_{\Sigma(t) \cap \mathscr{O}} |\nabla f(m)|\right)^2 \\ &\leq \left(\int_{\Sigma(t) \cap \mathscr{O}} |\nabla f(m)|^2 \frac{1}{|\nabla t(m)|}\right)\left(\int_{\Sigma(t) \cap \mathscr{O}} |\nabla t(m)|\right). \end{aligned}$$

By the property (d) of the band, we have everywhere on ∂M that

$$\langle \nabla t, \nu \rangle = -\langle e_0, \nu \rangle = 0.$$

Fix numbers $t_1 < t_2$ so that $\alpha < t_1 < t_2 < \infty$. We have

$$\begin{aligned} \int_{\partial M(t_1, t_2)} \langle \nabla t, \nu \rangle &= \int_{\Sigma(t_2)} \langle \nabla t, \nu \rangle - \int_{\Sigma(t_1)} \langle \nabla t, \nu \rangle \\ &= \int_{\Sigma(t_2)} |\nabla t| - \int_{\Sigma(t_1)} |\nabla t| = \int_{M(t_1, t_2)} \Delta t = 0. \end{aligned}$$

Consequently, the integral

$$\mu = \int_{\Sigma(t)} |\nabla t|$$

is independent of t. Now we obtain

$$\begin{aligned}\frac{(p'-p)q^2}{\mu} &\le q^2 \int_{\tau+p}^{\tau+p'} \frac{ds}{\mu(s)} = q^2 \int_{\tau+p}^{\tau+p'} ds \Big/ \int_{\Sigma(s)} |\nabla t(m)| \\ &\le q^2 \int_{\tau+p}^{\tau+p'} ds \Big/ \int_{\Sigma(s)\cap\mathscr{O}} |\nabla t(m)| \\ &\le \int_{\tau+p}^{\tau+p'} ds \int_{\Sigma(s)\cap\mathscr{O}} |\nabla f(m)|^2 \frac{1}{|\nabla t(m)|} \le J(\tau+p', \mathscr{O}).\end{aligned}$$

Taking into account (3.32), we arrive at (3.31).

Now, we let $\{m \in M : f(m) > 0\}$ have either a finite or an infinite number of connected components $\mathscr{O}_1, \mathscr{O}_2, \dots$ of the type (p,q).

Clearly, for any given finite number of connected components $\mathscr{O}_1, \mathscr{O}_2, \dots, \mathscr{O}_N$ of $\{m \in M : f(m) > 0\}$, there are $p, q > 0$ such that all $\mathscr{O}_i$ are (p,q)-components.

We set

$$\tau_i = \tau(\mathscr{O}_i), \qquad \mu_i(t) = \mu(t, \mathscr{O}_i), \qquad i = 1, 2, \dots,$$

and define functions $\mu_i^*(t)$: $(\alpha, \infty) \to \mathbf{R}$, $i = 1, 2, \dots$, in the following way:

$$\mu_i^*(t) = \mu_i(t) \text{ for } t > \tau_i + p + 1 \qquad \text{and} \qquad \mu_i^*(t) = \infty \text{ for } t \in (\alpha, \tau_i + p + 1].$$

In Lemma 3.30, we choose $p' = p + 1$ and fix $t > \alpha$. Since (3.31) for any $i = 1, 2, \dots$ such that $\tau_i + p + 1 < t$, we can write

$$\frac{q^2}{\mu} \exp\Big\{\int_{\tau_i+p+1}^{t} \frac{ds}{\mu_i(s)}\Big\} \le \int_{\mathscr{O}_i \cap M(\alpha,t)} |\nabla f|^2,$$

and

$$\frac{q^2}{\mu} \sum_{i=1}^{N} \exp\Big\{\int_{\alpha}^{t} \frac{ds}{\mu_i^*(s)}\Big\} \le \sum_{i=1}^{N} \int_{\mathscr{O}_i \cap M(\alpha,t)} |\nabla f|^2 \le \int_{M(\alpha,t)} |\nabla f|^2. \tag{3.33}$$

3.34. Definition. The counting function $N_f(t; p, q)$ is equal to the number of all (p,q)-domains $\mathscr{O}_i$ for which $\tau(\mathscr{O}_i) + p + 1 < t$, and it vanishes for $t \le \inf_i \tau(\mathscr{O}_i) + p + 1$.

We use an inequality between arithmetic and geometric means. We have

$$\begin{aligned}\exp\left\{\frac{1}{N_f(t)}\sum_{i=1}^{N_f(t)}\int_\alpha^t\frac{ds}{\mu_i^*(s)}\right\} &= \left(\prod_{i=1}^{N_f(t)}\exp\left\{\int_\alpha^t\frac{ds}{\mu_i^*(s)}\right\}\right)^{1/N_f(t)}\\ &\le \frac{1}{N_f(t)}\sum_{i=1}^{N_f(t)}\exp\left\{\int_\alpha^t\frac{ds}{\mu_i^*(t)}\right\},\end{aligned}$$

where $N_f(t) = N_f(t; p, q)$ is the counting function.

Hence from (3.33) for any $t > \alpha$, we get

$$\frac{q^2}{\mu}N_f(t)\exp\left\{\frac{1}{N_f(t)}\sum_{i=1}^{N_f(t)}\int_\alpha^t\frac{ds}{\mu_i^*(s)}\right\} \le \int_{M(\alpha,t)}|\nabla f|^2. \tag{3.35}$$

Further, we note that

$$\begin{aligned}\sum_{i=1}^{N_f(t)}\int_\alpha^t\frac{ds}{\mu_i^*(s)} &= \int_\alpha^t\sum_{i=1}^{N_f(t)}\frac{ds}{\mu_i^*(s)} = \int_\alpha^t\sum_{i=1}^{N_f(s)}\frac{ds}{\mu_i^*(s)}\\ &\ge \int_\alpha^t N_f(s)\left(\prod_{i=1}^{N_f(s)}\frac{1}{\mu_i^*(s)}\right)^{1/N_f(s)}ds\\ &= \int_\alpha^t N_f(s)\,ds \Bigg/ \left(\prod_{i=1}^{N_f(s)}\mu_i^*(s)\right)^{1/N_f(s)}\\ &\ge \int_\alpha^t N_f^2(s)\,ds \Bigg/ \sum_{i=1}^{N_f(s)}\mu_i^*(s) \ge \int_\alpha^t N_f^2(s)\,ds \Bigg/ \sum_{i=1}^{N_f(s)}\int_{\Sigma(s)\cap\mathcal{O}_i}|\nabla t|\\ &\ge \int_\alpha^t N_f^2(s)\,ds \Bigg/ \int_{\Sigma(s)}|\nabla t| \ge \frac{1}{\mu}\int_\alpha^t N_f^2(s)\,ds,\end{aligned}$$

and, consequently,

$$\sum_{i=1}^{N_f(t)}\int_\alpha^t\frac{ds}{\mu_i^*(s)} \ge \frac{1}{\mu}\int_\alpha^t N_f^2(s)\,ds. \tag{3.36}$$

Combining (3.35) and (3.36), we obtain the following estimate

$$\frac{q^2}{\mu}N_f(t)\exp\left\{\frac{1}{\mu N_f(t)}\int_\alpha^t N_f^2(s)\,ds\right\} \le \int_{M(\alpha,t)}|\nabla f|^2, \tag{3.37}$$

where $N_f(t) = N_f(t; p, q)$.

