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LENGTH FUNCTIONS AND PARAMETERIZATIONS OF TEICHMÜLLER SPACE FOR SURFACES WITH CUSPS

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Abstract. For $g \geq 0$ and $m \geq 1$ such that $2g-2+m \geq 1$ let $\mathscr{T}_{g,m}$ be the Teichmüller space of hyperbolic metrics on a surface of genus g with m punctures, and let $\partial\mathscr{T}_{g,m}$ be its Thurston boundary. Using geodesic length functions, we construct a homeomorphism of $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$ onto a convex finite-sided polyhedron in $\mathbf{R}P^{6g-6+2m}$.

1. Introduction

A *Riemann surface of finite type* is a closed Riemann surface from which a finite number $m \geq 0$ of points, the so-called *punctures*, have been deleted. Such a surface S_0 is topologically determined by its *genus* $g \geq 0$ and the number m of its punctures. In the sequel we only consider surfaces S_0 of negative Euler characteristic with at least one puncture which are different from the thrice-punctured sphere. Then S_0 carries a nontrivial family of complete hyperbolic metrics of finite volume.

The *Teichmüller space* $\mathscr{T}_{g,m}$ of *marked* hyperbolic metrics on S_0 is the set of all pairs (f,h) where h is a hyperbolic metric on a surface S and f is the homotopy class of a homeomorphism $F\colon S_0 \to S$ of S_0 onto S. With respect to a natural topology, the space $\mathscr{T}_{g,m}$ is homeomorphic to an open cell of dimension $6g-6+2m$.

A *geodesic length function* on $\mathscr{T}_{g,m}$ is defined by the choice of a closed curve γ on the base surface S_0 which is not nullhomotopic and not puncture parallel, i.e. which cannot be homotoped into one of the punctures. For every hyperbolic metric h on a surface S which is marked by the homotopy class of a homeomorphism $F\colon S_0 \to S$ the curve $F(\gamma)$ is then freely homotopic to a unique closed geodesic with respect to the metric h. The length $l_\gamma(S)$ of this geodesic depends on h and the marking and defines a smooth (in fact real analytic) function l_γ on $\mathscr{T}_{g,m}$ which we call the *length function* of γ.

It is well known [FLP] that Teichmüller space can be parameterized by finitely many of these length functions. A natural problem is then to find a collection

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$\gamma_1, \dots, \gamma_k$ of closed geodesics on our base surface S_0 of minimal cardinality whose length functions define an embedding of $\mathscr{T}_{g,m}$ into $\mathbf{R}^k$. For surfaces with cusps, i.e. if $m \geq 1$, this problem was solved by Seppälä and Sorvali [SS2] who showed that $\mathscr{T}_{g,m}$ can be parameterized by $6g-6+2m$ length functions of simple closed geodesics. For closed surfaces it is known that no embedding of $\mathscr{T}_{g,0}$ into $\mathbf{R}^{6g-6}$ by $6g-6$ geodesic length functions exists. In this case the optimal answer was given by Schmutz Schaller [S1] who found an embedding of $\mathscr{T}_{g,0}$ by $6g-5$ geodesic length functions. In Section 4 of this note we construct a new and simpler such embedding of $\mathscr{T}_{g,0}$ into $\mathbf{R}^{6g-5}$.

The *Thurston boundary* $\partial\mathscr{T}_{g,m}$ of $\mathscr{T}_{g,m}$ consists of projective classes of measured geodesic laminations and is homeomorphic to a sphere of dimension $6g-7+2m$. It defines naturally a compactification of $\mathscr{T}_{g,m}$ in such a way that $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$ is homeomorphic to a closed ball.

To construct parameterizations of $\mathscr{T}_{g,m}$ which extend continuously to a homeomorphism of the Thurston boundary we have to projectivize our k-tuple of length functions. Namely, if the curves $\gamma_1, \dots, \gamma_k$ *fill up*, i.e. if every closed geodesic on our surface intersects at least one of the curves γ_i transversely, then the map which assigns to a surface $S \in \mathscr{T}_{g,m}$ the projectivized k-tuple

$$\bigl[l_{\gamma_1}(S), \dots, l_{\gamma_k}(S)\bigr] \in \mathbf{R}\,P^{k-1}$$

of length functions extends continuously to the Thurston boundary by mapping a measured lamination η to its projectivized k-tuple $\bigl[i(\eta,\gamma_1), \dots, i(\eta,\gamma_k)\bigr] \in \mathbf{R}\,P^{k-1}$ of intersection numbers. We call a map of $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$ into $\mathbf{R}\,P^{k-1}$ *geometric* if it is defined in this way by geodesic length functions.

Call a subset P of the real projective space $\mathbf{R}\,P^{k-1}$ a *finite-sided convex polyhedron* if it is the projection of an intersection of finitely many closed halfspaces in $\mathbf{R}^k$. We show:

Theorem. *For every $g \geq 0$ and $m \geq 1$ there is a geometric homeomorphism of $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$ onto a finite-sided convex polyhedron in $\mathbf{R}\,P^{6g-6+2m}$.*

The case of closed surfaces seems to be much more difficult. A geometric embedding of $\mathscr{T}_{g,0}$ into a real projective space of minimal dimension is only known for $g=2$ [S2]. In any case it can be easily computed that the projections into $\mathbf{R}\,P^{6g-6}$ of the known embeddings of $\mathscr{T}_{g,0}$ into $\mathbf{R}^{6g-5}$ do not extend to injective maps on the Thurston boundary. On the other hand, there are $6g-5$ length functions of simple closed geodesics which define a homeomorphism of the Thurston boundary $\partial\mathscr{T}_{g,0}$ onto the boundary of a finite-sided convex polyhedron in $\mathbf{R}\,P^{6g-5}$ [H]. We do not know whether this embedding extends to an embedding of $\mathscr{T}_{g,0}$ into $\mathbf{R}\,P^{6g-5}$.

