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NONLINEAR HARMONIC MEASURES ON TREES

Robert Kaufman, José G. Llorente, and Jang-Mei Wu

University of Illinois, Department of Mathematics
1409 West Green Street Urbana, Illinois 61801, U.S.A.; rpkaufma@math.uiuc.edu

Universitat Autònoma de Barcelona, Department de Matemàtiques
ES-08193 Bellaterra, Barcelona, Spain; gonzalez@mat.uab.es

University of Illinois, Department of Mathematics
1409 West Green Street, Urbana, Illinois 61801, U.S.A.; wu@math.uiuc.edu

Abstract. We show that nonlinear harmonic measures on trees lack many desirable properties of set functions encountered in classical analysis. Let F be an averaging operator on $\mathbf{R}^\kappa$ and ω_F be the F-harmonic measure on a κ-regular forward branching tree. Unless F is the usual average, ω_F is not a Choquet capacity; union of sets of ω_F measure zero can have positive ω_F measure when F is permutation invariant; and there exist sets of full ω_F measure having "small" dimension. Let A be a monotone operator on $\mathbf{R}^\kappa$, then A-harmonic functions on trees need not obey the strong maximum principle unless the ratio of the ellipticity constants is close to 1.

We show that nonlinear harmonic measures on trees lack many desirable properties of set functions encountered in classical analysis.

Let T be a directed tree with regular κ-branching; in this paper we continue earlier work on p-harmonic functions on trees in [CFPR] and [KW]. We treat the p-Laplacian as a special case of a nonlinear averaging operator $F\colon \mathbf{R}^\kappa \to \mathbf{R}^1$ studied by Alvarez, Rodríguez and Yakubovich in [ARY]. Then the F-potential theory on T is the discrete version of the nonlinear potential theory in the Euclidean space structured on the nonlinear Euler equation of the variational integral $\int \mathscr{F}(x,\nabla u)\,dx$ with $\mathscr{F}(x,h) \approx |h|^p$; see [GLM] and [HKM].

Each averaging operator F leads to a harmonic measure on the boundary ∂T of the tree. Except when F is the usual average, there exists a number $d(\kappa,F)$ strictly less than the dimension of ∂T, so that every compact set on ∂T of dimension $< d(\kappa,F)$ must have zero F-harmonic measure, and there exist sets of dimension $\le d(\kappa,F)$ having full F-harmonic measure. If we could show that every Borel set on ∂T of dimension $< d(\kappa,F)$ has zero F-harmonic measure, then the statement "the dimension of F-harmonic measure is $d(\kappa,F)$" would

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follow; here dimension of F-harmonic measure is defined to be the infimum of the dimensions of those Borel sets on ∂T having full F-harmonic measure. When F is the p-Laplacian operator, $d(\kappa, F)$ is introduced in [KW] and used to study the sizes of Fatou sets and sets of finite radial variations for bounded p-harmonic functions on trees. Here again $d(\kappa, F)$ is the critical dimension of Fatou sets for bounded F-harmonic functions.

We also prove that when F is permutation invariant and is not the usual average, there exist two sets on ∂T, of zero F-harmonic measure, whose union has positive F-harmonic measure. This answers a question raised by Martio on trees ([ARY], [M]). Our work is motivated by [ARY], in which it is proved that for certain F's, there exist congruent sets $B_1, B_2, \ldots, B_\kappa$ of arbitrarily small positive F-harmonic measure, whose union is ∂T. Our construction starts from the root of the tree and theirs starts from the boundary.

We also show that when F is permutation invariant and is not the usual average, F-harmonic measure is not a Choquet capacity; consequently, it is not left continuous on increasing sequences of sets.

While these results show that F-harmonic measure lacks many desirable properties of set functions in the linear theory, many problems remain: some concern inner approximation of Borel sets by compact sets, others are about the behavior of monotone sequences A_n such that $\bigcup_1^\infty A_n = \partial T$.

In another direction, we study the analogue of the quasilinear elliptic equation $\operatorname{div}(A\nabla u) = 0$ ([HKM]) on trees. We examine the notion of A-harmonic functions on trees, when A is a monotone operator on $\mathbf{R}^\kappa$. We find that A-harmonic functions do not always satisfy the strong maximum principle defined in Section 1; this shows that some caution is necessary in arguing from elliptic operators in the Euclidean space to potential theory on trees. When an A-operator is close to the p-Laplacian, the strong maximum principle holds and a Fatou type theorem is valid; our sufficient condition is sharp when $p = 2$. We also show that the critical dimension $d(\kappa, A)$ for the A-operator approaches the critical dimension $d(\kappa, p)$ for the p-Laplacian uniformly as the ellipticity constants of A approach that of the p-Laplacian.

Finally, we comment on the meanings of 1-Laplacian and ∞-Laplacian on trees.

Our operators on trees may be considered as simple analogues of the p-Laplacian $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, on the unit disk D. Martio ([HKM], [M]) asked whether the p-harmonic measure on the unit circle ∂D is subadditive, or whether the union of two null sets must be null when $p \neq 2$; for work in this direction, see [AM], [GLM]. The Fatou set $\mathscr{F}(u)$ of a function u in D is the set on ∂D where the radial limits exist. The classical theorem of Fatou from 1906 states that $\mathscr{F}(u)$ has length 2π for bounded harmonic functions. When $p \neq 2$, the Fatou set $\mathscr{F}(u)$ of a bounded p-harmonic function u can have length zero; examples are given by Wolff ([W]) for $2 < p < \infty$ and by Lewis ([L]) for $1 < p < 2$.

It is known that $\dim \mathscr{F}(u) \geq \delta(p) > 0$, see ([MW], [FGMS]). The best value of $\delta(p)$ is unknown when $p \neq 2$, in particular whether $\delta(p)$ is equal to 1.

1. Preliminaries

Let $\kappa > 1$ be a natural number and T a directed tree with regular κ-branching. That is, T consists of the empty set ϕ and all finite sequences $(v_1, v_2, \dots, v_r)$ of lengths $r = 1, 2, 3, \dots$, whose coordinates are chosen from $\{1, 2, 3, \dots, \kappa\}$. The elements in T are called vertices. Each vertex v has κ successors, obtained by adding another coordinate. These are abbreviated by $(v, 1)$, $(v, 2)$, $\dots$, (v, κ) and have length one more than the length of v.

A branch b of T is an infinite sequence $(b_1, b_2, \dots)$ with coordinates in $\{1, 2, \dots, \kappa\}$. And b can be regarded as an infinite sequence of vertices (b_1), (b_1, b_2), $\dots$, $(b_1, b_2, \dots, b_r), \dots$, each followed by its immediate successor. The set of all branches forms the boundary ∂T of the tree.

A metric on $T \cup \partial T$ is defined as follows. The distance between two sequences (finite or infinite) $b = (b_1, b_2, \dots)$ and $b' = (b'_1, b'_2, \dots)$ is κ^{-N+1} when N is the first index n such that $b_n \neq b'_n$; but when $b = (b_1, b_2, \dots, b_N)$ and $b' = (b_1, b_2, \dots, b_N, b'_{N+1}, \dots)$, the distance is κ^{-N}. Hausdorff measure and Hausdorff dimension are defined using this metric. The tree T and the boundary ∂T then have diameter one, and ∂T has Hausdorff dimension one.

We define a *probability measure* λ on ∂T through the mapping $g(b) = \sum_1^\infty \kappa^{-r}(b_r - 1)$ onto $[0,1]$. The λ-measure of a set E is the Lebesgue measure of $g(E)$; this is the infinite product of uniform distributions on each factor $\{1, 2, \dots, \kappa\}$. λ is sometimes called *Lebesgue measure*.

Let F: $\mathbf{R}^\kappa \to \mathbf{R}^1$ be a continuous function. We call F an *averaging operator* if it satisfies the following:

(i) $F(0, 0, \dots, 0) = 0$ and $F(1, 1, \dots, 1) = 1$;

(ii) $F(tx_1, tx_2, \dots, tx_\kappa) = tF(x_1, x_2, \dots, x_\kappa)$ for $t \in \mathbf{R}^1$;

(iii) $F(t + x_1, t + x_2, \dots, t + x_\kappa) = t + F(x_1, x_2, \dots, x_\kappa)$ for $t \in \mathbf{R}^1$;

(iv) $F(x_1, x_2, \dots, x_\kappa) < \max\{x_1, x_2, \dots, x_\kappa\}$ if not all x_j's are equal;

(v) F is nondecreasing with respect to each variable.

The definition is adopted from [ARY].

Property (iv) is the *strong maximum principle*, which implies that if

$$F(x_1, x_2, \dots, x_\kappa) = 0,$$

unless all x_j's are zero, there must be a sign change among the entries. Property (v) is the *monotonicity property* which gives the comparison principle for the Dirichlet problem needed in developing potential theory.

It follows from (i) $\sim$ (iv) that

(vi) $F(1-x_1, 1-x_2, \dots, 1-x_\kappa) = 1 - F(x_1, x_2, \dots, x_\kappa)$;

and

(vii) there is a number $b > 0$ such that whenever $F(x_1, x_2, \dots, x_\kappa) \geq 0$ and $\max x_j \leq 1$, then $\min x_j \geq -b$.

To verify (vii), let S be the set in $\mathbf{R}^\kappa$ defined by $\max x_j \leq 0$ and $\min x_j = -1$. Then by (ii) and (iv), $F(x_1, x_2, \dots, x_\kappa) < 0$ on S. By continuity, there is an $\varepsilon > 0$ so that $F < 0$ on the set $\{(x_1, x_2, \dots, x_\kappa) : \max x_j \leq \varepsilon \text{ and } \min x_j = -1\}$. Hence we can choose $b = 1/\varepsilon$.