Therefore, we obtain the following assertion which is important for applications.

3.38. Lemma. *Let $f(m)$ be a C^2-solution of* (3.18) *on $\mathscr{M}$ satisfying* (3.17) *and* (3.21). *Let $p, q > 0$ and let $\mathscr{O}_1, \mathscr{O}_2, \dots$ be (p,q)-components of $\{m \in M : f(m) > 0\}$. Then for any $t > \alpha$, the counting function $N_f(t) = N_f(t; p, q) < \infty$ is nondecreasing, and* (3.37) *is true.*

For the *proof* it is sufficient to show that the counting function is locally finite. By Lemmas 3.19 and 3.20 for each connected component $\mathscr{O}_i$, and for any $t > \tau(\mathscr{O}_i)$, it is true that $\mathscr{O}_i \cap \Sigma(t) \neq \emptyset$. Thus the function $N_f(t)$ is nondecreasing as $t \to \infty$. The relation (3.37) implies locally the boundedness of $N_f(t)$.

3.39. Theorem. *Let $f(m)$ be a C^2-solution of* (3.18) *on $\mathscr{M}$ satisfying* (3.17) *and* (3.21). *Let $p, q > 0$ and let $\mathscr{O}_1, \mathscr{O}_2, \dots$ be (p,q)-components of $\{m \in M : f(m) > 0\}$. Then for any $\tau > \alpha$, it is true that*

$$N_f(\tau) \exp\left\{ \frac{1}{\mu N_f(\tau)} \int_\alpha^\tau N_f^2(t)\, dt \right\} \leq \frac{\mu^2}{q^2} \max_{m \in \Sigma(\tau+1)} f^2(m), \tag{3.40}$$

where $N_f(t) = N_f(t; p, q)$.

Proof. Let $\phi(m) = \xi \circ t(m)$, and let $\xi(t) = 1$ for $\alpha < t < \tau$ and $\xi(t) = \tau + 1 - t$ for $\tau \leq t \leq \tau + 1$. By (3.17), (3.18) and (3.21), we can write

$$\begin{aligned}
\int_{M(\alpha,\tau)} |\nabla f|^2 &\leq \int_{M(\alpha,\tau+1)} \phi^2 |\nabla f|^2 \\
&= \int_{\partial M(\alpha,\tau+1)} f\phi^2 \langle \nabla f, \nu \rangle - 2\int_{M(\alpha,\tau+1)} f\phi \langle \nabla f, \nabla \phi \rangle - \int_{M(\alpha,\tau+1)} \phi^2 f \Delta f \\
&\leq -2 \int_{M(\alpha,\tau+1)} f\phi \langle \nabla f, \nabla \phi \rangle \\
&\leq 2 \left(\int_{M(\alpha,\tau+1)} f^2 |\nabla \phi|^2 \right)^{1/2} \left(\int_{M(\alpha,\tau+1)} \phi^2 |\nabla f|^2 \right)^{1/2}.
\end{aligned}$$

Thus, we find

$$\int_{M(\alpha,\tau+1)} \phi^2 |\nabla f|^2 \leq 4 \int_{M(\alpha,\tau+1)} f^2 |\nabla \phi|^2$$

and

$$\begin{aligned}
\int_{M(\alpha,\tau)} |\nabla f|^2 &\leq 4 \max_{m \in \Sigma(\tau+1)} f^2(m) \int_{M(\tau,\tau+1)} |\nabla t|^2 \\
&= 4 \max_{m \in \Sigma(\tau+1)} f^2(m) \int_\tau^{\tau+1} dt \int_{\Sigma(t)} |\nabla t| \\
&\leq 4\mu \max_{m \in \Sigma(\tau+1)} f^2(m).
\end{aligned}$$

Using (3.37), we arrive at (3.40).

Proof of Theorem 2.3. At first we recall that the function

$$l^2(m) = |u(m) - \chi_0|^2$$

satisfies the inequality $\Delta l > 0$.

Let $k \in \mathbf{R}_1^n$ be a fixed vector, and let $\mathscr{M} \subset \mathbf{R}_1^{n+1}$ be a two-dimensional surface. For an arbitrary point $m \in M$, we denote by $T = T(m)$ the tangent space to $\mathscr{M}$ at this point and the projection of k onto $T(m)$ by $k^T = k^T(m)$. It is not difficult to see that

$$\Delta l^2(m) = -2|e_0^T|^2 + 2\sum_{i=1}^{n} |e_i^T|^2 = 4.$$

If $\{m \in M : f(m) > 0\} = M^+$ is not empty, then $l^2 \Delta l^2 \geq 0$ on M^+, and (2.2) implies (3.40) for any $p, q > 0$.

Proof of Theorem 2.5. Here $h(m)$ satisfies $\Delta h = 0$ on M and $h\Delta h \geq 0$ on M^+. By Theorem 3.39, property (2.4) implies (3.21) and it follows that (3.40) and (2.6) also hold.

4. Neighborhood of isolated singularity

Let $\mathscr{M} = (M, u)$ be a tube with a projection (α, β) defined by an immersion $u \colon M \to \mathbf{R}_1^3$. We say that the surface $\mathscr{M}$ has a singularity at $\chi_0 \in \mathbf{R}_1^3$ if $\Sigma(t) \to \chi_0$ as $t \to \alpha + 0$.

4.41. Lemma. *Let M be a two-dimensional, doubly-connected, spacelike tube of zero mean curvature in $\mathbf{R}_1^3$ with a projection $(0, \beta)$. Then $\mathscr{M}$ can be defined by an immersion $w = (x_1, x_2, t) \colon K(1, R) \to \mathbf{R}_1^3$ of an annulus*

$$K(1,R) = \{\zeta \in \mathbf{C} : 1 < |\zeta| < R\}, \qquad \zeta = \xi + i\eta,$$

such that

$$\begin{aligned} x_1 &= \frac{\mu}{4\pi} \operatorname{Re} \int_{\zeta_0}^{\zeta} \frac{1}{iz}\left(\frac{1}{g(z)} - g(z)\right) dz, \\ x_2 &= \frac{\mu}{4\pi} \operatorname{Re} \int_{\zeta_0}^{\zeta} \frac{1}{z}\left(\frac{1}{g(z)} + g(z)\right) dz, \\ t &= \frac{\mu}{2\pi} \log|\zeta|, \end{aligned} \tag{4.42}$$

where $g(z)$ is a holomorphic function on $K(1, R)$ for which

$$\operatorname{Re} \oint \frac{i}{z}\left(\frac{1}{g} - g\right) dz = \operatorname{Re} \oint \frac{1}{z}\left(\frac{1}{g} + g\right) dz = 0. \tag{4.43}$$