2. Triangulations and laminations on surfaces with cusps

In this section we construct a homeomorphism of the space of measured laminations on a surface $S \in \mathscr{T}_{g,m}$ with $m \geq 1$ cusps onto the boundary of a convex cone in $\mathbf{R}^{6g-5+2m}$.

First we look briefly at closed surfaces. A *signed geodesic current* on a closed surface S of genus $g \geq 2$ is a locally finite signed Borel-measure on the space of unoriented geodesics in the hyperbolic plane $\mathbf{H}^2$ which is invariant under the action of the fundamental group $\pi_1(S)$ of S. The space $\mathscr{S}\mathscr{C}$ of signed geodesic currents for S, equipped with the weak*-topology, is a topological vector space which only depends on the topological type of S. It contains the space $\mathscr{C}$ of *geodesic currents* which consists of all nonnegative elements of $\mathscr{S}\mathscr{C}$ as a closed subcone.

There is a bilinear form i on $\mathscr{S}\mathscr{C}$, the so called *intersection form*, whose restriction to $\mathscr{C}$ is continuous with respect to the weak*-topology [B], but it is not continuous globally as a form on $\mathscr{S}\mathscr{C}$. The subset $\mathscr{L}$ of $\mathscr{C}$ of all geodesic currents $\mu \in \mathscr{C}$ with vanishing self-intersection $i(\mu, \mu) = 0$ is the closed cone of *measured geodesic laminations* and is homeomorphic to $\mathbf{R}^{6g-6}$ [FLP]. However, $\mathscr{L}$ is not contained in any finite-dimensional linear subspace of $\mathscr{S}\mathscr{C}$. The projectivization $\mathscr{P}\mathscr{L}$ of the space of nonzero measured geodesic laminations defines a compactification of the Teichmüller space $\mathscr{T}_{g,0}$ which is called the *Thurston boundary*. Every closed geodesic ψ on S can naturally be viewed as a geodesic current and hence via $\mu \to i(\psi, \mu)$ it defines a linear functional on $\mathscr{S}\mathscr{C}$ whose restriction to $\mathscr{C}$ and hence to $\mathscr{L}$ is continuous.

We can also consider the space $\mathscr{L}$ of measured geodesic laminations on hyperbolic surfaces with cusps. By definition, a measured geodesic lamination for such a surface S with $m \geq 1$ cusps is a *compact* subset of S which is foliated by geodesics and equipped with a transverse invariant measure.

Now let $m \geq 1$ and let $S \in \mathscr{T}_{g,m}$. Fix one of the cusps of S and denote it by O. Choose $6g - 5 + 2m$ simple mutually disjoint geodesics $\tilde{\eta}_1, \ldots, \tilde{\eta}_{6g-5+2m}$ on S whose two ends go into the cusp O and which decompose S into $4g - 3 + m$ ideal triangles and $m - 1$ once-punctured discs (see [S3]).

If ψ is any closed geodesic on S then ψ is contained in a compact subset of S and hence it intersects each of the geodesics $\tilde{\eta}_i$ transversely in a finite number of points. We denote by $i(\psi, \tilde{\eta}_i)$ the number of intersections of ψ with $\tilde{\eta}_i$. Since measured laminations on S have compact support, intersection of closed geodesics with one of the curves $\tilde{\eta}_i$ extends to a continuous convex-linear functional $i(\,\cdot\,, \tilde{\eta}_i)$ on the space $\mathscr{L}$.

The intersection of each closed geodesic ψ on S with one of the ideal triangles T cut out by the geodesics $\tilde{\eta}_i$ consists of a finite number of simple arcs. Each of these arcs has its endpoints on two different sides of T. In other words, the number of intersections of ψ with a fixed side of T is not smaller than the sum of the number of intersections with the two other sides. In particular, the $6g -$

$5+2m$-tuple of intersection numbers of ψ with the geodesics $\tilde{\eta}_i$ is contained in the intersection of a collection of $10g-8+3m$ halfspaces bounded by a linear hyperplane through the origin. The boundary $\mathscr{A} \subset \mathbf{R}^{6g-5+2m}$ of this convex cone equals the set of all $6g-5+2m$-tuples $(a_1, \dots, a_{6g-5+2m})$ of nonnegative real numbers with the following properties:

(1) $a_i \le a_j + a_k$ if the geodesics $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ are the sides of an ideal triangle in S.

(2) There is at least one ideal triangle in S with sides $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ and such that $a_i = a_j + a_k$.

In particular, $\mathscr{A}$ is a cone with vertex at the origin over the boundary ∂P of a convex polyhedron P in the unit sphere $S^{6g-6+2m} \subset \mathbf{R}^{6g-5+2m}$. Since the boundary of a convex polyhedron in $S^{6g-6+2m}$ is homeomorphic to a sphere of dimension $6g-7+2m$, our set $\mathscr{A}$ is homeomorphic to $\mathbf{R}^{6g-6+2m}$.

We summarize our discussion in the following lemma.