Property (vii) is used in the proof of Theorem 1, more precisely, in proving (iii) in Lemma 1. When the strong maximum principle is replaced by

(iv)′ $F(x_1, x_2, \dots, x_\kappa) \leq \max\{x_1, x_2, \dots, x_\kappa\}$,

then Lemma 1(iii) fails and the proof of Theorem 1 will be considerably more tedious.

In certain theorems, we require, in addition, F to be *permutation invariant*:

(viii) $F(x_1, x_2, \dots, x_\kappa) = F(x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(\kappa)})$ for each permutation τ of $\{1, 2, \dots, \kappa\}$.

From now on, we assume F is an averaging operator not necessarily permutation invariant, unless otherwise mentioned. We let

$$\mathbf{0} = (0, 0, \dots, 0), \qquad \mathbf{1} = (1, 1, \dots, 1),$$

and use $X = (x_1, x_2, \dots, x_\kappa)$ to denote vectors in $\mathbf{R}^\kappa$.

Remark. If F is differentiable at $\mathbf{0}$, then $F(X) = \Sigma \lambda_j x_j$ for some $\lambda_j \in [0,1]$ with $\Sigma \lambda_j = 1$; if F is also permutation invariant then $F(X) \equiv \Sigma x_j / k$ is the usual average. To see this, let $\lambda_j = (\partial / \partial x_j) F(\mathbf{0})$, and $X \neq \mathbf{0}$. Then $tF(X) = F(tX) = \Sigma \lambda_j t x_j + o(t)$ as $t \to 0$. This implies that $F(X) = \Sigma \lambda_j x_j$.

We call $X \in \mathbf{R}^\kappa$ an F*-harmonic vector* (or F*-superharmonic vector*) if $F(X) = 0$ (or $F(X) \leq 0$).

Given a function u on T, the *gradient* of u at a vertex v is

$$\nabla u(v) = \big(u(v,1) - u(v), u(v,2) - u(v), \dots, u(v,\kappa) - u(v)\big).$$

We say u is an F*-harmonic function* on tree T, if $\nabla u(v)$ is F-harmonic at each vertex v, i.e.

$$F\big(u(v,1), u(v,2), \dots, u(v,\kappa)\big) = u(v) \qquad \text{for all } v \in T.$$

F-superharmonicity is defined analogously.

For $E \subseteq \partial T$, let $E^c = \partial T \backslash E$,

$$U = \Big\{ u : F\text{-superharmonic on } T \text{ so that } \liminf_{v \to b} u(v) \geq \chi_E(b) \text{ for all } b \in \partial T \Big\}$$

the upper class of E, and

$$\omega_F(v, E) = \inf\{u(v) : u \in U\}$$

the F-harmonic measure function for E; and call $\omega_F(\phi, E)$ the *F-harmonic measure* of E, in short $\omega_F(E)$. Theorem 3 below shows that ω_F is not an outer measure unless F is the usual average.

Following the arguments in [HKM] for the continuous case, we have

(i) $0 \leq \omega_F(\,\cdot\,, E) \leq 1$ on T;

(ii) $\omega_F(E) \leq \omega_F(G)$ when $E \subseteq G$;

(iii) if E is compact, then $\lim_{v \to b} \omega_F(v, E) = 0$ for $b \in E^c$;

(iv) $\omega_F(\,\cdot\,, E)$ is F-harmonic on T;

(v) if E and G are disjoint compact sets on ∂T and $\omega_F(E) = \omega_F(G) = 0$, then $\omega_F(E \cup G) = 0$;

(vi) if E is compact, then $\omega_F(E) + \omega_F(E^c) = 1$;

(vii) if $E_1 \supseteq E_2 \supseteq \cdots \supseteq E_j \supseteq \cdots$ are compact sets then $\lim \omega_F(E_j) = \omega_F(\cap E_j)$.

Examples. (1) For $1 < p < \infty$, the p-Laplacian Δ_p of a vector X in $\mathbf{R}^\kappa$ is

$$\sum_j x_j |x_j|^{p-2},$$

and X is said to be p-harmonic (or p-superharmonic) if $\Delta_p X = 0$ (or ≤ 0). The operator $p(X) = t$ from $\mathbf{R}^\kappa$ to $\mathbf{R}^1$ defined implicitly by

$$\Delta_p(X - t\mathbf{1}) = \Sigma(x_j - t)|x_j - t|^{p-2} = 0$$

is an averaging operator; and $\Delta_p(X - t\mathbf{1}) = 0$ if and only if $\sum_1^\kappa |x_j - x|^p$ attains its minimum at $x = t$. Thus p-harmonic functions and F_p-harmonic functions are the same, and they are the discrete analogues of the solutions to $\operatorname{div}(\nabla u |\nabla u|^{p-2}) = 0$. We use ω_p to denote the p-harmonic measure.

(2) The discrete analogue to the quasilinear elliptic equation $\operatorname{div} A(\nabla u) = 0$, which contains p-Laplacian as a special case, gives rise to another host of averaging operators, see Section 5.

2. The critical dimension $d(\kappa, F)$

Given a vector $X = (x_1, x_2, \dots, x_\kappa)$, we identify it with a random variable, again called X, with probability $P(X = x_j) = \kappa^{-1}$ for $1 \le j \le \kappa$. When X contains both positive and negative entries, with $\mathscr{E}$ denoting expectation,

$$\beta(X) = \min\{\mathscr{E}(e^{tX}) : t \in \mathbf{R}\}$$

is less than 1 and is attained at some $t(X)$, with $t(X) > 0$ if $\Sigma x_j < 0$ and $t(X) < 0$ if $\Sigma x_j > 0$ ([KW]).

Let

$$m(\kappa, F) = \min\left\{\sum_1^\kappa e^{x_j} : F(X) = 0\right\}$$

and observe that the *minimum is attained.* In fact, $x_j < \log\kappa$ as soon as the sum is less than κ. It follows from property (vii) of the averaging operators that $\min x_j \ge -b\log\kappa$. The minimum is attained by continuity.

Define

$$d(\kappa, F) = \log m(\kappa, F)/\log\kappa.$$

Then $0 < d(\kappa, F) < 1$ unless F is the usual average, in which case $d(\kappa, F) = 1$, see Lemma 3 below.

The Fatou set $\mathscr{F}(u)$ of a function u is the set of branches $b = (b_1, b_2, \dots)$ on which $\lim_{n\to\infty} u(b_1, b_2, \dots, b_n)$ exists and is finite, and $\mathrm{BV}(u)$ is the set of branches b on which u has finite variation $\Sigma|u(b_1, b_2, \dots, b_{n+1}) - u(b_1, b_2, \dots, b_n)|$. Recall that F_p is the averaging operator associated with the p-Laplacian, and let $m(\kappa, p) = m(\kappa, F_p)$, $d(\kappa, p) = d(\kappa, F_p)$. It is proved in [KW] that

$$\min_{\mathscr{H}_p} \dim \mathscr{F}(u) = \min_{\mathscr{H}_p} \dim \mathrm{BV}(u) = d(\kappa, p)$$

with $\mathscr{H}_p$ being the set of bounded p-harmonic functions on T. Values of $d(\kappa, p)$ can be estimated by Lagrange multipliers, and asymptotics can be found for large p or large κ ([KW]). The same argument gives the following.

Theorem A. *Let F be an averaging operator and $\mathscr{H}_F$ be the set of bounded F-harmonic functions on T. Then*

$$\min_{\mathscr{H}_F} \dim \mathscr{F}(u) = \min_{\mathscr{H}_F} \dim \mathrm{BV}(u) = d(\kappa, F).$$

Remark. The monotonicity of the averaging operators is not used in the proof. Thus Theorem A is valid for the class of operators satisfying (i) $\sim$ (iv) only.

Theorem A suggests that $\dim\omega_F = d(\kappa, F)$ might be true.

Theorem 1. *Let F be an averaging operator. Suppose that E is a compact set on ∂T with $\dim E < d(\kappa, F)$ then $\omega_F(E) = 0$.*

The *entropy* of a probability measure μ on $\{1, 2, \dots, \kappa\}$ is $H(\mu) = -\Sigma \mu_j \log \mu_j$, where $\mu_j = \mu(\{j\})$.

Lemma 1. *Let $F(x_1, x_2, \dots, x_\kappa) = 0$. Then there is a probability measure ν on $\{1, 2, \dots, \kappa\}$ such that*

(i) $\Sigma \nu_j x_j = 0$,

(ii) $H(\nu) \geq \log m(\kappa, F)$,

(iii) $\min \nu_j \geq c(\kappa, F) > 0$.

Proof. We follow the proof in [KW]. If $\Sigma x_j = 0$, choose $\nu_j = 1/\kappa$ for all j. We may assume then $\Sigma x_j < 0$ and recall that $(1/\kappa)\Sigma e^{t x_j}$ attains a minimum $\beta(X)$ at some $\tau > 0$. Then $\kappa \beta(X) \geq m(\kappa, F)$ and $\Sigma x_j e^{\tau x_j} = 0$. Define $\nu_j = e^{\tau x_j}/\kappa \beta(X)$, and observe that $\Sigma \nu_j = 1$, $\Sigma \nu_j x_j = 0$ and $H(\nu) = \log \kappa \beta(X) \geq \log m(\kappa, F)$. To prove (iii), note that $\tau x_j \leq \log \kappa$ for all j; then by property (vi) of averaging operators, $\min \tau x_j \geq -b \log \kappa$. This gives (iii).

In the following, we let S_r be the class of subsets of branches of the tree, defined by the first r coordinates. This is a finite field whose atoms are called cylinders of rank r. Now S_r contains κ^r cylinders $\mathscr{C}_r$ of Lebesgue measure κ^{-r} each and the same diameter. The cylinder $\mathscr{C}_0$ is ∂T.