Here $R = e^{2\pi\beta/\mu}$ *and*

$$\mu = \int_{\Sigma(t)} |\nabla t|.$$

Proof. Because the function $t(m)$ is harmonic with respect to the metric of $\mathscr{M}$ by the Stokes formula, we conclude that the integral

$$\int_{\Sigma(t)} |\nabla t|$$

does not depend on t, that is, $\mu \equiv \text{const}$, and the conjugate form $*dt$ has a period

$$\int_{\Sigma(t)} *dt = \mu.$$

There exists a multifunction $h(m)$ such that $dh = *dt$, and the mapping

$$\zeta(m) = \exp \frac{2\pi}{\mu}(t + i\ h)$$

establishes a one-to-one holomorphic correspondence between $\mathscr{M}$ and $K(1, R)$ (see [27]). By $m = m(\zeta)$ we denote the inverse mapping to $\zeta(m)$. Clearly,

$$t \circ m = \frac{\mu}{2\pi} \log |\zeta|.$$

Using the arguments of [37, Chapter 3, Section 3], we obtain a Weierstrass representation for spacelike surfaces of zero mean curvature in $\mathbf{R}^3_1$

$$\begin{aligned} x_1 &= \frac{1}{2} \operatorname{Re} \int_{\zeta_0}^{\zeta} f(z)\big(1 - g^2(z)\big)\, dz, \\ x_2 &= \frac{1}{2} \operatorname{Re} i \int_{\zeta_0}^{\zeta} f(z)\big(1 + g^2(z)\big)\, dz, \\ t &= \operatorname{Re} \int_{\zeta_0}^{\zeta} if(z)\, g(z)\, dz, \end{aligned} \tag{4.44}$$

where $f(z)$ and $g(z)$ are holomorphic functions on $K(1, R)$ satisfying the conditions

$$\operatorname{Re} \oint f(1 - g^2)\, dz = \operatorname{Re} \oint if(1 + g^2)\, dz = \operatorname{Re} \oint ifg\, dz = 0.$$

These conditions provide a tubular type of $\mathscr{M}$.

On the other hand,

$$\operatorname{Re} \int_{\zeta_0}^{\zeta} if(z)g(z)\, dz = \frac{\mu}{2\pi} \log |\zeta|;$$

therefore, we can put

$$f(z) = \frac{\mu}{2\pi i z g(z)}.$$

Substituting the expression in (4.44), we obtain what is needed.

A similar result was obtained for minimal tubes in [35] and [36].

4.45. Lemma. *Let $\mathscr{M} \subset \mathbf{R}^3_1$ be a spacelike tube of zero mean curvature having a singularity at the origin. Then it can be defined by an immersion* (4.42) *of the annulus $K(1, e^{2\pi\beta/\mu})$ with some holomorphic function $g(z)$ such that*

$$|g(z)| < 1, \quad z \in K(1, e^{2\pi\beta/\mu}) \quad \text{and} \quad |g(e^{i\phi})| \equiv 1.$$

Proof. We note that $\mathscr{M}$ is spacelike if and only if $|g(z)| \neq 1$, $z \in K(1, R)$. Below we shall suppose that $|g(z)| < 1$ because the substitution $g(z) \to 1/g(z)$ reflects the surface $\mathscr{M}$ with respect to the plane $x = 0$. On the other hand, it is known [18] that $\mathscr{M}$ is conelike in a neighborhood of the singularity, and also the totality of the tangent rays to $\mathscr{M}$ at this point forms a light cone. This means that $|g(z)| \equiv 1$ on the interior circle of $K(1, R)$.

Now it is sufficient to verify (4.43). In fact, if $g(e^{i\varphi}) = e^{i\theta(\varphi)}$, then

$$\begin{aligned} \operatorname{Re} \frac{1}{i} \oint \frac{1}{z}\left(\frac{1}{g} - g\right) dz &= \operatorname{Re} \frac{1}{i} \int_0^{2\pi} e^{-i\varphi}(e^{-i\theta(\varphi)} - e^{i\theta(\varphi)}) i e^{i\varphi}\, d\varphi \\ &= -2 \operatorname{Re} i \int_0^{2\pi} \sin\theta(\varphi)\, d\varphi = 0. \end{aligned}$$

The second condition of (4.43) is verified similarly.

Let $x_1(m)$ and $x_2(m)$ be the coordinate functions of an immersion $w(\zeta)\colon K(1, R) \to \mathbf{R}^3_1$ of a spacelike zero mean curvature tube with a projection $(0, \beta)$. Both functions are harmonic with respect to the metric of the surface $\mathscr{M}$ [21, Note 14]. Therefore, the Stokes formula implies that quantities

$$\mu_1 = \int_{\Sigma(t)} \langle \nabla x_1, \nabla t\rangle \frac{1}{|\nabla t|}, \qquad \mu_2 = \int_{\Sigma(t)} \langle \nabla x_2, \nabla t\rangle \frac{1}{|\nabla t|}$$

do not depend on t. According to this, we define the vector $Q = \mu e_0 + \mu_1 e_1 + \mu_2 e_2$, which will be called the *flow vector* of the tube $\mathscr{M}$ [36].

We let $K(\zeta)$ denote the Gaussian curvature of $\mathscr{M} = \big(K(1, R), w\big)$ at the point $w(\zeta)$.

4.46. Lemma. *The Gaussian curvature $K(s, t)$ of a conformal metric*

$$dl^2 = \lambda(s, t)(ds^2 \pm dt^2)$$

is expressed by the following formula:

$$K(t, s) = -\frac{1}{2\lambda}\left[\frac{\partial}{\partial s}\left(\frac{\lambda_s}{\lambda}\right) \pm \frac{\partial}{\partial t}\left(\frac{\lambda_t}{\lambda}\right)\right].$$

Proof. We put $E_1 = \partial/\partial s$ and $E_2 = \partial/\partial t$. Then

$$|E_1|^2 = \lambda, \quad |E_2|^2 = \pm\lambda \quad \text{and} \quad \langle E_1, E_2\rangle = 0,$$

$$\nabla_{E_1} E_2 - \nabla_{E_2} E_1 = \frac{\partial^2}{\partial s \partial t} - \frac{\partial^2}{\partial t \partial s} = 0.$$

It is known [6, Addition A] that

$$K(s,t) = \langle R(e_1, e_2)e_2, e_1\rangle = \pm\lambda^{-2}\langle R(E_1, E_2)E_2, E_1\rangle,$$

where $e_i = E_i/\sqrt{\lambda}$ and $R(\,\cdot\,,\cdot\,)\cdot$ is the curvature tensor of the given metric. If the connection of the metric is denoted by ∇, then