Lemma 2.1. *$\mathscr{A}$ is a cone with vertex at the origin over the boundary of a convex finite-sided polyhedron in the sphere $S^{6g-6+2m}$. In particular, $\mathscr{A}$ is homeomorphic to $\mathbf{R}^{6g-6+2m}$.*

Lemma 2.2. *The map $\mu \in \mathscr{L} \to \big(i(\tilde{\eta}_1, \mu), \dots, i(\tilde{\eta}_{6g-5+2m}, \mu)\big) \in \mathbf{R}^{6g-5+2m}$ is a homeomorphism of $\mathscr{L}$ onto $\mathscr{A}$.*

Proof. For $\mu \in \mathscr{L}$ write $\Phi(\mu) = \big(i(\tilde{\eta}_1, \mu), \dots, i(\tilde{\eta}_{6g-5+2m}, \mu)\big)$. We show first that the map Φ is injective. By continuity, for this it suffices to show that every simple closed geodesic multicurve γ (i.e. a finite union of pairwise disjoint simple closed geodesics, possibly multiple covered) is determined by $\Phi(\gamma)$.

By assumption, the arcs $\tilde{\eta}_i$ define a decomposition of S into $4g-3+m$ ideal triangles with vertices at the cusp O and $m-1$ punctured discs. Each arc is either the common side of exactly two triangles or it is the common side of one triangle and one once-punctured disc. Let γ be a simple closed geodesic multicurve on S and let T be a triangle from our triangulation of S with sides $\beta_1, \beta_2, \beta_3$. Write $j_i = i(\beta_i, \gamma)$ and assume that $j_1 \ge j_2 \ge j_3$. Since the interior of T is contractible in the compactification of S, the total intersection number $j_1 + j_2 + j_3$ of γ with the boundary of T is even and hence $j_2 + j_3 - j_1$ is even as well. Moreover we have $j_1 \le j_2 + j_3$. Draw $\frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_2 and β_3, $j_2 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and β_2, $j_3 - \frac{1}{2}(j_2 + j_3 - j_1)$ simple arcs connecting the sides β_1 and β_3 in such a way that all these arcs are disjoint. The configuration of these arcs in T is uniquely determined up to homotopy by j_1, j_2, j_3.

If $\tilde{\eta}_i$ is the boundary of a once-punctured disc D then each connected component of $\gamma \cap D$ has its two endpoints on $\tilde{\eta}_i$ and therefore $i(\gamma, \tilde{\eta}_i) = 2k$ for some $k \ge 0$. Draw k simple arcs in D with endpoints on $\tilde{\eta}_i$ and the additional property that each of these arcs separates the puncture in the interior of D from the puncture on the boundary. Once again, this configuration is determined uniquely up to homotopy by the number $2k$.

The thus constructed arcs in the $4g-3+m$ triangles and $m-1$ punctured discs of our triangulation can be connected in a unique way to a simple closed multicurve ψ on S. This multicurve is determined up to isotopy by the intersection numbers with the geodesics $\tilde\eta_1, \dots, \tilde\eta_{6g-5+2m}$.

To show that ψ is isotopic to γ we just have to verify that each connected component of the intersection of γ with a punctured disc D separates the puncture in the interior from the puncture on the boundary. But this follows from the fact that those arcs are the only simple arcs in D which are not freely homotopic relative to the boundary of D to an arc contained in the boundary. Thus γ is uniquely determined by $\Phi(\gamma)$ (compare the discussion in [FLP]) and Φ is injective.

We are left with showing that the image of $\mathscr{L}$ under the map Φ equals the cone $\mathscr{A}$. We show first that $\Phi\mathscr{L}$ is contained in $\mathscr{A}$. For this recall from our consideration above that every $6g-5+2m$-tuple $(b_1, \dots, b_{6g-5+2m})$ of even nonnegative integers with the additional property that $b_i \le b_j + b_k$ whenever the geodesics $\tilde\eta_i, \tilde\eta_j, \tilde\eta_k$ are the sides of an ideal triangle determines uniquely a (possibly multiple covered) simple closed multicurve (compare [FLP]). However, not every such multicurve is freely homotopic to a simple geodesic multicurve. Namely, a simple closed curve which is parallel to the cusp O corresponds to the $6g-5+2m$-tuple $(2, \dots, 2)$. Since the homotopy class of this curve cannot be represented by a simple closed geodesic, a $6g-5+2m$-tuple of equal even positive integers does not occur as the intersection tuple of a geodesic lamination.

Denote by $\mathscr{B} \subset \mathbf{R}^{6g-5+2m}$ the set of all $6g-5+2m$-tuples $(b_1, \dots, b_{6g-5+2m})$ of nonnegative numbers with the additional property that $b_i \le b_j + b_k$ if $\tilde\eta_i, \tilde\eta_j, \tilde\eta_k$ are the sides of a triangle in S. If $(b_1, \dots, b_{6g-5+2m}) \in \mathscr{B}$ is such that for each triangle in S with sides $\tilde\eta_i, \tilde\eta_j, \tilde\eta_k$ the strict inequality $b_i < b_j + b_k$ holds, then there are $\beta > 0$ and $(a_1, \dots, a_{6g-5+2m}) \in \mathscr{A}$ such that $(b_1, \dots, b_{6g-5+2m}) = (a_1, \dots, a_{6g-5+2m}) + \beta(2, \dots, 2)$. As before, if the numbers a_i are even integers, then the vector $(a_1, \dots, a_{6g-5+2m})$ defines a unique simple closed multicurve μ in S. Let γ be a simple closed curve which is parallel to the cusp O and which does not intersect μ. Then $\mu \cup \beta\gamma$ is a weighted multicurve whose intersection tuple equals $(b_1, \dots, b_{6g-5+2m})$. By uniqueness, $(b_1, \dots, b_{6g-5+2m})$ does not define a measured lamination. From continuity we therefore conclude that the image of Φ is contained in $\mathscr{A}$.