Proof of Theorem 1. Let $u(v) = \omega_F(v, E)$, then $0 \leq u \leq 1$ on T. Since E is compact, $\lim_{v \to b} u(v) = 0$ for all $b \in \partial T \backslash E$. We now apply Lemma 1 to the gradient of u at each vertex v, to find a probability measure μ on ∂T so that u is a martingale with respect to μ. The process resembles the linearization of a solution of a nonlinear operator in the continuous situation, see [CFPR] and [KW]. To define μ, let $\mu(C_0) = 1$ and assume μ has been defined on all cylinders of rank $\leq r$. Let C_r be a cylinder of rank r represented by $(b_1, b_2, \dots, b_r)$. The gradient $(x_1, \dots, x_\kappa)$ of u at the vertex $v = (b_1, \dots, b_r)$ forms an F-harmonic vector. Let ν be the probability measure on $\{1, 2, \dots, \kappa\}$ associated with the present $(x_1, x_2, \dots, x_\kappa)$ as in Lemma 1, and define μ on the κ cylinders C_{r+1} of rank $r+1$ contained in C_r by $\mu(C_{r+1}) = \nu_j \mu(C_r)$ if C_{r+1} is represented by $(b_1, b_2, \dots, b_r, j)$. We have defined μ on all cylinders of rank $r+1$, and then on ∂T by σ-additivity.

Properties (ii) and (iii) in Lemma 1 yield that $\mu(C_{r+1}) > c(\kappa, F)\mu(C_r)$ whenever $C_{r+1} \subseteq C_r$, and $H(\mu \mid C_r) \geq \log m(\kappa, F)$. Then by a theorem on entropy and the dimension associated with a measure, known as early as Besicovitch and Eggleston and stated in the form used here in [KW], the measure μ is zero on any subset of ∂T of dimension less than $\log m(\kappa, F)/\log \kappa$.

Note from (i) of Lemma 1, u is a bounded martingale with respect to μ, and therefore by the martingale convergence theorem, $\lim_{v\to b} u(v) = u^*(b)$ exists μ-a.e. on ∂T and

$$u(\phi) = \int_{\partial T} u^*(b)\, d\mu(b) = \int_E u^*(b)\, d\mu(b) = 0.$$

This proves that E has zero F-harmonic measure and thus the theorem.

We need another expression for the quantity $m(\kappa, F)$ in the next section.

Lemma 2. *Let F be an averaging operator, not equal to the usual average. Then there exists $Y = (y_1, y_2, \dots, y_\kappa)$ so that $F(Y) = 1$, $\prod y_j > 1$ and $y_j > 0$ for all $1 \le j \le \kappa$.*

Proof. Since F is not the usual average, there is some vector $X = (x_1, \dots, x_\kappa)$ such that $F(X) = 0$ but $\Sigma x_j > 0$. Now we can take $Y = \mathbf{1} + tX$ for some small positive t.

Let

$$m^*(\kappa, F) = \inf\Big\{\min_{t\le 0} \Sigma y_j^t : F(Y) = 1 \text{ and } y_j > 0 \text{ for all } 1 \le j \le \kappa\Big\},$$

and

$$m^{**}(\kappa, F) = \inf\Big\{\min_{t\le 0} \Sigma y_j^t : F(Y) = 1,\ \prod y_j > 1 \text{ and } y_j > 0 \text{ for all } 1 \le j \le \kappa\Big\}.$$

Lemma 3. $m(\kappa, F) = m^*(\kappa, F) = m^{**}(\kappa, F)$. *The minimum $m(\kappa, F)$ is attained. When F is not the usual average, $m^*(\kappa, F)$ and $m^{**}(\kappa, F)$ are not attained by vectors defining them, and $0 < m(\kappa, F) < \kappa$; when F is the usual average, $m(\kappa, F) = \kappa$.*

Proof. First suppose $F(X) = 0$. Then $F(\mathbf{1} + tX) = 1$ and $1 + tx_1$, $1 + tx_2, \dots, 1 + tx_\kappa > 0$ for small $t < 0$. Since $\lim_{t\to 0^-} \Sigma(1 + tx_j)^{1/t} = \Sigma e^{x_j}$, then $m^*(\kappa, F) \le m(\kappa, F)$.

Conversely, suppose $y_1, y_2, \dots, y_\kappa > 0$ and $F(y_1, y_2, \dots, y_\kappa) = 1$. Hence $\log y_j \le y_j - 1$ and $F(y_1 - 1, y_2 - 1, \dots, y_\kappa - 1) = 0$. Thus $m(\kappa, F) \le \Sigma e^{t(y_j - 1)} \le \Sigma y_j^t$ for $t \le 0$. This proves $m(\kappa, F) \le m^*(\kappa, F)$, and $m(\kappa, F) = m^*(\kappa, F)$.

Suppose F is the usual average, then a simple calculation gives $m(\kappa, F) = \kappa$. Otherwise, there exists an F-harmonic vector $(x_1, x_2, \dots, x_\kappa)$ with $\Sigma x_j < 0$. Choosing $t > 0$ and sufficiently small, we have $m(\kappa, F) \le \Sigma e^{tx_j} < \kappa$. We proved earlier that $m(\kappa, F)$ is attained and therefore strictly positive. Thus $0 < m(\kappa, F) < \kappa$.

Suppose $\prod_1^\kappa y_j \le 1$ and $t \le 0$. By the arithmetic-geometric mean inequality, we have

$$\kappa^{-1}\Sigma y_j^t \ge \exp(\kappa^{-1}\Sigma t \log y_j) \ge 1.$$

Since $0 < m(\kappa, F) < \kappa$, $m^*(\kappa, F) = m^{**}(\kappa, F)$.

Because $m(\kappa, F) < \kappa$, we only need to consider $t < 0$ and $Y \ne \mathbf{1}$ in the second paragraph of the proof. Since log is strictly concave, $m(\kappa, F) < \Sigma y_j^t$. This shows that $m^*(\kappa, F)$ and $m^{**}(\kappa, F)$ are not attained, and completes the proof of Lemma 3.

3. F-harmonic measure—dimension and null sets

When F is not the arithmetic mean, we show in Theorem 2 that there exists a set of dimension at most $d(\kappa, F)$, having full ω_F measure; and in Theorem 3 that the union of two sets of zero ω_F measure can have positive ω_F measure.

If we were able to prove Theorem 1 for all Borel sets on ∂T, then from Theorem 2,

$$\text{dimension of } \omega_F = d(\kappa, F),$$

here dimension of ω_F is $\inf\{\dim E : E \text{ Borel on } \partial T,\ \omega_F(E) = 1\}$. The problem is nontrivial, in view of Theorem 5.

Proposition 1. *Let F be an averaging operator not equal to the usual average, and Y be a vector in $\mathbf{R}^\kappa$ satisfying $F(Y) = 1$, $\prod y_j > 1$ and $y_j > 0$ for all $1 \le j \le \kappa$. Let $X = (\log y_1, \log y_2, \dots, \log y_\kappa)$, then there exists a set $E \subseteq \partial T$ so that $\omega_F(E) = 0$, $\omega_F(E^c) = 1$ and $\dim(E^c) \le 1 + \log\beta(X)/\log\kappa$.*

Proof. Define an F-harmonic function u on T as follows: let $u(\phi) = 1$; suppose u has been defined at a vertex v, define u at its immediate successors $(v,1), (v,2), \dots, (v,\kappa)$ by $y_1 u(v), y_2 u(v), \dots, y_\kappa u(v)$, respectively. It is clear that u is positive F-harmonic. And suppose that $v = (v_1, v_2, \dots, v_n)$ with $v_j \in \{1, 2, \dots, \kappa\}$, then

$$u(v) = \prod_{j=1}^n y_{v_j}.$$

Let m and n be positive integers, define

$$E(m,n) = \left\{ b : \prod_{j=1}^n y_{b_j} > m \right\}, \quad E_m = \bigcup_{n\ge 1} E(m,n) \quad \text{and} \quad E = \bigcap_{m=1}^\infty E_m.$$

It is clear that E_m decreases as m increases, and E is exactly the set of b along which u is unbounded.

To calculate $\dim E^c$, we note that $E^c = \bigcup_{m=1}^{\infty} E_m^c$ and $E_m^c \subseteq E(m,n)^c$ for every $n \geq 1$. We shall calculate $\dim E_m^c$. Denote by X also the random variable defined by probability $P(X = \log y_j) = 1/\kappa$ for $1 \leq j \leq \kappa$; then the expectation $\mathscr{E}(e^{tX}) = \mathscr{E}(Y^t)$ attains its minimum $\beta(X)$ at $t(X) < 0$.

Let S_n be the sum $X_1 + X_2 + \cdots + X_n$ of independent identically distributed random variables with the same law as X. Then the Lebesgue measure

$$\begin{aligned}\lambda\big(E(m,n)^c\big) &= P(S_n \leq \log m) = P\big(e^{t(X)S_n} \geq m^{t(X)}\big)\\ &\leq m^{-t(X)}\big(\mathscr{E}(e^{t(X)X})\big)^n = m^{-t(X)}\beta(X)^n.\end{aligned}$$

Therefore $E(m,n)^c$ is contained in $\beta(X)^n m^{-t(X)}\kappa^n$ many balls of diameter κ^{-n}. From this we see that

$$\dim E_m^c \leq 1 + \log\beta(X)/\log\kappa \qquad \text{for all } m \geq 1.$$

Therefore

$$\dim E^c \leq 1 + \log\beta(X)/\log\kappa,$$

where

$$\beta(X) = \min_t \sum_j e^{t\log y_j} = \min_t \sum_j y_j^t.$$

We now prove $\omega_F(E) = 0$. For each $m > 1$, define u_m, an F-harmonic function by stopping time argument. Let $u_m(\phi) = u(\phi) = 1$, and suppose u_m has been defined at a certain vertex v. If $u_m(v) < m$, then let $u_m(v,j) = u(v,j)$ for $1 \leq j \leq \kappa$; if $u_m(v) \geq m$, then u_m stops and takes the value $u_m(v)$ at all successors (v,j) of v. It is clear that u_m is positive F-harmonic, and that $\lim_{v\to b} u_m(v) \geq m$ for $b \in E_m$. It follows that $\omega_F(E) \leq \omega_F(E_m) \leq u_m(\phi)/m = 1/m$ for each $m > 1$. Therefore $\omega_F(E) = 0$.