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Using the well-known properties of connections, we obtain

$$\langle \nabla_{E_2} E_2, E_2\rangle = \tfrac{1}{2}\nabla_{E_2}|E_2|^2 = \pm\tfrac{1}{2}\lambda_t,$$
$$\langle \nabla_{E_2} E_2, E_1\rangle = -\langle E_2, \nabla_{E_2} E_1\rangle = \langle E_2, \nabla_{E_1} E_2\rangle \tfrac{1}{2}\nabla_{E_1}|E_2|^2 = \mp\tfrac{1}{2}\lambda_s.$$

Therefore,

$$\nabla_{E_2} E_2 = \mp\frac{\lambda_s}{2\lambda}E_1 + \frac{\lambda_t}{2\lambda}E_2.$$

In the same way we find

$$\nabla_{E_1} E_2 = \frac{\lambda_t}{2\lambda}E_1 + \frac{\lambda_s}{2\lambda}E_2.$$

Now we get

$$\begin{aligned}
\langle R(E_1, E_2)E_2, E_1\rangle &= \langle \nabla_{E_1}\nabla_{E_2} E_2, E_1\rangle - \langle \nabla_{E_2}\nabla_{E_1} E_2, E_1\rangle \\
&= \left\langle \nabla_{E_1}\left(\mp\frac{\lambda_s}{2\lambda}E_1 + \frac{\lambda_t}{2\lambda}E_2\right), E_1\right\rangle \\
&\quad - \left\langle \nabla_{E_2}\left(\frac{\lambda_t}{2\lambda}E_1 + \frac{\lambda_s}{2\lambda}E_2\right), E_1\right\rangle \\
&= \mp\left(\frac{\lambda_s}{2\lambda}\right)'_t \lambda \mp \left(\frac{\lambda_s}{2\lambda}\right)\langle \nabla_{E_1} E_1, E_1\rangle + \left(\frac{\lambda_t}{2\lambda}\right) \\
&\quad - \left(\frac{\lambda_t}{2\lambda}\right)'_t \lambda - \left(\frac{\lambda_t}{2\lambda}\right)\langle \nabla_{E_2} E_1, E_1\rangle - \left(\frac{\lambda_s}{2\lambda}\right)\langle \nabla_{E_2} E_2, E_1\rangle \\
&= -\frac{-\lambda}{2}\left[\pm\left(\frac{\lambda_s}{2\lambda}\right)'_s + \left(\frac{\lambda_t}{2\lambda}\right)'_t\right],
\end{aligned}$$

because $\langle \nabla_{E_1} E_1, E_1\rangle = \frac{1}{2}\lambda_s$. Therefore,

$$K(t,s) = -\frac{1}{2\lambda}\left[\frac{\partial}{\partial s}\left(\frac{\lambda_s}{\lambda}\right) \pm \frac{\partial}{\partial t}\left(\frac{\lambda_t}{\lambda}\right)\right],$$

and the lemma is proved.

Let $\zeta = \xi + i\eta$. Then the Laplacian in coordinates (ξ, η) is denoted by the term $4\partial^2/\partial\zeta\partial\bar{\zeta}$. From Lemma 4.46 it is not difficult to calculate

$$K(\zeta) = 4\frac{\partial^2}{\partial\zeta\partial\bar{\zeta}}\lambda(\zeta) = \frac{64\pi^2|g'(\zeta)|^2|\zeta|^2|g(\zeta)|^2}{\mu^2(1-|g(\zeta)|^2)^4}, \tag{4.47}$$

where from Lemma 4.45

$$\lambda(\zeta) = \frac{\mu^2}{16\pi^2|\zeta|^2|g(\zeta)|^2}\bigl(1-|g(\zeta)|^2\bigr)^2.$$

We denote by $\sinh\alpha(\zeta)$ a sine of hyperbolic angle between the normal vector to the surface $\mathscr{M}$ and the vector e_0. Using Lemma 4.45, we can write

$$\sinh\alpha(\zeta) = \frac{2|g(\zeta)|}{1-|g(\zeta)|^2}.$$

Below we show that it is convenient to describe a geometric structure of a tube in a neighborhood of an isolated singularity in terms of the quantity $\kappa(\zeta) = K(\zeta)/\sinh^4\alpha(\zeta)$, called *the specific curvature* of the surface $\mathscr{M}$.

Using (4.47), we obtain

$$\kappa(\zeta) = \frac{4\pi^2}{\mu^2}\left|\frac{\zeta}{g(\zeta)}\right|^2|g'(\zeta)|^2. \tag{4.48}$$

Our immediate aim is to show that the flow vector and the specific curvature of $\mathscr{M}$ are characteristics of the first and second orders of a deviation of a tube from the light cone in a neighborhood of a singularity.

In order to accomplish this goal, we prove the following auxiliary statement about the asymptotic decomposition of the coordinate functions.

4.49. Lemma. *Let $\mathscr{M}$ be a spacelike zero mean curvature tube with a projection $(0,\beta)$ defined by an immersion (4.44). We put*

$$x_1(e^{i\varphi}) = a_0(\varphi), \qquad \frac{\partial x_1}{\partial r}(e^{i\varphi}) = a_1(\varphi),$$
$$x_2(e^{i\varphi}) = b_0(\varphi), \qquad \frac{\partial x_2}{\partial r}(e^{i\varphi}) = b_1(\varphi),$$

and suppose that a_0, a_1, b_0, b_1 are real analytic functions. Then

$$\begin{aligned} x_1(\zeta) &= \sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k)!}\left[a_0^{(2k)}(\varphi) - \frac{\log r}{2k+1}a_1^{(2k)}(\varphi)\right](\log r)^{2k}, \\ x_2(\zeta) &= \sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k)!}\left[b_0^{(2k)}(\varphi) - \frac{\log r}{2k+1}b_1^{(2k)}(\varphi)\right](\log r)^{2k}. \end{aligned} \tag{4.50}$$

If $\mathscr{M}$ has a singularity, then

$$a_1^2(\varphi) + b_1^2(\varphi) = \frac{\mu^2}{4\pi^2}, \qquad a_1'^2(\varphi) + b_1'^2(\varphi) = \frac{\mu^2}{4\pi^2}|g'(e^{i\varphi})|^2, \tag{4.51}$$

$$\int_0^{2\pi} a_1(\varphi)\,d\varphi = \mu_1, \tag{4.52}$$

$$\int_0^{2\pi} b_1(\varphi)\,d\varphi = \mu_2. \tag{4.53}$$

Proof. We put

$$x_1(\zeta) = x_1(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k(\varphi)(\log r)^k.$$

If $\zeta = \xi + i\,\eta$ and

$$D = \left(\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}\right),$$

then

$$\langle D\varphi, Dr\rangle = 0, \qquad \Delta\,\varphi = 0, \qquad \Delta\log r = 0, \qquad |D\varphi|^2 = \frac{1}{r^2}.$$

Since the mapping $w(\zeta)$ is holomorphic, $\Delta x(\zeta) = 0$. Therefore,

$$0 = \Delta x_1(\zeta) = \sum_{k=0}^{\infty}\left\{a_k''(\varphi)\frac{1}{r^2}(\log r)^k + a_k(\varphi)k(k-1)\frac{1}{r^2}(\log r)^{k-2}\right\}.$$

We get a system of differential equations

$$a_k''(\varphi) + (k+1)(k+2)a_{k+2}(\varphi) = 0, \qquad k = 0, 1, 2, \ldots,$$

which gives

$$a_{2k}(\varphi) = (-1)^{k+1}a_0^{(2k)}(\varphi)\frac{1}{(2k)!} \qquad \text{and} \qquad a_{2k+1}(\varphi) = (-1)^k a_1^{(2k)}(\varphi)\frac{1}{(2k+1)!}.$$

We have obtained the necessary decomposition.