To show that Φ maps $\mathscr{L}$ onto $\mathscr{A}$, recall that a tuple $(m_1, \dots, m_{6g-5+2m}) \in \mathscr{A}$ with even integers as coefficients determines uniquely up to isotopy a simple closed multicurve μ which does not contain any puncture parallel component. In particular, μ is isotopic to a geodesic multicurve γ. Our discussion above (compare also [FLP]) shows that the $6g-5+2m$-tuple of intersection numbers for γ with the geodesics $\tilde\eta_i$ coincides with $(m_1, \dots, m_{6g-5+2m})$. But this just means that we can define a map from $\mathscr{A}$ to $\mathscr{L}$ which is inverse to Φ. This finishes the proof of the lemma. ▫

As a corollary, we find the following well-known fact [FLP].

Corollary 2.3. *The space $\mathscr{L}$ of measured geodesic laminations on a surface of genus $g \geq 1$ with $m \geq 1$ cusps is homeomorphic to an open cell of dimension $6g-6+2m$.*

We assign now to each of the geodesics $\tilde{\eta}_i$ a closed geodesic with two self-intersections as follows. For each i the geodesic $\tilde{\eta}_i$ is contained in a unique once-punctured annulus $A \subset S$ with geodesic boundary. Let ψ_i be the closed geodesic in A with two self-intersections which intersects the perpendicular ν between the boundary geodesics of A twice and is invariant under reflection along ν. The next lemma shows that measured laminations on a surface of genus g with $m \geq 1$ punctures can be parameterized by their intersections with the $6g-5+2m$ closed geodesics $\psi_1, \ldots, \psi_{6g-5+2m}$.

Lemma 2.4. *Let $\mu \in \mathscr{L}$ be a measured geodesic lamination. Then $i(\mu, \psi_i) = 2i(\mu, \tilde{\eta}_i)$ for all i; in particular, the map*

$$\Psi \colon \mu \to \big(i(\mu, \psi_1), \ldots, i(\mu, \psi_{6g-5+2m})\big)$$

is a homeomorphism of $\mathscr{L}$ onto $\mathscr{A}$.

Proof. As before, it is enough to show the statement of the lemma for simple closed geodesic multicurves. Consider again the once-punctured annulus A containing $\tilde{\eta}_i$ and ψ_i. Its boundary consists of simple closed geodesics $\sigma, \hat{\sigma}$. There is no simple closed geodesic contained in the interior of A, so the intersection with A of each simple closed geodesic multicurve is a finite collection of simple arcs with endpoints on the boundary. There are only 3 different free homotopy classes of such simple arcs in A relative to the boundary. For each of these classes, it can be checked explicitly that the number of its intersections with the geodesic ψ_i is twice the number of its intersections with $\tilde{\eta}_i$. ▫

3. Length functions for surfaces with cusps

In Section 2 we constructed for every $g \geq 0$ and $m \geq 1$ such that $2g-2+m \geq 1$ a collection of $6g-5+2m$ free homotopy classes on a surface of genus g with m cusps which parameterize the space of measured geodesic laminations $\mathscr{L}$ via intersection. The projectivization of this parameterization defines a homeomorphism of the Thurston boundary of $\mathscr{T}_{g,m}$ onto the boundary of a finite-sided convex polyhedron P in $\mathbf{R}P^{6g-6+2m}$. The purpose of this section is to show that the length functions of these curves define a homeomorphism of Teichmüller space onto the interior of P. For this we continue to use the assumptions and notations from Section 2.

As in Section 2 we fix one of the punctures of a surface $S \in \mathscr{T}_{g,m}$ and call it O. Consider once again a collection of $6g-5+2m$ simple geodesics $\tilde{\eta}_1, \ldots, \tilde{\eta}_{6g-5+2m}$ whose two ends go into the cusp O and which decompose S into $4g-3$ ideal triangles and $m-1$ once-punctured discs. We call this decomposition the *preferred triangulation* of our surface S, and we call O the *preferred puncture*. For each i the length of the geodesic $\tilde{\eta}_i$ is infinite; however, we can assign a relative length $l_{\tilde{\eta}_i}(S)$ to it as follows. The puncture O of S admits a neighborhood in S which is isometric to a standard cusp C [Bu]. By definition, such a standard cusp can be identified with the cylinder $[-\log 2, \infty) \times S^1$ with the metric $d\varrho^2 + e^{-2\varrho}\, dt^2$. We define the *height* of a point in C to be the value of the ϱ-coordinate in this representation.

Let $\triangle_\infty$ be an ideal triangle in $\mathbf{H}^2$. It contains a unique finite equilateral triangle T with vertices on the sides of $\triangle_\infty$ and which is invariant under all isometries of $\triangle_\infty$. For every ideal vertex ξ of $\triangle_\infty$ there is a unique horocircle H at ξ which passes through two of the vertices of T. Choose a number $\varrho_0 > 0$ in such a way that $2\pi e^{-2\varrho_0}$ is smaller than the length of the intersection of the horocircle H with $\triangle_\infty$. Explicit computation shows that this is the case if and only if we have $\varrho_0 \geq -\log\bigl(\sinh\bigl(\frac{1}{2}\operatorname{arccosh}\bigl(\frac{3}{2}\bigr)\bigr)/\pi\bigr) \sim 1.838$. Every geodesic going into the cusp meets the circles $\varrho = \text{const}$ orthogonally and hence, since both ends of the geodesic $\tilde{\eta}_i$ go into the cusp, each choice of a height cuts from $\tilde{\eta}_i$ a unique compact arc of finite length. Denote by $l_{\tilde{\eta}_i}(S)$ the length of the subarc of $\tilde{\eta}_i$ which corresponds to the height ϱ_0.