Recall, from property (vi) stated after the definition of F-harmonic functions, that for compact sets S, $\omega_F(S) + \omega_F(S^c) = 1$. Since the identity is unknown for general sets, we need to show $\omega_F(E^c) = 1$. Note that $E^c = \bigcup_m E_m^c$ and that $1 - u_m/m \leq 1$ on T with boundary values $\lim_{v\to b} 1 - u_m(v)/m \leq 0$ on E_m. It follows from the definition of F-harmonic measure and the monotonicity property of F that $\omega_F(E^c) \geq \omega_F(E_m^c) \geq 1 - u_m(\phi)/m = 1 - 1/m$. This says that $\omega_F(E^c) = 1$.

This completes the proof of Proposition 1.

Theorem 2. *Let F be an averaging operator on $\mathbf{R}^\kappa$. Then there exists a set E on ∂T such that $\omega_F(E) = 0$, $\omega_F(E^c) = 1$ and $\dim E^c \leq d(\kappa,F)$.*

The fact that $m(\kappa,F)$ is not attained by $\min_{t\leq 0}\Sigma y_j^t$ for any vector Y satisfying $F(Y) = 1$, $\prod y_j > 1$ and $y_j > 0$ for all $1 \leq j \leq \kappa$, complicates the proof of the theorem. For otherwise, we could have used an extremal Y in Proposition 1 and achieved the dimension $d(\kappa,F)$.

Proof. Assume that F is not the usual average; otherwise take $E^c = \partial T$.

Let X be an F-harmonic vector such that $\Sigma e^{x_j} = m(\kappa, F)$, then $\Sigma x_j < 0$. Let $\psi(n) = (\log^+ \log^+ n)^{-1}$ when $n > e^e$ and $\psi(n) = 1$ otherwise. Let $Y^{(n)} = (y_1^{(n)}, y_2^{(n)} \cdots y_\kappa^{(n)})$ be a sequence of vectors defined by

$$y_j^{(n)} = 1 - \psi(n) x_j$$

provided that n is so large that $y_j^{(n)} > \frac{1}{2}$ for all j; and let $Y^{(n)} = \mathbf{1}$ otherwise. Then $F(Y^{(n)}) = 1$.

Define now a positive F-harmonic function u on T by the multiplicative process in Proposition 1 with respect to the varying sequence $\{Y^{(n)}\}$. That is $u(\phi) = 1$, after $u(v)$ has been defined at a vertex $v = (v_1, v_2, \ldots, v_n)$, let $u(v, j) = y_j^{(n+1)} u(v)$ for $1 \le j \le \kappa$. So, for $v = (v_1, v_2, \ldots, v_n)$,

$$u(v) = \prod_{r=1}^{n} \bigl(1 - \psi(r) x_{v_r}\bigr).$$

Let $\eta(n) = \psi(n)^{-1}$, so $\eta(n) = \log\log n$ for $n > e^e$. We want to estimate the average M_n of $u(v)^{-\eta(n)}$ over all vertices of length n. For n large and $\sqrt{n} \le r \le n$, we have $1 \le \psi(r)\eta(n) \le 1 + 2\psi(n)$. For these r's we have a formula for the expected value

$$\begin{aligned} \mathscr{E}\bigl(1 - \psi(r)X\bigr)^{-\eta(n)} &= \mathscr{E}\bigl(e^{-\eta(n)\log(1-\psi(r)X)}\bigr) \\ &= \mathscr{E}\bigl(e^{(1+O(\psi(n)))X}\bigr) = \kappa^{-1} m(\kappa, F) + 0\bigl(\psi(n)\bigr). \end{aligned}$$

Here X is the random variable defined by $P(X = x_j) = \kappa^{-1}$. For $1 \le r < \sqrt{n}$, we use $y_j^{(r)} = 1 - \psi(r) x_j > \frac{1}{2}$.

Writing $\gamma = \log m(\kappa, F) - \log \kappa$, and applying the product formula for u, we can summarize

$$\begin{aligned} M_n &= \mathscr{E}\bigl(u(v_1, v_2, \ldots, v_n)^{-\eta(n)}\bigr) \\ &= \mathscr{E}\Biggl(\prod_{1}^{\sqrt{n}-1} \bigl(1 - \psi(r)X\bigr)^{-\eta(n)}\Biggr) \mathscr{E}\Biggl(\prod_{\sqrt{n}}^{n} \bigl(1 - \psi(r)X\bigr)^{-\eta(n)}\Biggr) \\ &= e^{\gamma n + o(n)}. \end{aligned}$$

Let

$$E_n = \Biggl\{ b : u(b_1, b_2, \ldots, b_n) = \prod_{r=1}^{n} \bigl(1 - \psi(r) x_{b_r}\bigr) > n \Biggr\},$$

and

$$E = \limsup E_n.$$

Note from the estimate in the last paragraph, the Lebesgue measure

$$\lambda(E_n^c) \le M_n n^{\eta(n)} = \exp\bigl(\gamma n + o(n)\bigr);$$

therefore E_n^c is contained in at most $\kappa^n \exp\bigl(\gamma n + o(n)\bigr)$ many balls of diameter κ^{-n} each. Since $E^c = \liminf E_n^c$, $\dim E^c \le 1 + \gamma/\log\kappa = d(\kappa, F)$.

Note that u is unbounded at each branch in E; unlike Proposition 1, the set E here does not contain all branches b along which u is unbounded. However arguing as in Proposition 1, we can show that $\omega_F(E) = 0$ and $\omega_F(E^c) = 1$. This proves Theorem 2.

Theorem 3. *Let F be a permutation invariant averaging operator, not equal to the usual average. Then there exist congruent sets $E_1, E_2, \dots, E_\kappa$ with $\cup E_j = \partial T$ such that $\omega_F(E_j) = 0$ for $1 \le j \le \kappa$. And there exist sets A and B on ∂T such that $\omega_F(A) = \omega_F(B) = 0$, however $\omega_F(A \cup B) > 0$.*

Proof. Let τ be the permutation of $1, 2, 3, \dots, \kappa$ defined by $\tau 1 = 2$, $\tau 2 = 3, \dots, \tau\kappa = 1$. Then for each state j, the images j, τj, $\tau^2 j$, $\tau^{\kappa-1} j$ are just the κ states in a different order. The powers of τ operate on the vertices of T in the obvious way, and also on ∂T. Sets A and B in ∂T are congruent if $B = \tau^q A$ for some $q = 0, 1, 2, \dots, \kappa - 1$. Clearly congruent sets have the same F-harmonic measure. Let E be the set defined in Proposition 1. Then we claim that $E \cup \tau E \cup \tau^2 E \cup \dots \cup \tau^{\kappa-1} E = \partial T$. After that the first assertion in the theorem follows. Now let $q_0 = \min\bigl\{q : F - \omega(E \cup \tau E \cup \tau^2 E \cup \dots \cup \tau^q E) > 0\bigr\}$, $A = E \cup \tau E \cup \dots \tau^{q_0 - 1} E$ and $B = \tau^{q_0} E$, and thus the second assertion follows.

To verify the claim, let Y be the vector in Proposition 1, $\delta = \prod y_j > 1$, and u be the function in the proof of Proposition 1. Note from the definition of u that for a vertex v of length n,

$$u(v) u(\tau v) u(\tau^2 v) \cdots u(\tau^{\kappa-1} v) = \delta^n.$$

Hence there is some $q = 0, 1, \dots, \kappa - 1$ such that $u(\tau^q v) \ge \delta^{n/\kappa}$. Taking an infinite branch $b = (b_1, b_2, \dots)$, we apply this to each segment $(b_1, b_2, \dots, b_n)$ obtaining a number $q(b, n)$ in $\{0, 1, 2, \dots, \kappa - 1\}$ so that

$$u\bigl(\tau^{q(b,n)}(b_1, b_2, \dots, b_n)\bigr) \ge \delta^{n/\kappa}.$$

For each b, one of the numbers q in $\{0, 1, 2, \dots, \kappa - 1\}$ occurs infinitely often in $\{q(b, n) : n \ge 1\}$. For that number q, u is unbounded on $\tau^q b$ and therefore $b \in \tau^{\kappa - q} E$. This proves the claim and thus the theorem.

Our next theorem says that there exists $E \subseteq \partial T$ of $\omega_p(E) = 0$ and $\omega_p(E^c) = 1$ for all numbers p in an interval. When $\kappa = 2$, p-harmonic functions are 2-harmonic; hence we assume $\kappa \geq 3$.

Theorem 4. *Given $p_0 > 2$, there exists $E \subseteq \partial T$ with $\dim E^c < 1$ so that $\omega_p(E) = 0$ and $\omega_p(E^c) = 1$ for all $p \geq p_0$; given $1 < p_0 < 2$, there exists $E \subseteq \partial T$ with $\dim E^c < 1$ so that $\omega_p(E) = 0$ and $\omega_p(E^c) = 1$ for all $1 < p \leq p_0$.*

Proof. Fix $p_0 \neq 2$, $1 < p_0 < \infty$; let $a = F_{p_0}(1, 0, \dots, 0)$, i.e.

$$(1-a)^{p_0-1} - (\kappa - 1)a^{p_0-1} = 0.$$

Note that $1/\kappa < a < \frac{1}{2}$ when $p_0 > 2$, and $0 < a < 1/\kappa$ when $p_0 < 2$. Follow the proof of Lemma 2; let $y_1 = 1 + (1-a)t$, $y_j = 1 - at (2 \leq j \leq k)$ and $Y = (y_1, y_2, \dots, y_\kappa)$ with t fixed ($t < 0$ when $p_0 > 2$ and $t > 0$ when $p_0 < 2$) so that $F_{p_0}(Y) = 1$, $\prod y_j > 1$ and $y_j > 0$ for $1 \leq j \leq \kappa$.