Further, we have

$$Dx_1 = \frac{\mu}{4\pi}\overline{\left(\frac{1}{i\zeta}\left(\frac{1}{g(\zeta)} - g(\zeta)\right)\right)} = \frac{\mu}{4\pi}\frac{i}{\bar\zeta\bar g(\zeta)} - \frac{\mu}{4\pi}\frac{i\bar g(\zeta)}{\bar\zeta},$$
$$Dx_2 = \frac{\mu}{4\pi}\overline{\left(\frac{1}{\zeta}\left(\frac{1}{g(\zeta)} + g(\zeta)\right)\right)} = \frac{\mu}{4\pi}\frac{1}{\bar\zeta\bar g(\zeta)} + \frac{\mu}{4\pi}\frac{\bar g(\zeta)}{\bar\zeta}.$$

Therefore,

$$-Dx_1 + iDy_1 = i\frac{\mu}{2\pi}\frac{\bar{g}(\zeta)}{\bar{\zeta}}.$$

On the other hand, as

$$Dx_1(e^{i\varphi}) = a_1(\varphi)e^{i\varphi} \qquad \text{and} \qquad Dx_2(e^{i\varphi}) = b_1(\varphi)e^{i\varphi},$$

we find

$$g(e^{i\varphi}) = -\frac{2\pi i}{\mu}\big(a_1(\varphi) + ib_1(\varphi)\big) \qquad \text{and} \qquad g'(e^{i\varphi}) = -\frac{2\pi i}{\mu}\big(a_1'(\varphi) + ib_1'(\varphi)\big),$$

and we obtain (4.51).

In order to prove (4.52) and (4.53), we note the mapping $w(\zeta)$ is holomorphic and, consequently,

$$\int_{\Sigma(t)} \langle \nabla x_1, \nabla t\rangle \frac{1}{|\nabla t|} = \int_{u^{-1}\circ\Sigma(t)} \langle Dx_1, Dr\rangle = \int_0^{2\pi} a_1(\varphi)\, d\varphi.$$

The equality (4.53) can be proved similarly.

We note that from (4.51), (4.52), (4.53) and Cauchy's inequality we get

4.54. Corollary. *We have*

$$-\mu^2 + \mu_1^2 + \mu_2^2 \le 0.$$

This means that the flow vector $Q = (\mu, \mu_1, \mu_2)$ is not spacelike.

The following statement gives a geometric interpretation of coefficients at the decomposition of coordinate functions $x_1(\zeta)$ and $x_2(\zeta)$.

4.55. Lemma. *Let $\mathscr{M} \subset \mathbf{R}_1^3$ be a spacelike tube of zero mean curvature having an isolated singularity at the origin. Then the function $\kappa(\zeta)$ has real analytic values $\kappa(\varphi)$ on the unit circle $\zeta = e^{i\varphi}$, $0 \le \varphi \le 2\pi$, and also*

$$\left(\frac{\mu^2}{4\pi^2}\right)^2 \kappa(\varphi) = -a_1(\varphi)a_1''(\varphi) - b_1(\varphi)b_1''(\varphi).$$

Proof. By (4.48), we have

$$\kappa(\zeta) = \frac{4\pi^2}{\mu^2}\left|\frac{\zeta}{g(\zeta)}\right|^2 |g'(\zeta)|^2.$$

As $|g(e^{i\varphi})| \equiv 1$, by the symmetry principle, $g(\zeta)$ can be holomorphically extended on the annulus

$$K\left(\frac{1}{R}, R\right) = \left\{\zeta \in \mathbf{C} : \frac{1}{R} < |\zeta| < R\right\}, \qquad R = e^{2\pi\beta/\mu}.$$

Therefore, $\kappa(\varphi)$ is a real analytic function, and from (4.51) it follows that

$$\begin{aligned} \kappa(\varphi) &= \frac{4\pi^2}{\mu^2}|g'(e^{i\varphi})|^2 = \big(a_1'^2(\varphi) + b_1'^2(\varphi)\big)\left(\frac{4\pi^2}{\mu^2}\right)^2 \\ &= \left(\frac{4\pi^2}{\mu^2}\right)^2 \big(-a_1(\varphi)a_1''(\varphi) - b_1(\varphi)b_1''(\varphi)\big). \end{aligned} \tag{4.56}$$

Proof of Theorem 2.9. Let

$$h(m) = \frac{1}{2}\log\frac{\cosh\alpha - 1}{\cosh\alpha + 1},$$

where $\cosh\alpha$ is a cosine of the hyperbolic angle between the unit normal vector to $\mathscr{M}$ and the time axis. Because the Gauss mapping of the surface $\mathscr{M}$ is holomorphic, $h(m)$ is harmonic with respect to the metric of $\mathscr{M}$, and also

$$\lim_{f(m)\to 0} h(m) = 0.$$

Therefore, the sets

$$H_\tau = \{m : h(m) = \tau\}$$

are compact for small τ. The quantity

$$c = \int_{H_\tau} |\nabla h|$$

does not depend on τ, and

$$c = \int_{\Sigma(t)} \langle \nabla h, \nabla t\rangle \frac{1}{|\nabla t|}.$$

On the other hand, the last integral is a value of integral curvature of the curve $\Sigma(t)$, which equals 2π. Therefore,

$$\begin{aligned} 2\pi &= \int_{\Sigma(\tau)} \langle \nabla h, \nabla t\rangle \frac{1}{|\nabla t|}\, ds \\ &= \int_{\{(\mu/2\pi)\log|\zeta| = \tau\}} \langle Dh, Dr\rangle = \int_{\{(\mu/2\pi)\log|\zeta| = 0\}} |Dh|, \end{aligned}$$

that is,

$$\int_{|\zeta|=1} |g'(\zeta)| = \int_0^{2\pi} \sqrt{a_1'^2(\varphi) + b_1'^2(\varphi)}\, d\varphi = 2\pi. \tag{4.57}$$

As $a_1^2 + a_2^2 \equiv \mu^2/\pi^2$, there exists a function $\theta(\varphi)$ such that

$$a_1(\varphi) = \frac{\mu}{2\pi}\cos\theta(\varphi) \qquad \text{and} \qquad a_2(\varphi) = \frac{\mu}{2\pi}\sin\theta(\varphi).$$

