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THE HYPERBOLIC METRIC IN A RECTANGLE II

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Abstract. We obtain precise results on the hyperbolic distance in a rectangle. In particular, we give formulae for the hyperbolic distances along the lines of symmetry of the rectangle.

1. Introduction

The function $z \mapsto \log z - \pi i/2$ maps the upper half-plane $\mathbf{H}$ conformally onto the infinite strip $S = \{x+iy : |y| < \pi/2\}$, and when we transfer the hyperbolic metric $|dz|/y$ from $\mathbf{H}$ to S we find that the hyperbolic density $\lambda_S(z)$ in S satisfies $\lambda_S(x) = 1$ for all real x. Thus the Euclidean and hyperbolic distances coincide on the real axis in S. This paper contains a discussion of the hyperbolic metric, and a comparison of Euclidean and hyperbolic distances, in the rectangle

$$R(l) = \{x+iy : |x| < l,\ |y| < \pi/2\} = (-l,l) \times (-\pi/2, \pi/2)$$

in the complex plane $\mathbf{C}$, and it is a continuation of earlier work in [3] and [5]. For brevity, we shall often use R for $R(l)$, and we use $\lambda_R(z)$ and $d_R(z,w)$ for the hyperbolic density and distance, respectively, in R; thus R, λ_R and d_R all depend implicitly on l. When l is large the results for R are similar to those for S, but here we obtain an explicit formula for $d_R(0,x)$ that is valid for all l.

Let σ and τ be the hyperbolic geodesics joining the vertices of $R(l)$ (as illustrated in Figure 1), and let u and iv be the points where τ meets the real axis, and σ meets the imaginary axis, respectively. Our first result gives asymptotic values of the hyperbolic distance $d_R(0,u)$, and the Euclidean distances $|u-0|$ and $|l-u|$, when l is large.

Theorem 1.1. *Let $R(l)$, u and v be as above. Then, as $l \to +\infty$,*

(a) $l = u + \log\bigl(1+\sqrt{2}\,\bigr) + o(1)$;
(b) $d(0,u) = l - \log 2 + o(1)$;
(c) $d(0,u) = u + \log\bigl(1+\sqrt{2}\,\bigr)/2 + o(1)$.

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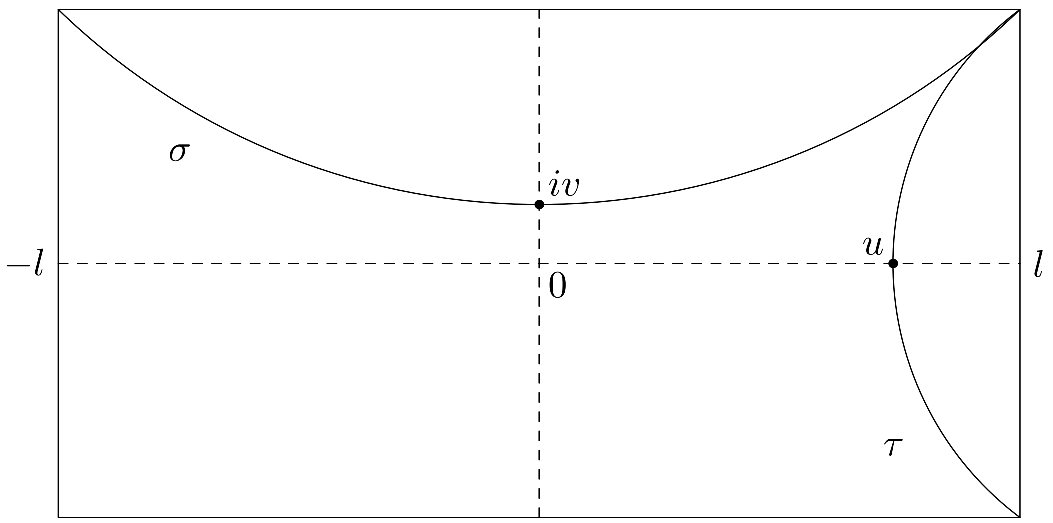

Figure 1.