Proposition 3.1. *The map*

$$\Lambda\colon S \in \mathscr{T}_{g,m} \to \bigl(l_{\tilde{\eta}_1}(S), \ldots, l_{\tilde{\eta}_{6g-5+2m}}(S)\bigr) \in \mathbf{R}^{6g-5+2m}$$

is a diffeomorphism of $\mathscr{T}_{g,m}$ onto a hypersurface in $\mathbf{R}^{6g-5+2m}$.

Proof. Consider again an ideal triangle $\triangle_\infty$ in $\mathbf{H}^2$ with sides a_1, a_2, a_3. We parameterize the geodesics a_i by arc length in such a way that the origin corresponds to the vertex z_i of the subtriangle $T \subset \triangle_\infty$ on a_i and that the orientations of a_i define the boundary orientation of $\triangle_\infty$. The distinguished points z_i, z_{i+1} lie on a common horocircle through the point $a_i(\infty)$.

For given numbers $l_1, l_2, l_3 > 0$ define

$$x_i = a_i\bigl(\tfrac{1}{2}(-l_i + l_{i+1} - l_{i+2})\bigr), \quad y_i = a_i\bigl(\tfrac{1}{2}(l_i + l_{i+1} - l_{i+2})\bigr).$$

The distance between x_i and y_i equals l_i, and the points y_i and x_{i+1} are contained in a common horocircle at $a_i(\infty)$. If we identify the sides of $\triangle_\infty$ with $\mathbf{R}$ in the above way, then the assignment which maps (l_1, l_2, l_3) to the triple (x_1, x_2, x_3) of points on the three different sides of $\triangle_\infty$ is a real analytic diffeomorphism onto its image.

Now let $(l_1, \ldots, l_{6g-5+2m})$ be any $6g-5+2m$-tuple of positive numbers. If $i, j, k \leq 6g-5+2m$ are such that the geodesics $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ are the sides of a triangle of our preferred triangulation, then the triple (l_i, l_j, l_k) determines uniquely three arcs of length l_i, l_j, l_k on the three sides of an ideal triangle $\triangle_\infty$. Moreover, there is a distinguished point on the boundary geodesic of a once-punctured disc which is just the orthogonal projection of the puncture to the boundary. Each number $l \geq 0$ then determines a compact arc of length l on the boundary of a once-punctured disc whose midpoint is this distinguished point.

Each geodesic $\tilde{\eta}_i$ is either a common side of exactly two ideal triangles from our triangulation, or a common side of one triangle and a once-punctured disc. Each such pair of sides can be glued with an isometry which identifies the distinguished arcs of length l_i and reverses the orientation. The result of these $6g-5+2m$ glueings is a complete hyperbolic surface with m cusps and a distinguished horocircle going around one of the cusps. The length of this horocircle is the sum of the lengths of all the horocyclic arcs connecting pairs of endpoints of our distinguished boundary segments in each of the triangles and once-punctured discs.

Our construction defines a real analytic map μ of $\mathbf{R}_+^{6g-5+2m}$ into $\mathscr{T}_{g,m}$ which satisfies $\mu \circ \Lambda = \mathrm{Id}$ and hence is surjective. A point $(l_1, \ldots, l_{6g-5+2m}) \in \mathbf{R}_+^{6g-5+2m}$ belongs to the image of Λ if and only if the total length of the distinguished horocircle in the surface constructed from $(l_1, \ldots, l_{6g-5+2m})$ in the above way equals $2\pi e^{-2\varrho_0}$. In particular, the map Λ is a real analytic diffeomorphism of $\mathscr{T}_{g,m}$ onto a smooth hypersurface in $\mathbf{R}^{6g-5+2m}$. □

Let Π: $\mathbf{R}^{6g-5+2m} - \{0\} \to \mathbf{R}\,P^{6g-6+2m}$ be the canonical projection. The map $\Pi \circ \Lambda$: $\mathscr{T}_{g,m} \to \mathbf{R}\,P^{6g-6+2m}$ is real analytic.

Corollary 3.2. *The map $\Pi \circ \Lambda$ is a diffeomorphism of $\mathscr{T}_{g,m}$ onto the interior of a finite-sided closed convex polyhedron P in $\mathbf{R}\,P^{6g-6+2m}$ which extends to a homeomorphism of $\mathscr{T}_{g,m} \cup \partial \mathscr{T}_{g,m}$ onto P.*

Proof. Let again $\triangle_\infty$ be an ideal triangle in $\mathbf{H}^2$ and consider a horocyclic arc through one of its ideal vertices ζ which passes through the two distinguished points on the sides adjacent to ζ. By the choice of ϱ_0, the length of this arc is bigger than $2\pi e^{-2\varrho_0}$. It therefore follows from the construction in the proof of Proposition 3.1 that for every $(l_1, \ldots, l_{6g-5+2m}) \in \Lambda \mathscr{T}_{g,m}$ and for every triple i, j, k which defines a triangle of our triangulation we have $l_i < l_j + l_k$.

Denote by a_1, a_2, a_3 the sides of the ideal triangle $\triangle_\infty$ and let (l_1, l_2, l_3) be a triple of positive reals such that $l_i < l_{i+1} + l_{i+2}$ for all i (indices are taken mod 3). In the proof of Proposition 3.1 we constructed from this triple for each i a subsegment α_i of length l_i of the side a_i of $\triangle_\infty$. For $\varepsilon > 0$ denote by $\tilde{\alpha}_i$ the subsegment of a_i which is induced by the triple $\big((1+\varepsilon)l_1, (1+\varepsilon)l_2, (1+\varepsilon)l_3\big)$. Since $l_i < l_{i+1} + l_{i+2}$, our explicit construction of the segments $\alpha_i, \tilde{\alpha}_i$ shows that α_i is a subarc of $\tilde{\alpha}_i$. But this means that the length of each of the three horocyclic arcs in $\triangle_\infty$ connecting pairs of endpoints of the segments $\tilde{\alpha}_i$ as above is strictly shorter than the length of the corresponding horocyclic arc connecting endpoints of the segments α_i. Similarly, the horocyclic arc connecting points on the boundary geodesic of a once-punctured disc which is determined by the length $(1+\varepsilon)l$ is shorter than the arc determined by the length l.