Then the equality (4.57) can be rewritten

$$\int_0^{2\pi} |\theta'(\varphi)|\, d\varphi = 2\pi. \tag{4.58}$$

We note that the representation (4.44) is invariant under rotations of the plane $\mathbf{C}$. It means that the transformation of the variables $\zeta \to e^{i\alpha}\zeta$ retains $\mathscr{M}$. By this fact and without losing generality, we will suppose that $\theta(0) = 0$. By (4.58) the function θ is monotone on $[0, 2\pi]$ and $\theta(2\pi) = 2\pi$. Moreover, it is obvious that $\theta'(\varphi) > 0$ by the maximum principle. Therefore,

$$\sqrt{\kappa(\varphi)} = \theta'(\varphi) 2\pi/\mu.$$

Now let $f(x, y)$ be a solution of (2.7) having an isolated singularity at $(0, 0)$. We put $x + iy = \varrho e^{i\psi}$. As in Lemma 4.41 $\log r = 2\pi t/\mu$, the decomposition (2.10) follows directly from analyticity and monotonicity of $\theta(\lambda)$.

Next, we have

$$\frac{x(re^{i\varphi})^2 + y(re^{i\varphi})^2 - f^2\big(x(re^{i\varphi}), y(re^{i\varphi})\big)}{\big(x(re^{i\varphi})^2 + y(re^{i\varphi})^2\big)^2} = \frac{1}{3}\kappa\big((re^{i\varphi})\big) + o(\log r).$$

We fix ψ, and suppose that φ satisfies $\psi = \theta(\varphi)$. By Lemma 4.49 we conclude that

$$\varrho e^{i\psi} = \frac{\mu}{2\pi}\log r e^{i\theta(\varphi)} + o(\log r).$$

Therefore,

$$\lim_{\varrho \to 0} \kappa^*(\varrho e^{i\theta(\varphi)}) = \kappa(\varphi), \tag{4.59}$$

which implies real analyticity of the function κ^*.

From the last equality and (4.56), we obtain

$$\int_0^{2\pi} \frac{d\psi}{\sqrt{\kappa^*(\psi)}} = \int_0^{2\pi} \frac{\theta'(\varphi)\, d\varphi}{\sqrt{\kappa^*\big(\theta(\varphi)\big)}} = \int_0^{2\pi} \frac{\theta'(\varphi)\, d\varphi}{\sqrt{\kappa(\varphi)}} = \mu.$$

By (4.56) it is not difficult to get the equality (2.13), from which it follows that

$$\max_{[0,2\pi]} \kappa^* \geq \frac{4\pi^2}{\mu^2} \qquad \text{and} \qquad \min_{[0,2\pi]} \kappa^* \leq \frac{4\pi^2}{\mu^2}.$$

The theorem is proved.

The characteristic κ^* introduced for the behavior of a solution of the spacelike zero mean curvature surfaces equation in the neighborhood of the singularity is complete.

4.60. Theorem. *Let* $f_1(x,y)$ *and* $f_2(x,y)$ *be two solutions of* (2.7) *defined in a neighborhood of its common isolated singular point* $(0,0)$. *We suppose that the limit values*

$$\kappa_i^*(\psi) = 6 \lim_{\varrho\to 0} \frac{\varrho - f_i(\varrho e^{i\psi})}{\varrho^3}, \qquad i = 1, 2,$$

of these solutions are equal. Then $f_1(x,y) \equiv f_2(x,y)$.

Proof. By (4.59) and the equivalence $\kappa_1^* \equiv \kappa_2^*$, we can conclude that $\theta_1'(\varphi) \equiv \theta_2'(\varphi)$. Hence, $\theta_1(\varphi) \equiv \theta_2(\varphi)$. Therefore, $g_1(e^{i\varphi}) \equiv g_2(e^{i\varphi})$. Using the uniqueness theorem for holomorphic functions, we obtain $g_1(\zeta) \equiv g_2(\zeta)$. The representation (4.44) leads to the equality $f_1 \equiv f_2$.

5. Timelike surfaces

In this section we investigate timelike tubular surfaces of zero mean curvature in a neighborhood of a singular point. Below we will suppose that the Cartesian coordinates $(x_1, \dots, x_n, t) \in \mathbf{R}_1^{n+1}$ are determined so that the scalar square of a vector $\chi = (x_1, \dots, x_n, t)$ is expressed by

$$|\chi|^2 = -t^2 + \sum_{i=1}^{n} x_i^2.$$

Let M be a two-dimensional connected orientable C^4-manifold without a boundary. We consider a surface $\mathscr{M} = (M, u)$ defined by a C^3-immersion $u\colon M \to \mathbf{R}_1^{n+1}$. A surface $\mathscr{M}$ is called *timelike* if each of its tangent planes contains timelike vectors. Because the metric of timelike surfaces is indefinite, we introduce the norm $\|\cdot\|$ by

$$\|X\| = \sqrt{|\langle X, X\rangle|}\,.$$

5.61. Lemma. *Let* $\mathscr{M}$ *be a two-dimensional timelike tube of zero mean curvature in* $\mathbf{R}_1^{n+1}$. *Then,*

(a) *with respect to the metric of the surface* $\mathscr{M}$, $\Delta x_i = 0$, $i = 0, 1, \dots, n$;

(b) *the quantity* $\mu = \int_{\Sigma(t)} \|\nabla t\|$ *does not depend on* t;

(c) *if* (t, s) *are local isothermal coordinates on* $\mathscr{M}$, *then*

$$\frac{\partial^2 x_i}{\partial t^2} = \frac{\partial^2 x_i}{\partial s^2} \quad \textit{for any } i = 0, 1, \dots, n.$$

Proof. The harmonicity of the coordinate functions of $\mathscr{M}$ is known as well as the harmonicity of the coordinate functions of minimal surfaces in the Euclidean space [21, Note 14].

Using the Stokes formula with $\Delta t = 0$ as well as the above one, we conclude that μ is independent of t.

The third statement follows from (a) and from the special expression of the Laplacian in local isothermal coordinates.

In fact, if the metric is $dl^2 = \lambda^2(ds^2 - dt^2)$ and $f(s,t)$ is a C^2-function, then

$$\nabla f = \left(\frac{1}{\lambda^2} f_s, -\frac{1}{\lambda^2} f_t\right)$$

and

$$\Delta f = \operatorname{div}(\nabla f) = \langle \nabla_{E_1} \nabla f, E_1 \rangle - \langle \nabla_{E_2} \nabla f, E_2 \rangle = \frac{1}{\lambda^2}(f_{ss} - f_{tt}),$$

where $E_1 = \lambda^{-1}\partial/\partial s$ and $E_2 = \lambda^{-1}\partial/\partial t$.

We will prove the following auxiliary statement.