As we have remarked above, when l is large, the Euclidean and hyperbolic distances between 0 and x, where $0 < x < l$, are similar providing that x is not too close to l. These distances were compared in [5, Lemma 6], where (after scaling the hyperbolic metric in [5] by a factor 2 so that it corresponds to $|dz|/y$ on $\mathbf{H}$) it was shown that if $0 \le x \le l - \pi/2$ then

$$x \le d_R(0,x) \le x + \pi/2 = x + 1.5708\dots. \tag{1.1}$$

This was improved in [3], where it was shown that if $0 < x < l - \pi/2$, then

$$x < d_R(0,x) < x + \frac{4}{e^{2(l-x)}} \le x + \frac{4}{e^{\pi}} = x + 0.17\dots. \tag{1.2}$$

Both of these inequalities require x to be outside the semi-disc

$$R(l) \cap \{|z - l| < \pi/2\},$$

and this is more restrictive than Theorem 1.1(c) because, as can be seen from Theorem 1.1(a), u is approximately $(l - \pi/2) + 0.68942$. Moreover, whereas the geodesic τ has an intrinsic significance in the geometry of R, the semi-circle does not, and this is probably the reason why Theorem 1.1(c) leads to sharper estimates than (1.1) and (1.2). Indeed, we shall see later that $d_R(0,x) - x$ is increasing, so that if $0 \le x \le u$, then, from Theorem 1.1(c),

$$x \le d_R(0,x) \le x + \log\bigl(1 + \sqrt{2}\,\bigr)/2 = x + 0.18823\dots. \tag{1.3}$$

Note that Theorem 1.1(c) also shows that this upper bound is asymptotically correct when $x = u$ and $l \to +\infty$.

Let us now consider a different problem associated with a long rectangle. Suppose that a simply connected domain D contains a long thin *channel*; by this we mean that there is a long rectangle R which is contained in D, and which is such that the two long sides of R lie in ∂D. It is intuitively clear that any hyperbolic geodesic γ in D that passes down the length of the rectangle must pass close to the centre, say O, of R, and that the distance (hyperbolic or Euclidean) between O and γ must tend to zero as rectangle gets longer (with a fixed width). We verify this, and even obtain an explicit estimate for the distance between O and γ.

Theorem 1.2. *Let $R(l)$ and v be as above. Then $d_R(0, iv) \sim 4/e^l$ as $l \to \infty$.*

Corollary 1.3. *Let D be a simply connected domain that contains the channel $R(l)$. Then any hyperbolic geodesic in D that joins the two short ends of $R(l)$ passes within a hyperbolic distance $4e^{-l}\big(1 + o(1)\big)$ as $l \to \infty$.*

Finally, we give an explicit formula for $d_R(0, x)$, where $0 < x < l$, in terms of x and Jacobi's elliptic function sn. We shall use the standard theory of elliptic functions as may be found in, for example, [1], [4], [6], [7] and [8]. We suppose that $0 < k < 1$, and (as usual) we put $k' = \sqrt{1 - k^2}$. Then F given by

$$F(z) = \int_0^z \frac{dw}{\sqrt{(1 - w^2)(1 - k^2 w^2)}}. \tag{1.4}$$

is the conformal map of $\mathbf{H}$ onto the rectangle $(-K, K) \times (0, K')$ with $-k^{-1}$, -1, 1, k^{-1} mapping to $-K + iK'$, $-K$, K, $K + iK'$, respectively [7, p. 280]. Jacobi's function sn is the inverse function F^{-1}, and this is an increasing map of $[-K, K]$ onto $[-1, 1]$. We remark that K is given in terms of k by an elliptic integral ([7, p. 281]), and also as an infinite series ([3, p. 403] and [4, p. 90]), and $K'(k) = K(k')$. Finally, given any positive number t there is a unique value of k such that $K(k)/K'(k) = t$.

Theorem 1.4. *Let $R(l)$ be as above. Then $d_R(0, x)$ is a convex function of x on $[0, l]$, and if $0 \le x < l$, then*

$$d_R(0, x) = \frac{1}{2} \log\left(\frac{1 + \mathrm{sn}(Kx/l)}{1 - \mathrm{sn}(Kx/l)}\right) = \frac{1}{2} \log\left(\frac{1 + \mathrm{sn}(2K'x/\pi)}{1 - \mathrm{sn}(2K'x/\pi)}\right), \tag{1.5}$$

where k is chosen so that $l = \pi K(k)/2K'(k)$.