Thus replacing a tuple $(l_1, \ldots, l_{6g-5+2m})$ in our construction above by a multiple with factor $1+\varepsilon > 1$ results in decreasing the length of our distinguished horocircle. In particular, $\Pi \circ \Lambda$ is injective. The proposition now follows from this and Lemma 2.2. □

In Section 2 we associated to each of the geodesics $\tilde{\eta}_i$ on a surface $S \in \mathscr{T}_{g,m}$ a closed geodesic ψ_i $(i = 1, \ldots, 6g-5+2m)$ on S. The "length" $l_{\tilde{\eta}_i}(S)$ of $\tilde{\eta}_i$ on S can be computed from the length $l_{\psi_i}(S)$ of ψ_i as follows. In the hyperbolic plane $\mathbf{H}^2$ draw a geodesic segment ν of length $l_{\psi_i}(S)/4$. Let ζ_1, ζ_2 be the orthogonals of ν through the endpoints of ν. There is a unique geodesic line ζ_3 in $\mathbf{H}^2$ with one endpoint at $\zeta_1(\infty)$ which intersects the geodesic ζ_2 orthogonally in a point x. Through each point of ζ_3 passes a horocircle at $\zeta_3(\infty) = \zeta_1(\infty)$ which intersects the geodesic ζ_1. The length of the subarc of this horocircle with endpoints on ζ_1 and ζ_3 decreases exponentially along ζ_3. Thus there is a unique point y on ζ_3 such that this length equals $\pi e^{-2\varrho_0}$. Then $l_{\tilde{\eta}_1}(S)/2$ is the oriented distance between x and y. This observation is summarized in the following lemma.

Lemma 3.3. *There is a strictly increasing real analytic function $\varphi\colon [0,\infty) \to [0,\infty)$ such that for every $i \le 6g-5+2m$ and every surface $S \in \mathscr{T}_{g,m}$ we have $l_{\tilde{\eta}_i}(S) = \varphi\big(l_{\psi_i}(S)\big)$. In particular, the map*

$$\Psi\colon S \in \mathscr{T}_{g,m} \to \big(l_{\psi_1}(S), \ldots, l_{\psi_{6g-5+2m}}(S)\big) \in \mathbf{R}^{6g-5+2m}$$

is a diffeomorphism onto its image.

The fact that a surface $S \in \mathscr{T}_{g,m}$ is uniquely determined by the lengths of the geodesics ψ_i was earlier observed by Seppälä and Sorvali [SS2] (see also [S3]).

Recall from Section 2 that the image of the space $\mathscr{PL}$ of projective measured laminations under the map $[\mu] \to \big[i(\psi_1,\mu), \ldots, i(\psi_{6g-5+2m},\mu)\big]$ is the boundary ∂P of a compact convex polyhedron P in $\mathbf{R}P^{6g-6+2m}$ with finitely many sides.

Lemma 3.4. *The image of $\mathscr{T}_{g,m}$ under the map $\Pi \circ \Psi$ equals the interior of the convex polyhedron P.*

Proof. It follows from Lemma 2.4 that the map $\Pi \circ \Psi$ extends continuously to a homeomorphism of $\partial\mathscr{T}_{g,m}$ onto the boundary ∂P of the polyhedron P. Since $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$ is homeomorphic to a closed ball it is therefore enough to show that the image of $\mathscr{T}_{g,m}$ under $\Pi \circ \Psi$ is contained in the interior of P.

For $j \in \{1, \ldots, 6g-5+2m\}$ and a measured lamination μ on a surface of genus g with m punctures define $\alpha_j(\mu) = i(\psi_j, \mu)$. Then α_j is a linear functional on the space $\mathscr{L}$ of measured laminations. Each side B of codimension 1 of our polyhedron P is either the projection of a hyperplane defined by the equation $\alpha_j = 0$ for some $j \in \{1, \dots, 6g-5+2m\}$ or there is a triangle of our preferred triangulation with sides $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ and such that B is the intersection with P of the projection of the linear hyperplane defined by $\alpha_i + \alpha_j - \alpha_k = 0$. Thus it is enough to show that whenever $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ are the sides of a triangle of our preferred triangulation, then $l_{\psi_i}(S) + l_{\psi_j}(S) - l_{\psi_k}(S) > 0$ for every $S \in \mathscr{T}_{g,m}$.

By Lemma 3.3 there is a real analytic monotonously increasing function $f\colon (0,\infty) \to (0,\infty)$ such that $l_{\tilde{\eta}_i}(S)/2 = f\big(l_{\psi_i}(S)/4\big)$ for all $i \le 6g-5+2m$ and all $S \in \mathscr{T}_{g,m}$. The function f can be computed explicitly as follows (see [S3]). For some small $\varepsilon > 0$ replace our surface S with cusps by a surface S_ε with geodesic boundary and such that the length of each boundary component equals ε. Define the length $l_{\tilde{\eta}_i}(S_\varepsilon)$ as the length of the perpendicular to the boundary component of S_ε which corresponds to the preferred cusp O of S and whose free homotopy class relative to the boundary corresponds to $\tilde{\eta}_i$. The closed geodesic ψ_i on S_ε and its length $l_{\psi_i}(S_\varepsilon)$ is also defined. Hyperbolic trigonometry shows [S3] that

$$l_{\tilde{\eta}_i}(S_\varepsilon)/2 = \operatorname{arsinh}\big(\sinh(\varepsilon/2)^{-1}\cosh\big(l_{\psi_i}(S_\varepsilon)/4\big)\big).$$