5.62. Lemma. *Let $\mathscr{M} \subset \mathbf{R}_1^{n+1}$ be a two-dimensional doubly-connected timelike C^2-tube of zero mean curvature with a projection (α, β) and a flow μ. Then $\mathscr{M}$ can be represented by a C^2-immersion*

$$u(t,s) = \frac{1}{2}\big(r(s+t) + r(s-t)\big) + \frac{1}{2}\int_{s-t}^{s+t} h(\lambda)\, d\lambda + e_0 t,$$

of the string $(\alpha, \beta) \times (-\infty, +\infty)$.

Here $r, h\colon \mathbf{R} \to \mathbf{R}^n$ are vector functions such that for any $s \in \mathbf{R}$

$$|r'(s)|^2 + |h(s)|^2 = 1, \qquad \langle r'(s), h(s) \rangle = 0, \tag{5.63}$$

and

$$r'(s+\mu) + h(s+\mu) = r'(s) + h(s). \tag{5.64}$$

Proof. With respect to the metric of $\mathscr{M}$, the function $t(m)$ satisfies the following differential equation:

$$\Delta t = \operatorname{div} \nabla t = 0.$$

Therefore, the differential form $*dt$ is closed. Hence, there exists a multifunction $\delta(m)$ such that $d\delta = *dt$ and

$$\int_{\Sigma(t)} d\delta = \mu = \int_{\Sigma(t)} \|\nabla t\|.$$

We consider the multivalued mapping

$$F\colon M \to (\alpha, \beta) \times (-\infty, +\infty), \qquad \text{where } m \to \big(t(m), \delta(m)\big).$$

Note that $u(t,s)$ is μ-periodic by the variable s.

We denote the inverse mapping to F by $u(t,s)$. We assume that

$$\overline{R}(t,s) = u(t,s) - e_0 t.$$

We note that the coordinates (t,s) are isothermal on $\mathscr{M}$. Because the mean curvature is equal to zero by Lemma 5.61, we conclude

$$\frac{\partial^2 \overline{R}}{\partial t^2} = \frac{\partial^2 \overline{R}}{\partial s^2}.$$

Using d'Alambert's formula [38, Section 13], we get

$$u(t,s) = \frac{1}{2}\big(r(s+t) + r(s-t)\big) + \frac{1}{2}\int_{s-t}^{s+t} h(\lambda)\, d\lambda + e_0 t, \tag{5.65}$$

where $r, h\colon \mathbf{R} \to \mathbf{R}^n$ are some vector functions. Since the coordinates (t,s) are isothermal,

$$|u_t|^2 = -|u_s|^2, \qquad \langle u_t, u_s\rangle = 0.$$

For vector functions $r(s)$ and $h(s)$, these equalities can be rewritten in the form

$$\begin{aligned} &|r'(s+t) + h(s+t)|^2 + |r'(s-t) - h(s-t)|^2 = 2,\\ &|r'(s+t) + h(s+t)|^2 - |r'(s-t) - h(s-t)|^2 = 0, \end{aligned}$$

which is the same as (5.63) and (5.64).

From (5.65) we conclude that if and only if the function $r'(s) + h(s)$ is μ-periodic, $\mathscr{M}$ is a tube. The lemma is proved.

Below we give a representation of a timelike surface of zero mean curvature having a singularity in $\mathbf{R}^3_1$.

5.66. Theorem. *Let $\mathscr{M} \subset \mathbf{R}^3_1$ be a two-dimensional doubly-connected timelike C^2-tube of zero mean curvature with projection $(0,\beta)$. Then there exists a C^1-function $\theta\colon (-\infty,+\infty) \to \mathbf{R}$ such that for some integer k, it is true that $\theta(s+\mu) = \theta(s) + 2\pi k$, and $\mathscr{M}$ can be represented in the form*

$$\begin{aligned} x(\tau,s) &= \frac{1}{2}\int_{s-\tau}^{s+\tau} \cos\theta(\lambda)\, d\lambda;\\ y(\tau,s) &= \frac{1}{2}\int_{s-\tau}^{s+\tau} \sin\theta(\lambda)\, d\lambda;\\ t &= \tau. \end{aligned} \tag{5.67}$$

Proof. Because $\mathscr{M}$ has a singularity at the origin, we have $r(s) \equiv 0$. From Lemma 5.62, we obtain $|h(s)| \equiv 1$ and $h(s+\mu) = h(s)$.

Therefore, there exists a function $\theta(\lambda)$ such that

$$h(\lambda) = \bigl(\cos\theta(\lambda), \sin\theta(\lambda)\bigr), \qquad \theta(\lambda+\mu) = \theta(\lambda) + 2\pi k, \qquad k = 0, 1, 2, \dots .$$

We denote the Gaussian curvature by $K(s,t)$ and by $\cosh\alpha$ a hyperbolic cosine of an angle between a unit normal to $\mathscr{M}$ and the time axis. The quantity

$$\kappa = \frac{K}{\cosh^4\alpha}$$

is said to be a *specific curvature* of $\mathscr{M}$.

5.68. Lemma. *The following formulas are true*:

$$\begin{aligned}\cosh^2\alpha &= \left(\sin^2\frac{\theta(s+t)-\theta(s-t)}{2}\right)^{-1},\\ \kappa(t,s) &= \theta'(s+t)\theta'(s-t).\end{aligned}$$

Proof. By Theorem 5.66, the metric of $\mathscr{M}$ has the form

$$ds^2_{\mathscr{M}} = \sin^2\omega(ds^2 - dt^2), \qquad \text{where } \omega = \tfrac{1}{2}\bigl(\theta(s+t) - \theta(s-t)\bigr).$$

Therefore,

$$\cosh^2\alpha = \|e_0^T\|^2 = \|\nabla t\|^2 = \frac{1}{\sin^2\omega}.$$

From Lemma 4.46, we have

$$K(s,t) = -\frac{1}{\sin^2\omega}\Delta'\log\sin\omega = -\frac{1}{\sin^2\omega}\left(\frac{\Delta'\sin\omega}{\sin\omega} - \frac{|D\sin\omega|^2}{\sin^2\omega}\right),$$

where

$$\Delta' = \frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial t^2}, \qquad D = \left(\frac{\partial}{\partial s}, -\frac{\partial}{\partial t}\right).$$

Since

$$|D\omega|^2 = \left(\frac{\partial\omega}{\partial s}\right)^2 - \left(\frac{\partial\omega}{\partial t}\right)^2 \qquad \text{and} \qquad \Delta'\omega = 0,$$

the Gaussian curvature can be rewritten as

$$K(s,t) = \frac{|D\omega|^2}{\sin^4\omega}.$$

Hence, the specific curvature is expressed by the equality

$$\kappa(s,t) = |D\omega|^2 = \theta'(s+t)\theta'(s-t).$$

Proof of Theorem 2.15. Let $\Sigma(\tau)$ be a section of $\mathscr{M}$ by the plane $t=\tau$. We note that the curve $\Sigma(\tau)$ is closed and can be defined by

$$x(\tau,s) = \frac{1}{2}\int_{s-\tau}^{s+\tau} \cos\theta(\lambda)\,d\lambda \qquad \text{and} \qquad y(\tau,s) = \frac{1}{2}\int_{s-\tau}^{s+\tau} \sin\theta(\lambda)\,d\lambda.$$