Simple calculations show that $Y - \mathbf{1}$ is p-superharmonic for $p > p_0$ when $p_0 > 2$, and p-superharmonic for $1 < p < p_0$ when $p < 2$.

Assume for now that $p_0 > 2$. Follow the construction of E and u in Proposition 1 with $F = F_{p_0}$ and the vector Y chosen above. Then u is p_0-harmonic and is p-superharmonic for all $p > p_0$. Since F-harmonic measure is defined to be the infimum of an upper class of F-superharmonic functions, the proof in Proposition 1 shows $\omega_p(E) = 0$ and $\omega_p(E^c) = 1$ for all $p \geq p_0$.

The case $p_0 < 2$ is similar.

4. Choquet capacity

Theorem 5. *Suppose F is a permutation invariant averaging operator on $\mathbf{R}^\kappa$, not equal to the usual average. Then there exists an increasing sequence of sets $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_j \subseteq \cdots \subseteq \partial T$ so that $\lim \omega_F(B_j) < \omega_F(\cup B_j)$. In other words, ω_F is not a Choquet capacity.*

A *Choquet capacity* $\mathscr{C}$ on a metric space Ω is a set function defined on all subsets of Ω into $[0, \infty]$ with the following properties:

(i) $\mathscr{C}$ is monotone: $\mathscr{C}(A) \leq \mathscr{C}(B)$ when $A \subseteq B$;

(ii) $\mathscr{C}$ is right continuous on compact sets: if $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_j \supseteq \cdots$ are compact sets then $\lim \mathscr{C}(A_j) = \mathscr{C}(\cap A_j)$;

(iii) $\mathscr{C}$ is left continuous on arbitrary sets: if $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_j \subseteq \cdots$ are arbitrary sets then $\lim \mathscr{C}(B_j) = \mathscr{C}(\cup B_j)$.

The *Capacitability Theorem of Choquet* ([C], [D]) asserts that when Ω is a complete separable metric space then all Borel subsets E of Ω are capacitable:

$$\mathscr{C}(E) = \sup\bigl\{\mathscr{C}(A) : A \text{ compact},\ A \subseteq E\bigr\}.$$

Proof of Theorem 5. Assume $\mathscr{C}$ is a Choquet capacity on a complete separable metric space Ω. Later, we let $\Omega = \partial T$ and $\mathscr{C} = \omega_F$.

Let M be another complete separable metric space and ψ a continuous function of M into Ω. Then the set function $\mathscr{C}^*(S) \stackrel{\text{def}}{=} \mathscr{C}\big(\psi(S)\big)$ is easily seen to be a Choquet capacity on M.

Let A, B be Borel sets in Ω, then A and $B\backslash A$ are continuous images $\psi_1(\mathscr{N}), \psi_2(\mathscr{N})$ of the space $\mathscr{N} = N^N$, the set of sequences of positive integers; ([K, p. 446]). The discrete union $\mathscr{N}_1 \cup \mathscr{N}_2$ of two copies of $\mathscr{N}$ is homeomorphic to $\mathscr{N}$ and we can map $\mathscr{N}_1 \cup \mathscr{N}_2$ into Ω by a continuous ψ (induced by ψ_1, and ψ_2) so that $\psi(\mathscr{N}_1) = A$ and $\psi(\mathscr{N}_2) = B\backslash A$.

Let $M = \mathscr{N}_1 \cup \mathscr{N}_2$; when L is compact in M then $L \cap \mathscr{N}_1$ and $L \cap \mathscr{N}_2$ are both compact. Applying the capacitability theorem to $\mathscr{C}^*$ on M, we see that

$$\begin{aligned}\mathscr{C}(A \cup B) = \mathscr{C}\big(\psi(M)\big) = \mathscr{C}^*(M) \\ &= \sup\{\mathscr{C}^*(L_1 \cup L_2) : L_1,\ L_2 \text{ compact and } L_1 \subseteq \mathscr{N}_1, L_2 \subseteq \mathscr{N}_2\} \\ &\le \sup\{\mathscr{C}(G_1 \cup G_2) : G_1,\ G_2 \text{ compact and } G_1 \subseteq A \text{ and } G_2 \subseteq B\backslash A\}.\end{aligned}$$

Suppose that ω_F is a Choquet capacity, and let $\Omega = \partial T$, $\mathscr{C} = \omega_F$, A and B be the sets in Theorem 3 chosen with $\omega_F(A) = \omega_F(B) = 0$ but $\omega_F(A \cup B) > 0$. Note that $\omega_F(G_1) = \omega_F(G_2) = 0$ for compact $G_1 \subseteq A$ and $G_2 \subseteq B\backslash A$, thus $\omega_F(G_1 \cup G_2) = 0$ by property (v) of F-harmonic measures stated in Section 1. The inequalities in the last paragraph would show that $\omega_F(A \cup B) = 0$. The contradiction says that ω_F can not be a Choquet capacity. Since the first two properties of Choquet capacity hold for ω_F, property (iii) must fail for ω_F. This completes the proof of Theorem 5.

5. Ellipticity and the strong maximum principle

In this section, we study the analogue of the quasilinear elliptic equation $\operatorname{div} A(\nabla u) = 0$ on a tree T.

Let $1 < p < \infty$, $q = p/(p-1)$, and $0 < \alpha \le \beta$. Let A: $\mathbf{R}^\kappa \to \mathbf{R}^\kappa$ be a continuous function satisfying the following structural conditions:

(i) $\langle AX, X\rangle \ge \alpha\|X\|_p^p$,

(ii) $\|AX\|_q \le \beta\|X\|_p^{p-1}$,

(iii) $\langle AX - AY, X - Y\rangle > 0$ for all $X \ne Y$,

(iv) $A(\lambda X) = \lambda|\lambda|^{p-2}AX$ for all $\lambda \in \mathbf{R}$;

A is an example of *monotone operator.*

Example. Let $1 < p < \infty$ and $AX = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \dots, x_\kappa|x_\kappa|^{p-2})$. Then A satisfies (i) $\sim$ (iv) with $\alpha = \beta = 1$; and $\langle AX, \mathbf{1}\rangle$ is the p-Laplacian of X.

A vector $X \in \mathbf{R}^\kappa$ is called an *A-harmonic vector* if

$$\langle AX, \mathbf{1}\rangle = 0.$$

Each monotone operator A defines an operator F_A from $\mathbf{R}^\kappa$ to $\mathbf{R}^1$ with $F_A(X) = t$ provided that

$$\langle A(X - t\mathbf{1}), \mathbf{1}\rangle = 0,$$

or $X - t\mathbf{1}$ is A-harmonic.

From (iii), we see that this equation has at most one solution. To prove that there is a solution, we show that $\lim\langle A(X - t\mathbf{1}), \mathbf{1}\rangle = \mp\infty$ as $t \to \pm\infty$; in fact $A(X + t\mathbf{1}) = t|t|^{p-2}A(\mathbf{1} + t^{-1}X)$ and A is continuous while $\langle A\mathbf{1}, \mathbf{1}\rangle > 0$.

The solution of $\langle A(X - t\mathbf{1}), \mathbf{1}\rangle = 0$ is a quasiminimizer for the p-Dirichlet sum $\sum_j |x_j - x|^p$ over all real x. To see this,

$$\begin{aligned}\|X - t\mathbf{1}\|_p^p &\le \alpha^{-1}\langle A(X - t\mathbf{1}), X - t\mathbf{1}\rangle = \alpha^{-1}\langle A(X - t\mathbf{1}), X - x\mathbf{1}\rangle\\ &\le \frac{\beta}{\alpha}\|X - t\mathbf{1}\|_p^{p-1}\|X - x\mathbf{1}\|_p\end{aligned}$$

and thus

$$\|X - t\mathbf{1}\|_p^p \le \left(\frac{\beta}{\alpha}\right)^p \|X - x\mathbf{1}\|_p^p.$$

It is easy to see that F_A has properties (i) $\sim$ (iii) of the averaging operators.

F_A satisfies (iv), the strong maximum principle, if and only if every nonzero A-harmonic vector changes sign. For each $p \in (1, \infty)$ and $\kappa > 1$, there are A's for which the strong maximum principle fails, see the second remark below.

When the ratio α/β of the ellipticity constants is close to 1, the strong maximum principle holds. Let

$$\gamma(\kappa, p) = \big((\kappa - 1)^{q-1}\big[1 + (\kappa - 1)^{q-1}\big]^{-1}\big)^{1/q}.$$

Theorem 6. *Under the assumption $\alpha/\beta > \gamma(\kappa, p)$, any nonzero A-harmonic vector changes sign, and the strong maximum principle holds for F_A.*

From Theorem A and the remark immediately after, we obtain the following.

Corollary. *Suppose that $\alpha/\beta > \gamma(\kappa, p)$. Then*

$$\min_{\mathscr{H}_A} \dim \mathscr{F}(u) = \min_{\mathscr{H}_A} \dim \mathrm{BV}(u) = d(\kappa, A),$$

where $\mathscr{H}_A$ is the set of bounded A-harmonic functions on T.

