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A FUCHSIAN GROUP PROOF OF THE HYPERELLIPTICITY OF RIEMANN SURFACES OF GENUS 2

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Abstract. The fact that compact Riemann surfaces of genus 2 are always hyperelliptic is usually presented as a corollary of the Riemann–Roch theorem. Here we give a proof which involves only the theory of Fuchsian groups that uniformize them.

The class of hyperelliptic Riemann surfaces in each genus $g > 1$, because of its particular simplicity, is often taken as a model to illustrate known results or to test new conjectures. These are the surfaces S that appear as double covers of the Riemann sphere. They are characterized by the property that they admit an automorphism J of order 2 whose quotient $S/\langle J\rangle$ has genus zero, the double cover being provided by the quotient map $S \to S/\langle J\rangle$. A basic fact, which is part of the widely developed theory of automorphism groups of Riemann surfaces, is that surfaces of genus 2 are always hyperelliptic.

Due to the uniformization theorem, an arbitrary Riemann surface of genus $g > 1$, S can be viewed as a quotient space $S = \mathbf{H}/K$, where $\mathbf{H}$ is the upper half-plane and K is the uniformizing group, a discrete subgroup of the group of conformal automorphisms of $\mathbf{H}$ isomorphic to the fundamental group of the surface. Such groups possess a very well known structure and are sometimes termed (*Fuchsian*) *surface groups.* From this point of view the hyperellipticity of surfaces of genus 2 is equivalent to the statement that a surface group K of genus 2 is automatically contained, with index 2, in a larger group Γ so that $\mathbf{H}/\Gamma \equiv \mathbf{CP}^1$, J being then induced by any element of $\Gamma \setminus K$. In most textbooks this result is presented as a consequence of the Riemann–Roch theorem, a result whose presentation is involved. The purpose of this note is to prove this fact in the framework of the elementary theory of Fuchsian groups.

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1. Notation and statement of the result

We now summarize some elementary facts from the theory of Riemann surfaces and Fuchsian groups. We refer to Beardon [1], Farkas–Kra [2] and Jones–Singerman [3] for general background.

1.1. We recall that the group of conformal automorphisms of $\mathbf{H}$ agrees with the group of orientation preserving isometries of $\mathbf{H}$ equipped with the *hyperbolic metric* d. This is the well-known group of real Möbius transformations $\mathrm{PSL}(2,\mathbf{R})$.

An element $C \in \mathrm{PSL}(2,\mathbf{R})$ is called *hyperbolic* if it fixes two points on the boundary $\partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$ and none in $\mathbf{H}$ (see e.g. [2]). In fact, for any $z \in \mathbf{H}$, these two points can be obtained as $\lim_{n\to\infty} C^n z$ (*attractive fixed point*) and $\lim_{n\to-\infty} C^n z$ (*repulsive fixed point*), and the *non Euclidean* or *hyperbolic* line connecting them is called the *axis* of C. The transformation $C \in \mathrm{PSL}(2,\mathbf{R})$ acts on its axis, $\mathrm{axis}(C)$, as a (hyperbolic) translation. Its *translation length* T_C is equal to $d(z, Cz)$ for any point z on its axis. Moreover, C is determined by its axis and its translation length. As a matter of fact, C can be written as the product of two *half-turns* R_z and R_w at any pair of points z and w on the axis of C with distance apart equal to $\frac{1}{2}T_C$. More precisely, $C = R_z \circ R_w$ or $C = R_w \circ R_z$ according to whether the ray from w to z ends at the attractive or at the repulsive fixed point of C. These facts can be easily proved by conjugation with a suitable isometry so that C fixes 0 and ∞ and hence becomes a transformation of the form

$$C = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \lambda > 0,$$

whose axis is simply the upper half imaginary axis (see [1, p. 174]).

1.2. The fundamental group $\pi_1(S,P)$ of a compact Riemann surface S of genus 2 at a point $P \in S$ is generated by the homotopy classes of simple loops α_1, β_1; α_2, β_2 enjoying the following properties:

(i) any pair of them intersect only at P;

(ii) this intersection is transversal (only) for the pairs α_1, β_1 and α_2, β_2;

(iii) they satisfy the single defining relation $\big[[\alpha_1],[\beta_1]\big]\big[[\alpha_2],[\beta_2]\big] = 1$, where the bracket $[\ ,\ ]$ stands for the commutator and $[\gamma]$ denotes the homotopy class of a loop γ.