By definition of our functions $l_{\tilde{\eta}_i}$ there is a sequence of constants $a_\varepsilon \to \infty$ such that $l_{\tilde{\eta}_i}(S_\varepsilon) - a_\varepsilon \to l_{\tilde{\eta}_i}(S)$ for all $S \in \mathscr{T}_{g,m}$; moreover, we have $l_{\psi_i}(S_\varepsilon) \to l_{\psi_i}(S)$. Since $\operatorname{arsinh}(t) - \log(t) + \log(2) \to 0$ ($t \to \infty$) we conclude that there is a constant $c \in \mathbf{R}$ such that

$$l_{\tilde{\eta}_i}(S)/2 = \log\cosh\big(l_{\psi_i}(S)/4\big) + c.$$

It follows from our explicit construction that we may adjust our normalization for the definition of the functions $l_{\tilde{\eta}_i}$ (i.e. the choice of the height ϱ_0) in such a way that $c = 0$. Then we have $f(t) = \log\cosh(t)$, or equivalently, $l_{\psi_i}(S)/4 = \operatorname{arcosh} e^{l_{\tilde{\eta}_i}(S)/2}$ for all $S \in \mathscr{T}_{g,m}$.

Now $l_{\tilde{\eta}_i}(S) + l_{\tilde{\eta}_j}(S) - l_{\tilde{\eta}_k}(S) > 0$ for every $S \in \mathscr{T}_{g,m}$ and all i,j,k such that the geodesics $\tilde{\eta}_i, \tilde{\eta}_j, \tilde{\eta}_k$ form a triangle of the preferred triangulation. Since the function $t \to \operatorname{arcosh}(e^t)$ is a strictly increasing concave diffeomorphism of the half-line $[0,\infty)$ we conclude that also $l_{\psi_i}(S) + l_{\psi_j}(S) - l_{\psi_k}(S) > 0$. This shows the lemma. □

We can use Lemma 3.3 and Lemma 3.4 to complete the proof of our theorem from the introduction.

Proposition 3.5. *The map* $\Pi \circ \Psi\colon \mathscr{T}_{g,m} \to \mathbf{R}P^{6g-6+2m}$ *is a diffeomorphism of* $\mathscr{T}_{g,0}$ *onto the interior of a finite-sided convex polyhedron* P *in* $\mathbf{R}P^{6g-6+2m}$ *which extends to a homeomorphism of* $\mathscr{T}_{g,m} \cup \partial\mathscr{T}_{g,m}$.

Proof. Since Ψ is a diffeomorphism of $\mathscr{T}_{g,m}$ onto a smooth hypersurface in $\mathbf{R}^{6g-5+2m}$ it is enough to show that for every $S \in \mathscr{T}_{g,m}$ the line through $\Psi(S)$ and the origin is not tangent to the hypersurface $\Psi(\mathscr{T}_{g,m})$ at $\Psi(S)$.

For this we assume to the contrary that there is a surface $S \in \mathscr{T}_{g,m}$ and a smooth curve $c(t) \subset \mathscr{T}_{g,m}$ through $c(0) = S$ such that

$$\frac{d}{dt}\Psi\big(c(t)\big)\Big|_{t=0} = \Psi\big(c(0)\big).$$

Hyperbolic trigonometry shows that the lengths of the geodesics ψ_i are bounded from below by $2\operatorname{arsinh}(4\pi) \sim 6.45$. Recall from the proof of Lemma 3.4 that for a suitable choice of a height at the cusp O we have $l_{\tilde\eta_i}(M)/2 = \log\cosh\big(l_{\psi_i}(M)/4\big)$ for all i and all $M \in \mathscr{T}_{g,m}$ and therefore

$$\frac{d}{dt} l_{\tilde\eta_i}\big(c(t)\big)/2\,\Big|_{t=0} = l_{\psi_i}(S)\tanh\big(l_{\psi_i}(S)/4\big)/4.$$

The function σ: $s \to \log\cosh(s) + \log(2) - s\tanh(s)$ is monotonously decreasing with s and tends to 0 as $s \to \infty$. Its value at 1.5 is smaller than $(\log 2)/3$. By the above, for $s = l_{\psi_i}(S)/4$ we have

$$0 < \sigma(s) = \frac{1}{2} l_{\tilde\eta_i}(S) + \log 2 - \frac{d}{dt}\left(\frac{1}{2} l_{\tilde\eta_i}\big(c(t)\big)\right)\Big|_{t=0} < \frac{\log 2}{3}$$

and, in particular,

$$\frac{1}{2} l_{\tilde\eta_i}(S) + \frac{2}{3}\log 2 < \frac{d}{dt} l_{\tilde\eta_i}\big(c(t)\big)\Big|_{t=0} < \frac{1}{2} l_{\tilde\eta_i}(S) + \log 2.$$

Lemma 3.4 and its proof show that for every triangle of our preferred triangulation with sides $\tilde\eta_i, \tilde\eta_j, \tilde\eta_k$ we have $l_{\tilde\eta_i}(S) < l_{\tilde\eta_j}(S) + l_{\tilde\eta_k}(S)$. From this and the above we conclude that

$$\frac{d}{dt} l_{\tilde\eta_i}\big(c(t)\big)\Big|_{t=0} + \frac{d}{dt} l_{\tilde\eta_j}\big(c(t)\big)\Big|_{t=0} - \frac{d}{dt} l_{\tilde\eta_k}\big(c(t)\big)\Big|_{t=0} > 0.$$

Our explicit construction in the proof of Lemma 3.1 then shows that the differential at 0 of the length of the distinguished horocircle in the surface $c(t)$ is negative. This contradicts the definition of the functions $l_{\tilde\eta_i}$ and therefore our hypersurface $\Psi(\mathscr{T}_{g,m})$ is nowhere tangent to the lines through the origin. This shows the proposition. □

4. Length functions on closed surfaces

The purpose of this section is to construct a simple parameterization of $\mathscr{T}_{g,0}$ by $6g-5$ geodesic length functions and hence to give a new and simpler proof of the result of Schmutz Schaller [S1].