Remark. When $p = 2$, $\gamma(\kappa, 2) = (1 - 1/\kappa)^{1/2}$, and this is sharp for Theorem 6. To see this, let $\mathbf{e} = (1, 0, \dots, 0)$, then $\mathbf{e} - (1/\kappa)\mathbf{1}$ is its projection on the hyperplane $\langle Y, \mathbf{1}\rangle = 0$ and $\mathbf{u} = (1 - 1/\kappa)^{-1/2}\big(\mathbf{e} - (1/\kappa)\mathbf{1}\big)$ has length 1. Let A be an orthogonal matrix A such that $A\mathbf{e}^T = \mathbf{u}^T$ and $\langle AX, X\rangle \ge \langle \mathbf{e}, \mathbf{u}\rangle\|X\|_2^2$ for all column vectors $X \in \mathbf{R}^\kappa$. Then $\langle \mathbf{e}, \mathbf{u}\rangle = (1 - 1/\kappa)^{1/2}$ and the structural conditions are satisfied with $p = q = 2$, $\alpha = (1 - 1/\kappa)^{1/2}$ and $\beta = 1$. Since $\mathbf{e}$ is A-harmonic and does not change sign, $\gamma(\kappa, 2)$ is sharp for Theorem 6.

Remark. When $p \neq 2$, we do not have examples to show the sharpness of $\gamma(\kappa,p)$. However for each $p \neq 2$, $1 < p < \infty$, there exists A satisfying the structural conditions, for which $\mathbf{e} = (1,0,\dots,0)$ is A-harmonic. To see this, let

$$\varepsilon_0(\kappa,p) = (\kappa-1)^{-1/p}[1+(\kappa-1)^{q-1}]^{-1/p}$$

and let $X = \big((1-(\kappa-1)\varepsilon_o^p)^{1/p}, -\varepsilon_0, -\varepsilon_0, \dots, -\varepsilon_0\big)$ $\big(\varepsilon_0 = \varepsilon_0(p,\kappa)\big)$, then $\|X\|_p = 1$ and X is p-harmonic. Let P be the p-harmonic operator such that $PY = (y_1|y_1|^{p-2}, y_2|y_2|^{p-2}, \dots, y_\kappa|y_\kappa|^{p-2})$ for all $Y \in \mathbf{R}^\kappa$, and B be an invertible matrix with $B\mathbf{e}^T = X^T$ and $B\mathbf{1}^T = \mathbf{1}^T$. Let $A = B^TPB$, then $\langle A\mathbf{e}^T, \mathbf{1}^T\rangle = \langle PB\mathbf{e}^T, B\mathbf{1}^T\rangle = \langle PX^T, \mathbf{1}^T\rangle = 0$, so $\mathbf{e}$ is A-harmonic. Clearly A has properties (i) $\sim$ (iv).

The following result controls the oscillation of A-harmonic vectors, from which Theorem 6 follows by taking $C = 0$.

Proposition 2. *Let $X \neq \mathbf{0}$ be A-harmonic and $C \geq 0$. Suppose that $\alpha/\beta \geq \lambda\gamma(\kappa,p)$ for some $\lambda \in [1, \gamma(\kappa,p)^{-1}]$. Then $\max x_j \leq C$ (or $\min x_j \geq -C$) implies that*

$$\|X\|_p \leq C/\psi^{-1}(\lambda),$$

where

$$\psi(t) = t + [1-(\kappa-1)t^p]^{1/p}.$$

The function ψ is increasing in $[0, \varepsilon_0(\kappa,p)]$, $\psi(0) = 1$ and $\psi\big(\varepsilon_0(\kappa,p)\big) = \gamma(\kappa,p)^{-1}$.

If X is p-harmonic, then from the a priori bound $x_j \leq C$ it is easy to deduce that $|x_j| \leq C(\kappa-1)^{1/(p-1)}$. Proposition 2 can be considered as a generalization of this fact in the A-harmonic setting. The proof of the proposition is based on the following.

Lemma 4. *For $0 \leq \varepsilon \leq 1$, let*

$$\phi(\varepsilon,\kappa,p) = \max\big\{\langle X,Y\rangle : \|X\|_p = \|Y\|_q = 1, \langle Y, \mathbf{1}\rangle = 0,\ x_j \geq -\varepsilon \text{ for all } 1 \leq j \leq \kappa\big\}.$$

Then

$$\phi(\varepsilon,\kappa,p) = \begin{cases} \gamma(\kappa,p)\big(\varepsilon + [1-(\kappa-1)\varepsilon^p]^{1/p}\big), & \text{if } 0 \leq \varepsilon \leq \varepsilon_0, \\ 1, & \text{if } \varepsilon_0 \leq \varepsilon \leq 1. \end{cases}$$

Remark. For $0 \leq \varepsilon \leq \varepsilon_0(p,\kappa)$, the maximum $\phi(\varepsilon,\kappa,p)$ is attained when $X = \big((1-(\kappa-1)\varepsilon^p)^{1/p}, -\varepsilon, \dots, -\varepsilon\big)$ and $Y = \big(\gamma, -\gamma/(\kappa-1), \dots, -\gamma/(\kappa-1)\big)$, $\big(\gamma = \gamma(\kappa,p)\big)$; and $\phi(\varepsilon,p,\kappa) \geq \gamma(p,\kappa)$. When $\varepsilon = 0$, $X = (1,0,\dots,0)$ does not change sign and $\langle X,Y\rangle = \gamma(p,\kappa)$. Observe that $\langle X,Y\rangle = 1$ when $\varepsilon = \varepsilon_0(\kappa,p)$; Hölder's inequality implies that $\phi(\varepsilon,\kappa,p) = 1$ when $\varepsilon_0(\kappa,p) \leq \varepsilon \leq 1$.

Proof of Lemma 4. Assume as we may that $0 \le \varepsilon \le \varepsilon_0(\kappa, p)$. Before going to the proof, we need some preliminary computations. Suppose that $X, Y \in R^n$, are normalized in the sense that $\|X\|_p = \|Y\|_q = 1$ and that $\langle Y, \mathbf{1}\rangle = 0$. Assume that $y_1, \dots, y_n \ge 0$ and $y_{n+1}, \dots, y_\kappa < 0$, for some n, $1 \le n \le \kappa - 1$. Then

$$\begin{aligned}\sum_{j=1}^n y_j^q &\le \Bigl(\sum_{j=1}^n y_j\Bigr)^q = \Bigl(\sum_{j=n+1}^{\kappa} |y_j|\Bigr)^q \\ &\le (\kappa-1)^{q-1}\sum_{j=n+1}^{\kappa} |y_j|^q = (\kappa-1)^{q-1}\Bigl(1 - \sum_{j=1}^n y_j^q\Bigr),\end{aligned}$$

so

$$\sum_{j=1}^n y_j^q \le (\kappa-1)^{q-1}[1 + (\kappa-1)^{q-1}]^{-1},$$

and, analogously,

$$\sum_{j=n+1}^{\kappa} |y_j|^q \le (\kappa-1)^{q-1}[1 + (\kappa-1)^{q-1}]^{-1},$$

so

$$\sum_{j=1}^n y_j^q \ge [1 + (\kappa-1)^{q-1}]^{-1}.$$

Now, assume that $x_j \ge -\varepsilon$ for all j, and let $J = \{j : n+1 \le j \le \kappa,\ -\varepsilon \le x_j \le 0\}$. Then

$$\begin{aligned}\langle X, Y\rangle &\le \sum_{j=1}^n x_j y_j + \sum_{j\in J} |x_j|\,|y_j| \\ &\le \Bigl(\sum_{j=1}^n |x_j|^p\Bigr)^{1/p}\Bigl(\sum_{j=1}^n y_j^q\Bigr)^{1/q} + \Bigl(\sum_{j\in J} |x_j|^p\Bigr)^{1/p}\Bigl(1 - \sum_{j=1}^n y_j^q\Bigr)^{1/q}.\end{aligned}$$

Now set

$$a = \sum_{j=1}^n |x_j|^p, \quad b = \sum_J |x_j|^p, \quad c = \sum_{j=1}^n y_j^q.$$

Observe that, if $b_1 = (\kappa-1)\varepsilon^p$, $c_1 = \gamma(\kappa,p)^q$, then $a, b, c \ge 0$, $a + b \le 1$, $0 \le b \le b_1 \le 1 - c_1 \le c \le c_1 < 1$, and $\gamma(\kappa,p)\varepsilon + \gamma(\kappa,p)[1 - (\kappa-1)\varepsilon^p]^{1/p} = (1-b_1)^{1/p}c_1^{1/q} + b_1^{1/p}(1-c_1)^{1/q}$. The above considerations show that Lemma 4 is a consequence of the following

Lemma 5. *Let* $f(a,b,c) = a^{1/p}c^{1/q} + b^{1/p}(1-c)^{1/q}$ *and suppose that* $0 \le b_1 \le 1-c_1 \le c_1 < 1$. *Then*

$$\begin{aligned}\max\bigl\{f(a,b,c) : a,b,c \ge 0,\ a+b \le 1,\ 0 \le b \le b_1,\ 1-c_1 \le c \le c_1\bigr\} \\ = (1-b_1)^{1/p}c_1^{1/q} + b_1^{1/p}(1-c_1)^{1/q}.\end{aligned}$$

Proof. Note that f is increasing in a and in b, therefore at the maximum, $a+b=1$. Let

$$g(b,c) = (1-b)^{1/p}c^{1/q} + b^{1/p}(1-c)^{1/q},$$

and claim that

$$\max\bigl\{g(b,c) : 0 \le b \le b_1, 1-c_1 \le c \le c_1\bigr\} = (1-b_1)^{1/p}c_1^{1/q} + b_1^{1/p}(1-c_1)^{1/q}.$$

Fix $c \in [1-c_1, c_1]$; then $g(b,c)$ is an increasing function of b over the interval $[0, 1-c]$; since $b_1 \le 1-c_1 \le c$,

$$\max\{g(b,c) : 0 \le b \le c_1\} = g(b_1, c).$$

Again $g(b_1,c)$ is an increasing function of c over the interval $[0, 1-b_1]$ and $c_1 \le 1-b_1$, we conclude that

$$\max\{g(b_1,c) : 1-c_1 \le c \le c_1\} = g(b_1, c_1).$$

This verifies the claim and the lemma.