Such generators are called *canonical.*

1.3. The uniformization theorem quoted above shows that S can be viewed as a quotient space $S = \mathbf{H}/K$, where K is a discrete subgroup of the group $\mathrm{PSL}(2,\mathbf{R})$, i.e. a *Fuchsian* group, acting freely on $\mathbf{H}$. In these circumstances K consists entirely of hyperbolic elements (see e.g. [1]).

According to elementary covering space theory, for any $p \in \mathbf{H}$ projecting to $P \in S$ via the covering map $\pi\colon \mathbf{H} \to S$, we have an isomorphism $\phi_p\colon \pi_1(S,P) \to K$

defined by sending $[\gamma] \in \pi_1(S,P)$ to the Möbius transformation which maps p to the endpoint of the lift of γ at p; this lift we denote by $\tilde{\gamma}_p$. By applying ϕ_p we obtain *canonical* generators for K, $A_i =: \phi_p([\alpha_i])$, $B_i =: \phi_p([\beta_i])$ subject to the same defining relation $[A_1,B_1][A_2,B_2] = 1$.

Let now γ be any non trivial simple loop based at P and put $C = \phi_p([\gamma])$. Let us consider the curve $l_p(\gamma)$ in $\mathbf{H}$ obtained by putting together the translates of the arc $\tilde{\gamma}_p$ by all powers of C, that is, $l_p(\gamma) = \bigcup_{n\in\mathbf{Z}} C^n\tilde{\gamma}_p$. This is clearly a connected curve since the arc $C^n\tilde{\gamma}_p$ ends where $C^{n+1}\tilde{\gamma}_p$ begins, namely at the point $C^{n+1}p$. Moreover, $l_p(\gamma)$ does not cross itself, for if we had $C^r x = C^m y$ for two different points $x, y \in \tilde{\gamma}_p$ then x and y would project to the same point of S. It would then follow that x and y must be the endpoints of $\tilde{\gamma}_p$, say $x = p$ and $y = Cp$; hence we would have $C^{m+1-r}p = p$ which, C being fixed point free, can only occur if $r = m+1$; in other words the arcs $C^r\tilde{\gamma}_p$ and $C^m\tilde{\gamma}_p$ intersect only if they are consecutive and then, at the correct point $C^{m+1}p$. This implies that $l_p(\gamma)$ divides $\mathbf{H}$ into two connected components for on the one hand $l_p(\gamma)$ is a topological line (in fact, the restriction of $\pi\colon \mathbf{H} \to S$ to $l_p(\gamma)$ provides a universal cover for the topological circle γ) and on the other hand $l_p(\gamma)$ hits the boundary of $\mathbf{H}$ at two points, namely $\lim_{n\to\infty} C^n p$ and $\lim_{n\to-\infty} C^n p$, that is, the fixed points of C.

We next observe that, since α_i, β_i intersect only at the point P, $l_p(\alpha_i)$ and $l_p(\beta_i)$ intersect only at $p \in \mathbf{H}$, for if there were another point $p' \in l_p(\alpha_i) \cap l_p(\beta_i)$, then, necessarily, $\pi(p') = P$ and we would have $p' = A_i^d(p) = B_i^m(p)$ for some integers d and m, but this, in turn, would imply $A_i^d = B_i^m$, a contradiction. Now the fact that $l_p(\alpha_i)$ and $l_p(\beta_i)$ intersect transversally at only one point is equivalent to saying that each of these lines separates the endpoints of the other one. Of course this argument can be equally applied to any pair of topological lines connecting these same endpoints and, in particular, to the axes of A_i and B_i. We conclude that the axes of A_i and B_i do intersect.

1.4. Let us consider an arbitrary uniformization $S = \mathbf{H}/K$. As in 1.1 we may assume, by suitable conjugation, that

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \lambda > 0,$$

with axis(A_1) equals the upper half imaginary axis. We can even perform an extra normalization and require B_1 to fix 1 so that

$$B_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a+b = c+d$ and axis(B_1) equals the semi-circle joining its two fixed points, namely 1 and $-b/c$. Since axis$(A_1) \cap$ axis$(B_1) \neq \emptyset$, a simple calculation shows that we must have $bc > 0$, the intersection point being $z_1 = i\sqrt{b/c}$.

Theorem 1. *Let $S = \mathbf{H}/K$ be an arbitrary compact Riemann surface of genus 2, and let*

$$A_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \qquad \lambda > 0,$$
$$B_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a+b = c+d, \quad bc > 0,$$
$$A_2 \text{ and } B_2,$$

be a set of (normalized) canonical generators for the uniformizing group K. Then, the Möbius transformation

$$R_1(z) = \frac{-b}{cz}$$

induces on $S = \mathbf{H}/K$ an automorphism J, the hyperelliptic involution, such that the quotient $S/\langle J\rangle$ has genus zero.