We denote the curvature of this curve by $k(s)$. Using the formula

$$k(s) = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}},$$

we get

$$k(s) = \frac{\theta'(s+t) + \theta'(s-t)}{2|\sin\omega|}.$$

Because the length element of $\Sigma(t)$ is $dl = |\sin\omega|\,ds$, we can write

$$\int_{\Sigma(\tau)} k(s)\,dl = \frac{1}{2}\int_0^{\mu} \theta'(s+\tau)\,ds + \frac{1}{2}\int_0^{\mu} \theta'(s-\tau)\,ds.$$

On the other hand, as $\Sigma(\tau)$ is closed,

$$\int_{\Sigma(\tau)} k(s)\,dl = 2\pi,$$

and, therefore,

$$2\pi = \int_0^{\mu} \theta'(s)\,ds = \theta(\mu) - \theta(0).$$

Hence, we conclude that the range of $\theta(s)$ is $[0, 2\pi]$.

We put

$$\xi_\tau = \Big(\tfrac{1}{2}\big(\cos\theta(s+\tau) + \cos\theta(s-\tau)\big), \tfrac{1}{2}\big(\sin\theta(s+\tau) + \sin\theta(s-\tau)\big), 1\Big).$$

Clearly, the vectors ξ_τ are orthogonal to $\Sigma(\tau)$ and tangent to $\mathscr{M}$. Let $\tau \to 0$. We find that the totality of the vectors

$$\xi_0 = \big(\cos\theta(s), \sin\theta(s), 1\big)$$

forms a light cone. This follows from the above properties of $\theta(s)$.

We put $\psi = \theta(s)$.

5.69. Theorem. *Let $f(x,y)$ be a solution of* (2.14) *having an isolated singularity at $(0,0)$ with $f(0,0)=0$. Then*

$$\lim_{\varrho\to 0}\frac{f(\varrho e^{i\psi})-\varrho}{\varrho^3}=\frac{1}{6}\kappa(s) \qquad \textit{and} \qquad \varrho e^{i\psi}=x+iy.$$

Proof. We have

$$\begin{aligned}
x'_t(s,t) &= \tfrac{1}{2}\big(\cos\theta(s+t)+\cos\theta(s-t)\big),\\
x''_{tt}(s,t) &= \tfrac{1}{2}\big(-\theta'(s+t)\sin\theta(s+t)+\theta'(s-t)\sin\theta(s-t)\big),\\
y'_t(s,t) &= \tfrac{1}{2}\big(\sin\theta(s+t)+\sin\theta(s-t)\big),\\
y''_{tt}(s,t) &= \tfrac{1}{2}\big(\theta'(s+t)\cos\theta(s+t)-\theta'(s-t)\cos\theta(s-t)\big).
\end{aligned}$$

Computing directly, we get

$$\lim_{t\to 0}\frac{x_t'^2+y_t'^2-1}{t^2}=-\theta'^2(s) \qquad \text{and} \qquad \lim_{t\to 0}\frac{x\,x''+y\,y''}{t}=-\theta'^2(s).$$

Using Lemma 5.68, we find that

$$\lim_{t\to 0}\frac{t-\sqrt{x^2+y^2}}{\big(\sqrt{x^2+y^2}\,\big)^3}=\frac{\kappa(s)}{6}.$$

6. Smooth pasting

In this section, we investigate the possibility of a smooth pasting of spacelike and timelike tubes in a neighborhood of their common singular point. To begin, we prove the following auxiliary statement.

6.70. Lemma. *Let $f(x,y)$ be a solution of* (2.14) *with an isolated singularity $(0,0)$. We assume that the function $\theta(\lambda)$ in* (5.67) *is real analytic and satisfies $\theta'(\lambda)>0$. Then there exist real analytic functions $h_k(\psi)$, $k=1,2,\dots$, defined on $[0,2\pi]$ so that*

$$f(\varrho e^{i\psi})=\varrho+\sum_{k=1}^{\infty}h_k(\psi)\varrho^{2k+1}. \tag{6.71}$$

Proof. We consider the mapping (ϱ,ψ) of the rectangle

$$Q=(0,\mu)\times(0,\beta)$$

in $\mathbf{R}^2$, defined by

$$\varrho = \frac{1}{2}\left(\left(\int_{s-t}^{s+t} \cos\theta(\lambda)\,d\lambda\right)^2 + \left(\int_{s-t}^{s+t} \sin\theta(\lambda)\,d\lambda\right)^2\right)^{1/2} \tag{6.72}$$

and

$$\psi = \operatorname{arctg}\left(\int_{s-t}^{s+t} \cos\theta(\lambda)\,d\lambda\right)\left(\int_{s-t}^{s+t} \sin\theta(\lambda)\,d\lambda\right)^{-1}. \tag{6.73}$$

For the Jacobian $J(s,t)$ of this mapping, we have

$$J(s,0) = \theta'(s) > 0.$$

We denote by $G(\varrho,\psi)$ the local inverse mapping to (6.72) and (6.73). As $\theta(s)$ is real analytic, the mapping G is also real analytic. Differentiating (6.72) directly, we see that

$$\left.\frac{\partial^{2k} t}{\partial \varrho^{2k}}\right|_{\varrho=0} = 0.$$

Therefore, there exist real analytic functions $h_k(\psi)$, $k = 1,2,\ldots$, such that

$$t = \varrho + \sum_{k=1}^{\infty} h_k(\psi)\varrho^{2k+1},$$

that is, the decomposition (6.71) holds.

Proof of Theorem 2.16. Let $f_1(x,y)$ be a solution of (2.7) with singularity at $(0,0)$ and a flow μ. Using (2.10), we obtain

$$f_1(\varrho e^{i\psi}) = \varrho - \frac{1}{6}\kappa^*(\psi)\varrho^3 + \sum_{k=2}^{\infty} c_k(\psi)\varrho^{2k+1}.$$

Consider a solution of (2.14) constructed by (5.67) with

$$\theta(s) = \int_0^{2\pi s/\mu} \sqrt{\kappa(\lambda)}\,d\lambda.$$

By (6.71) we find

$$f_2(\varrho e^{i\psi}) = -\varrho - \frac{1}{6}\kappa^*(\psi)\varrho^3 + \sum_{k=2}^{\infty} h_k(\psi)\varrho^{2k+1}.$$

Using the analyticity of the functions $c_k(\psi)$ and $h_k(\psi)$, we conclude that the function $\delta(\varrho e^{i\psi}) \in C^2$ in some neighborhood of $(0,0)$. The theorem is proved.

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