For this let $S \in \mathscr{T}_{g,0}$ be a closed hyperbolic surface and let ν_0 be a simple closed separating geodesic on S such that after cutting S open along ν_0 we obtain a bordered torus T_0 and a bordered surface S_0 of genus $g-1$. Let ν_1, ν_2, ν_3 be simple closed geodesics on T_0 which mutually intersect in a single point. If we cut S open along ν_1 then we obtain a connected surface S_1 of genus $g-1$ with two boundary circles. Denote one of the boundary circles by O.

We can use our definition of the geodesic arcs $\tilde{\eta}_i$ from Lemma 2.2 also for surfaces with geodesic boundary by requiring that these arcs meet the boundary circle O perpendicularly at both of their endpoints. This then defines a collection of $6g-7$ simple geodesic arcs $\tilde{\eta}_1, \ldots, \tilde{\eta}_{6g-7}$ on S_1 which decompose S_1 into $4g-5$ right-angled hexagons and one annulus A with piecewise geodesic boundary. One boundary component of A is the boundary circle of S_1 different from O. We choose the numbering of our geodesic arcs in such a way that the second boundary component of A contains the geodesic $\tilde{\eta}_{6g-7}$ as a subarc.

For each $i \leq 6g-8$ the geodesic $\tilde{\eta}_i$ is contained in a unique pair of pants $P \subset S_1$ whose geodesic boundary consists of the circle O and two additional simple closed geodesics $\sigma, \hat{\sigma}$. Let ψ_i be the closed geodesic in P with two self-intersections which intersects the perpendicular ν of the geodesics $\sigma, \hat{\sigma}$ twice and is invariant under reflection along ν. We view ψ_i as a geodesic on S. Write, moreover, $\psi_{6g-8+i} = \nu_i \subset T_0 \subset S$.

We can now show that the length functions of the family $\psi_1, \ldots, \psi_{6g-5}$ of $6g-5$ closed geodesics define an embedding of $\mathscr{T}_{g,0}$ into $\mathbf{R}P^{6g-5}$.

Proposition 4.1. *The map*

$$\Psi_0 \colon S \in \mathscr{T}_{g,0} \to \big(l_{\psi_1}(S), \ldots, l_{\psi_{6g-5}}(S)\big) \in \mathbf{R}^{6g-5}$$

is a diffeomorphism onto its image.

Proof. Consider the subtorus T_0 of the closed hyperbolic surface S of genus g. The simple closed geodesics $\nu_i = \psi_{6g-8+i} \subset T_0$ $(i = 1, 2, 3)$ mutually intersect in a single point. Seppälä and Sorvali ([SS1], compare also [SS5] and [S1]) showed that the length functions of the geodesics ν_1, ν_2, ν_3 determine the hyperbolic structure of T_0 and the length of its boundary geodesic.

Let again S_1 be the subsurface of S of genus $g-1$ with two boundary circles which we obtain by cutting S along the geodesic $\nu_1 = \psi_{6g-7} \subset T_0$. Recall from Section 2 the definition of the geodesic arc $\tilde{\eta}_{6g-7}$ in S_1; it is contained in the torus T_0 and therefore its length is determined by the lengths of the geodesics ν_i.

As above let $\tilde\eta_j$ $(j \le 6g-8)$ be the simple geodesic arcs in S_1 which decompose S_1 into $4g-5$ right-angled hexagons and one annulus and which we used to define our geodesics ψ_j. Hyperbolic trigonometry shows that

$$\cosh\bigl(l_{\psi_i}(S)/4\bigr) = \sinh\bigl(l_{\tilde\eta_i}(S)/2\bigr)\sinh(l_{\nu_1}(S)/2)$$

(compare [S3]). In particular, the lengths of the arcs $\tilde\eta_j$ $(1 \le j \le 6g-8)$ are determined by the lengths of the geodesics ψ_j and the length of the boundary geodesic $\nu_1 = \psi_{6g-7}$. By construction, the arcs $\tilde\eta_j$ $(j = 1,\dots,6g-7)$ determine the hyperbolic structure of a collection of $4g-5$ right-angled hexagons which can be glued in a unique way to form the complement $\widehat{S}_1$ of an annulus A in the surface S_1. The boundary of A consists of one closed geodesic and one right-angled geodesic bigon. The hyperbolic structure of the annulus is completely determined by the length $l_{\nu_1}(S)$ of the boundary circle and the length of the arc $\tilde\eta_{6g-7}$. Moreover, there is a unique way to glue A to our subsurface $\widehat{S}_1$ of S_1 which is composed of our right-angled hexagons and such that we obtain a smooth hyperbolic surface with two boundary components. But this just means that the hyperbolic structure of S_1 is determined.

The surface S is obtained from S_1 by glueing the two boundary geodesics with a suitable twist. Since the boundary of S_1 is contained in the torus T_0, the twist parameter for the glueing is determined by the hyperbolic structure on T_0. Thus the lengths of the geodesics ψ_j determine the hyperbolic structure on S. □

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