2. Proof

(i) First of all we observe that R_1 is the half-turn fixing the point $z_1 = i\sqrt{b/c}$ at which the axes of A_1 and B_1 meet. Now by 1.1 we can write

$$A_1 = R_3R_1, \qquad B_1 = R_1R_2, \tag{1}$$

where R_3 (respectively R_2) is the half-turn at a certain point $z_3 \in \operatorname{axis}(A_1)$ (respectively $z_2 \in \operatorname{axis}(B_1)$). Similarly we put

$$A_2 = R_6R_5, \qquad B_2 = R_4R_6, \tag{2}$$

where R_6 is the half-turn at the point $z_6 \in \operatorname{axis}(A_2) \cap \operatorname{axis}(B_2)$ and R_5 (respectively R_4) is the half-turn at a suitable point $z_5 \in \operatorname{axis}(A_2)$ (respectively $z_4 \in \operatorname{axis}(B_2)$).

(ii) Using identities (1) and (2) we can write $[A_1, B_1] = (R_3R_2R_1)^2$ and $[A_2, B_2]^{-1} = (R_4R_5R_6)^2$, thus the defining relation $[A_1, B_1] = [A_2, B_2]^{-1}$ becomes $(R_3R_2R_1)^2 = (R_4R_5R_6)^2$. Since these elements are both hyperbolic, we deduce that

$$(R_3R_2R_1) = (R_4R_5R_6) \tag{3}$$

or equivalently,

$$R_1R_2R_3R_4R_5R_6 = 1 \tag{4}$$

(iii) Next we claim that the rotations $R_j, j = 1, \dots, 6$, are all equivalent $\operatorname{mod} K$. By (1) and (2), the statement is clear if we consider the first three or the second three separately. To complete the argument we only need to observe that $R_1R_6 \in K$. Indeed we have $R_1R_6 = B_1^{-1}A_1^{-1}B_2A_2$ as it is readily seen using (3).

(iv) In order to prove that R_1 normalizes K and, hence, that its action (and, by the previous point, that of any other R_j) induces an automorphism J on $S = \mathbf{H}/K$ we merely need to check that

$$R_1 A_1 R_1 = A_1^{-1}, \quad R_1 B_1 R_1 = B_1^{-1}, \tag{5}$$

$$R_1 A_2 R_1 = (R_1 R_6) A_2^{-1} (R_1 R_6)^{-1}, \quad R_1 B_2 R_1 = (R_1 R_6) B_2^{-1} (R_1 R_6)^{-1}. \tag{6}$$

(v) Finally we prove that the quotient surface $X = S/\langle J \rangle$ has genus zero. Let us denote by $p\colon S \to X$ the natural quotient map. We have an induced commutative diagram of homomorphisms between homology groups

$$\begin{tikzcd} H_1(S) \arrow[rr, "J_*"] \arrow[dr, "p_*"'] & & H_1(S) \arrow[dl, "p_*"] \\ & H_1(X) & \end{tikzcd}$$

If X has genus > 0, then there is on it a loop γ which is homologically non trivial. Consider then a lift of γ^2 to S. Such a lift, let us call it c, is again a (closed) loop because p has degree 2, and so it defines a homology class which satisfies $p_*(c) = \gamma^2$. But on the other hand, since the homology group is the abelianization of the fundamental group, it follows from (5) and (6) that $J_* = -\,$identity. This contradicts the commutativity of the diagram above.

Remark 1. An alternative way to state our result is to say that the group K is an index 2 subgroup of the Fuchsian group Γ generated by the half-turns R_j, $j = 1, \ldots, 6$, which satisfy the relation (4), thus it is a group of *signature* $(0, 6)$ (see [3, p. 260]).

The projections $\pi(z_j)$ of the points z_j, centers of the rotations R_j, are the six points which, according to the Riemann–Hurwitz formula, J fixes. They are called the *Weierstrass* points of S. Note that they divide the smooth geodesics $\pi\big(\text{axis}(A_i)\big)$, $\pi\big(\text{axis}(B_i)\big)$ into two parts of equal length.

Remark 2. If the genus g is larger than 2 we can still obtain expressions analogous to (1) and (2) for any set of canonical generators $A_1, \ldots, A_g$; $B_1, \ldots, B_g$. However this way of reasoning will break down when we try to find relations of the kind (5) and (6). In fact it is well known that when $g > 2$, most surfaces are not hyperelliptic.

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