Proof of Proposition 2. Assume that $\alpha/\beta = \lambda\gamma(\kappa,p)$ with $1 \le \lambda \le \gamma(\kappa,p)^{-1}$. Let X be a nonzero A-harmonic vector, $X' = X/\|X\|_p$ and $Y = AX/\|AX\|_q$, then X' is also A-harmonic and

$$\|X'\|_p = \|Y\|_q = 1, \qquad \langle X', Y\rangle \ge \alpha/\beta.$$

Suppose now that $\max x_j \le C$ and let $\varepsilon = C/\|X\|_p$. We can assume that $\varepsilon \le \varepsilon_0(\kappa,p)$. Then Lemma 4, applied to X' and Y, gives

$$\lambda\gamma(\kappa,p) \le \langle X', Y\rangle \le \gamma(\kappa,p)\varepsilon + \gamma(\kappa,p)[1-(\kappa-1)\varepsilon^p]^{1/p},$$

so $\varepsilon = C/\|X\|_p \ge \psi^{-1}(\lambda)$ and Proposition 2 follows.

6. Ellipticity and the critical dimension

Fix p, q and let A be a monotone operator on $\mathbf{R}^\kappa$ and α, β the constants in (i) and (ii). Assume from now on that $\alpha/\beta > \gamma(\kappa, p)$; then F_A has properties (i) $\sim$ (iv) of the averaging operators, and $m(\kappa, A) \overset{\mathrm{def}}{=} m(\kappa, F_A) = \min\{\Sigma e^{x_j} : X \text{ is } A\text{-harmonic}\}$ and $d(\kappa, A) = \log m(\kappa, A)/\log\kappa$ are attained.

We shall compare the critical dimension $d(\kappa, A)$ with the critical dimension $d(\kappa, p)$ when the ratio α/β of the ellipticity constants is close to 1. The estimates obtained are not sharp but are enough to show that if α/β tends to 1, then $d(\kappa, A)$ tends to $d(\kappa, p)$ uniformly in A.

Theorem 7. *Fix p, q and A as above, and assume that $\alpha/\beta = \lambda\gamma(p, \kappa)$ for some $\lambda \in (1, \gamma^{-1}(p, \kappa)]$. Then*

$$\begin{aligned}\exp\{-M(p, 1-\alpha/\beta)\log\kappa/\psi^{-1}(\lambda)\} &\le m(\kappa, A)/m(\kappa, p)\\ &\le \exp\{M(p, 1-\alpha/\beta)\log\kappa/\varepsilon_0(\kappa, p)\}\end{aligned}$$

where

$$M(p,\delta) = \begin{cases} \delta^{1/p}(p^{1/p}+1), & \text{if } p \ge 2 \text{ and } 0 \le \delta \le \dfrac{q}{2p}, \\ \delta^{1/2}\left(\left(\dfrac{2q}{p}\right)^{1/2}+1\right), & \text{if } 1 < p < 2 \text{ and } 0 \le \delta \le \dfrac{p}{2q}, \\ 2, & \text{if } \delta > \min\left\{\dfrac{p}{2q}, \dfrac{q}{2p}\right\}. \end{cases}$$

Corollary. *Under the assumptions above, $d(\kappa, A) \to d(\kappa, p)$ uniformly in A as $\alpha/\beta \uparrow 1$.*

To prove Theorem 7, we associate to any A-harmonic vector X a p-harmonic vector Z (and conversely) in such a way that each z_j is close to x_j as soon as the ratio α/β is close to 1; we look at the "near equality" case in the arithmetic-geometric inequality and compare Σe^{x_j} with Σe^{z_j}.

Suppose that $0 \le x, z \le 1$. Then by the arithmetic-geometric inequality,

$$\frac{x^p}{p} + \frac{z^p}{q} - xz^{p-1} \ge 0$$

with equality if and only if $x = z$. The following two lemmas give estimates on $|x - z|$ and $x + z$ in the "near equality" situation.

Lemma 6. *Let $0 \le \delta \le 1$ and*

$$M^+(p,\delta) = \max\left\{|x-z| : 0 \le x, z \le 1,\ \frac{x^p}{p} + \frac{z^p}{q} - xz^{p-1} = \delta\right\}.$$

Then

$$M^+(p,\delta) \le \begin{cases} (\delta p)^{1/p}, & \text{when } p \ge 2, \\ \left(\dfrac{2\delta q}{p}\right)^{1/2}, & \text{when } 1 < p < 2. \end{cases}$$

Lemma 7. *Let* $0 \le \delta \le 1$ *and*

$$M^-(p,\delta) = \max\left\{x + z : 0 \le x, z \le 1,\ \frac{x^p}{p} + \frac{z^p}{q} + xz^{p-1} = \delta\right\}.$$

Then

$$M^-(p,\delta) \le \delta^{1/p}(\max\{p^{1/p}, q^{1/p}\} + 1).$$

Proof of Lemma 6. Case 1: $p \ge 2$. Suppose that $z \le x = z + h$. By differentiation on h, it follows that

$$\frac{(z+h)^p}{p} + \frac{z^p}{q} - (z+h)z^{p-1} \ge \frac{h^p}{p}.$$

If $x \le z = x + h$, then also by differentiation we get

$$\frac{x^p}{p} + \frac{(x+h)^p}{q} - x(x+h)^{p-1} \ge \frac{h^p}{q} \ge \frac{h^p}{p},$$

so

$$\frac{x^p}{p} + \frac{z^p}{q} - xz^{p-1} \ge \frac{|x-z|^p}{p}.$$

Case 2: $1 < p < 2$. Also by differentiation we get now that

$$\frac{x^p}{p} + \frac{z^p}{q} - xz^{p-1} \ge \frac{p-1}{2}(x-z)^2$$

which proves the lemma.

Proof of Lemma 7. If $x \ge z$, then $z^p \le xz^{p-1} \le \delta$ and $x \le (\delta p)^{1/p}$ so, $x + z \le (\delta)^{1/p}(1 + p^{1/p})$. Analogously, if $z \ge x$, we get $x + z \le \delta^{1/p}(1 + q^{1/p})$ and therefore

$$M^-(p,\delta) \le \delta^{1/p}(\max\{p^{1/p}, q^{1/p}\} + 1).$$

Lemma 8. *Let* X *and* Y *be vectors in* $\mathbf{R}^\kappa$ *which satisfy* $\|X\|_p = \|Y\|_q = 1$, $\langle Y, \mathbf{1}\rangle = 0$ *and* $\langle X, Y\rangle \ge 1 - \delta$ *for some* $\delta \in [0,1]$. *Let* $Z \in \mathbf{R}^\kappa$ *be chosen so that* $y_j = z_j|z_j|^{p-2}$ *for* $1 \le j \le \kappa$. *Then* $\max|x_j - z_j| \le M(p,\delta)$, *where* M *is the function in the statement of Theorem* 7.

Proof. Observe that Z is a p-harmonic vector and that

$$\|Z\|_p = 1, \qquad 1-\delta \le \Sigma x_j z_j |z_j|^{p-2} \le 1.$$

Suppose for now that $0 \le \delta \le \min\{p/2q, q/2p\}$ and let $J^+ = \{j : x_j z_j > 0\}$ and $J^- = \{1,2,\ldots,\kappa\}\backslash J^+$. Suppose $i \in J^+$, then

$$\begin{aligned} 1-\delta \le \langle X, Y\rangle &\le \sum_{J^+} |x_j|\,|z_j|^{p-1} \le |x_i|\,|z_i|^{p-1} + \sum_{J^+\backslash\{i\}} \left(\frac{|x_j|^p}{p} + \frac{|z_j|^p}{q}\right) \\ &\le |x_i|\,|z_i|^{p-1} + \frac{1-|x_i|^p}{p} + \frac{1-|z_i|^p}{q}, \end{aligned}$$

so

$$\frac{|x_i|^p}{p} + \frac{|z_i|^p}{q} - |x_i|\,|z_i|^{p-1} \le \delta;$$

and $|x_i - z_i| \le M^+(p,\delta)$. On the other hand, if $i \in J^-$, then the same argument shows that

$$\frac{|x_i|^p}{p} + \frac{|z_i|^p}{q} + |x_i|\,|z_i|^{p-1} \le \delta$$

and that $|x_i - z_i| = |x_i| + |z_i| \le M^-(p,\delta)$. The lemma follows from the estimates of M^+ and M^- in Lemmas 6 and 7.

Proof of Theorem 7. Let X be a nonzero A-harmonic vector and set $X' = X/\|X\|_p$ and $Y' = AX/\|AX\|_q$. Choose $Z' \in \mathbf{R}^\kappa$ so that $y_j = z'_j|z'_j|^{p-2}$ for all j, and define $Z = \|X\|_p Z'$. We apply Lemma 8 to X', Z', $\delta = 1-\alpha/\beta$ to get

$$\max|x'_j - z'_j| \le M(p, 1-\alpha/\beta),$$

so

$$\max|x_j - z_j| \le M(p, 1-\alpha/\beta)\|X\|_p;$$

consequently

$$\exp\{-M(p,1-\alpha/\beta)\|X\|_p\} \le \Sigma e^{x_j}/\Sigma e^{z_j} \le \exp\{M(p,1-\alpha/\beta)\|X\|_p\}.$$

Suppose now that $\alpha/\beta = \lambda\gamma(\kappa,p)$ and that X is an extremal vector for $m(\kappa,A)$. Since $\mathbf{0}$ is also A-harmonic, we must have $\Sigma e^{x_j} \le \kappa$ and $\max x_j \le \log\kappa$. Apply Proposition 2 to the extremal vector X, we obtain $\|X\|_p \le \log\kappa/\psi^{-1}(\lambda)$. Therefore

$$\exp\bigl\{-M(p,1-\alpha/\beta)\log\kappa/\psi^{-1}(\lambda)\bigr\} \le m(\kappa,A)/m(\kappa,p).$$

To reverse the argument, we need some topological properties of the mapping A. First observe that by the monotonicity (iii), A is one to one; and by the homogeneity (iv), the mapping

$$BX = AX/\|AX\|_q$$

is also one to one from $S_p = \{X : \|X\|_p = 1\}$ to $S_q = \{Y : \|Y\|_q = 1\}$. As a consequence of Borsuk's Antipodal Theorem [Du, Chapter 16], this implies that B is also surjective, and again from homogeneity, it follows that A is surjective too.

Now choose a p-harmonic vector Z, extremal for $m(\kappa,p)$ and define Y by letting $y_j = z_j|z_j|^{p-2}/\|Z\|_p^{p-1}$ for $1 \le j \le \kappa$. Then $\|Y\|_q = 1$ and $\langle Y, \mathbf{1}\rangle = 0$. Choose X' with $\|X'\|_p = 1$ such that $BX' = Y$ and set $X = \|Z\|_p X'$. Then $Y = AX/\|AX\|_q$. Apply Proposition 2 to the p-harmonic case (where $\alpha = \beta = 1$), we get $\|X\|_p = \|Z\|_p \le \log\kappa/\varepsilon_0(\kappa,p)$; therefore

$$m(\kappa,A)/m(\kappa,p) \le \exp\bigl\{M(p, 1-\alpha/\beta)\log\kappa/\varepsilon_0(\kappa,p)\bigr\}.$$

This completes the proof of Theorem 7.

7. 1-Laplacian and ∞-Laplacian

Suppose we consider 1-Laplacian Δ_1 as the limit of Δ_p when $p \to 1^+$, and define $F_1(X) = t$ when

$$\lim_{p\to 1^+} \sum_j (x_j - t)|x_j - t|^{p-2} = 0;$$

we note immediately that this definition leads to confusion even in the simple situations, e.g. $F_1(1,2,3)$ and $F_1(1,1,3)$.

Therefore we define $F_1(X)$ through minimizing 1-Dirichlet sums. Given a vector $X \in \mathbf{R}^\kappa$, the possible values of the average $F_1(X)$ are the numbers $x = t$ that minimize the sum $\sum_{j=1}^{\kappa} |x_j - x|$ over all $x \in \mathbf{R}^1$.

Suppose that t is the unique minimizer, and let n^+, n^0, n^- be the number of x_j's that are greater than, equal to, or less than t, respectively. Then, necessarily, $n^- + n^0 > n^+$ and $n^+ + n^0 > n^-$, or simply $n^0 > |n^+ - n^-|$; this is also a sufficient condition that t be the sole minimizer. In this case, define $F_1(X) = t$.

If there is more than one minimizer, then the minimizers occupy an interval $[a,b]$, whose endpoints are among the x_j's. This occurs if and only if κ is even and there are exactly $\frac{1}{2}\kappa$ terms $x_j \ge b$, $\frac{1}{2}\kappa$ terms $x_j \le a$. We then define $F_1(X) = \frac{1}{2}(a+b)$.

The operator F_1 has properties (i) $\sim$ (iii) for the averaging operators. Since $F_1(x,0,0,\dots,0) = 0$ for any $(x,0,\dots,0) \in \mathbf{R}^\kappa$ when $\kappa \ge 3$, F_1 lacks the strong maximum principle.

We say X is a 1-*harmonic vector* if $F_1(X) = 0$.

Proposition 3. *Let* $m(\kappa,1)=\inf\{\Sigma e^{x_i} : X$ *is* 1*-harmonic*$\}$ *and* $d(\kappa,1)=\log m(\kappa,1)/\log\kappa$. *Then* $m(\kappa,1)=\left[\frac{1}{2}\kappa\right]+1$ *and* $d(\kappa,1)=\log\left(\left[\frac{1}{2}\kappa\right]+1\right)/\log\kappa$.

In finding the infimum, we make the negative coordinates tend to $-\infty$; the infimum is not attained when $\kappa\ge 3$.

Proof. Let X be 1-harmonic. In the case that 0 is the sole minimizer, we saw that $n^+ + n^0 > n^-$. If κ is even, then $n^+ + n^0 \ge 1+\frac{1}{2}\kappa$ and $\Sigma e^{x_i}\ge 1+\frac{1}{2}\kappa$. Choosing X with $n^0=1+\frac{1}{2}\kappa$, $n^-=-1+\frac{1}{2}\kappa$ and $n^+=0$, we see that the infimum in this special case is $1+\frac{1}{2}\kappa$. Next if κ is odd, we have $n^+ + n^0\ge\frac{1}{2}(\kappa+1)$ and $\Sigma e^{x_j}\ge\frac{1}{2}(\kappa+1)$; choosing vectors X with $n^0=\frac{1}{2}(\kappa+1)$, $n^-=\frac{1}{2}(\kappa-1)$ and $n^+=0$, we see that the infimum is $\frac{1}{2}(\kappa+1)$ in this special case.

In the second case, there is a pair of x_j's equal to $a>0$ and $-a<0$, and there are in addition exactly $\frac{1}{2}(\kappa-2)$ positive x_i's. Since $e^a+e^{-a}\ge 2$, $\Sigma e^{x_j}\ge 2+\frac{1}{2}(\kappa-2)=1+\frac{1}{2}\kappa$. And the infimum of Σe^{x_j} in the second case is $1+\frac{1}{2}\kappa$.

Remark. Alternative definitions of $F_1(X)$ in the case there is more than one minimizer will give different values of $m(\kappa,1)$ and $d(\kappa,1)$.

As for the ∞-Laplacian, first we regard Δ_∞ as the limit of Δ_p when $p\to\infty$, and define $F_\infty(X)=t$ provided that

$$\lim_{p\to\infty}\sum_j (x_j-t)|x_j-t|^{p-2}=0.$$

This definition leads to problems when the number of coordinates equal to $\max x_j$ is not the same as the number of coordinates equal to $\min x_j$.

Therefore we define $F_\infty(X)=t$ if t is the minimizer for $\max_i|x_i-x|$ over all x. It is easy to see that

$$F_\infty(X)=\tfrac{1}{2}(\max x_j+\min x_j).$$

Clearly F_∞ has all properties (i) $\sim$ (iv) for averaging operators. However F_∞ is not strictly increasing with respect to each variable, because $F(1,-1,0,\dots,0)=F(1,-1,-1,\dots,-1)=0$ when $\kappa\ge 3$.

We say X is ∞*-harmonic* if $F_\infty(X)=0$. Simple calculations give the following.

Proposition 4. *Let* $m(\kappa,\infty)=\min\{\Sigma e^{x_i} : X$ *is* ∞*-harmonic*$\}$ *and*

$$d(\kappa,\infty)=\log m(\kappa,\infty)/\log\kappa.$$

Then the minimum $m(\kappa,\infty)$ *is attained at*

$$\Big(\frac{1}{2}\log(\kappa-1),-\frac{1}{2}\log(\kappa-1),\dots,-\frac{1}{2}\log(\kappa-1)\Big)$$

with value $2\sqrt{\kappa-1}$ *; and* $d(\kappa,\infty)=\log\big(2\sqrt{\kappa-1}\,\big)/\log\kappa$.

References

[ARY] Alvarez, V., J. M. Rodríguez, and D. V. Yakubovich: Estimates for nonlinear harmonic "measures" on trees. - Michigan Math. J. 49, 2001, 47–64.

[AM] Aviles, P., and J. J. Manfredi: On null sets of p-harmonic measures. - In: Partial Differential Equations with Minimal Smoothness and Applications (Chicago, IL 1990), edited by B. Dahlberg et al., Springer-Verlag, New York, 1992, 33–36.

[CFPR] Cantón, A., J. L. Fernández, D. Pestana, and J. M. Rodríguez: On harmonic functions on trees. - Potential Anal. 15, 2001, 199–244.

[C] Choquet, G.: Theory of capacities. - Ann. Inst. Fourier (Grenoble) 5, 1953–54, 131–295.

[D] Doob, J. L.: Classical Potential Theory and its Probabilistic Counterpart. - Springer-Verlag, New York, 1984.

[Du] Dugundji, J.: Topology. - Allyn and Bacon, Inc., Boston, 1966.

[FGMS] Fabes, E., N. Garofalo, S. Marín-Malavé, and S. Salsa: Fatou theorems for some nonlinear elliptic equations. - Rev. Mat. Iberoamericana 4, 1988, 227–251.

[GLM] Granlund, S., P. Lindqvist, and O. Martio: F-harmonic measure in space. - Ann. Acad. Sci. Fenn. Math. 7, 1982, 233–247.

[HKM] Heinonen, J., T. Kilpeläinen, and O. Martio: Nonlinear Potential Theory of Degenerate Elliptic Equations. - Oxford Science Publications, 1993.

[KW] Kaufman, R., and J.-M. Wu: Fatou theorem of p-harmonic functions on trees. - Ann. Probab. 28, 2000, 1138–1148.

[K] Kuratowski, K.: Topology I. - Academic Press, New York, 1966.

[L] Lewis, J. L.: Note on a theorem of Wolff. - In: Holomorphic Functions and Moduli, Vol. 1 (Berkeley, CA, 1986), edited by D. Drasin et al., Math. Sci. Res. Inst. Publ. 10, Springer-Verlag, 1988, 93–100.

[M] Martio, O.: Potential theoretic aspects of nonlinear elliptic partial differential equations. - Report 44, University of Jyväskylä, Jyväskylä, 1989.

[MW] Manfredi, J. J., and A. Weitsman: On the Fatou theorem for p-harmonic functions. - Comm. Partial Differential Equations 13, 1988, 651–658.

[W] Wolff, T.: Gap series constructions for the p-Laplacian. - Preprint, 1984

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