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2. The half-strip

We need detailed information on the hyperbolic metric in a half-strip. The function $\sin z$ maps the half-strip $H = (-\pi/2, \pi/2) \times (0, +\infty)$ conformally onto $\mathbf{H}$ taking $-\pi/2$ and $\pi/2$ to -1 and 1, respectively [7, p. 276]. It follows that the hyperbolic geodesic γ in H with endpoints $\pm\pi/2$ is given by those z for which $|\sin z| = 1$. Now if $z = x + iy$ then $|\sin z|^2 = \sin^2 x + \sinh^2 y$, so that γ is given by $\sin^2 x + \sinh^2 y = 1$. In particular, the 'highest' point of γ is iy_0, where $\sinh y_0 = 1$ or, equivalently,

$$y_0 = \sinh^{-1}(1) = \log\bigl(1 + \sqrt{2}\,\bigr) < \pi/2. \tag{2.1}$$

The equations of the other geodesics that 'cross' H are $\sin^2 x + \sinh^2 y = R$, where $R > 0$, but we shall not need these.

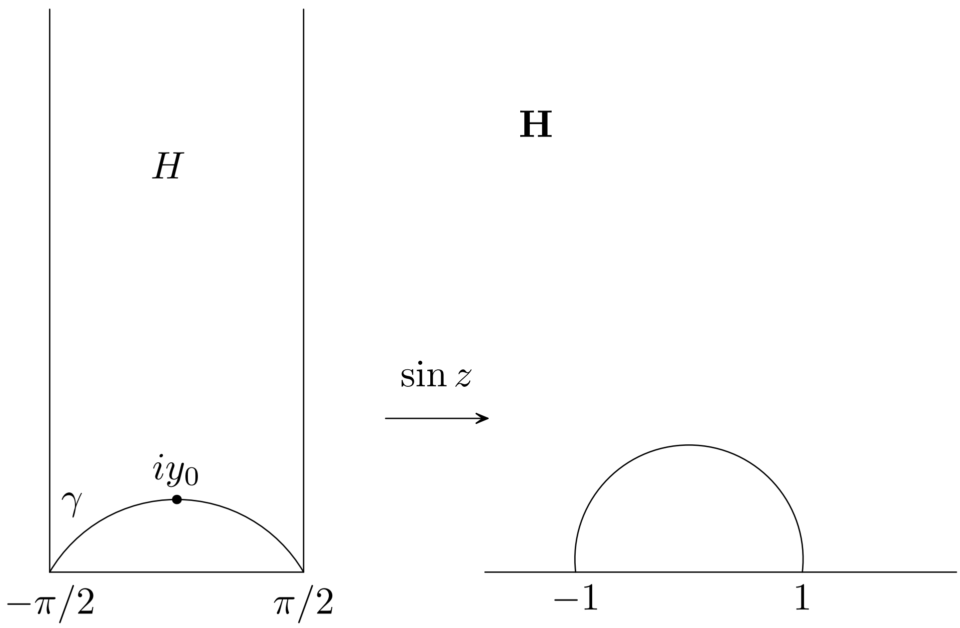

Figure 2.

We return to a discussion of the rectangle $R(l)$ and we shall show that, as $l \to +\infty$, the geodesic τ in Figure 1 converges (in the obvious sense) to the corresponding geodesic in the half-strip $(-\infty, l) \times (-\pi/2, \pi/2)$. In view of (2.1), this shows that

$$l - u \to y_0 = \log\bigl(1 + \sqrt{2}\,\bigr), \tag{2.2}$$

and this is Theorem 1.1(a). It remains to prove that τ converges to the geodesic in the half-strip as described above.

The proof of this is based on the following observation. The hyperbolic geodesic $(-1,1)$ in the unit disc $\mathbf{D}$ is the level curve $\omega(z) = 1/2$ of the harmonic measure ω of the upper (or lower) semi-circle of $\partial\mathbf{D}$. By the conformal invariance of both the harmonic measure and hyperbolic geometry, an analogous result holds for all simply connected domains (for which the Dirichlet problem is solvable).

Now consider the half-strip H and the rectangle $R(l)$, the rectangle $R(l)$ having been translated to the left by a distance l, and the half-strip H having been rotated by an angle $\pi/2$; see Figure 3. We shall continue to use the same notation as earlier despite these transformations of $R(l)$ and H.

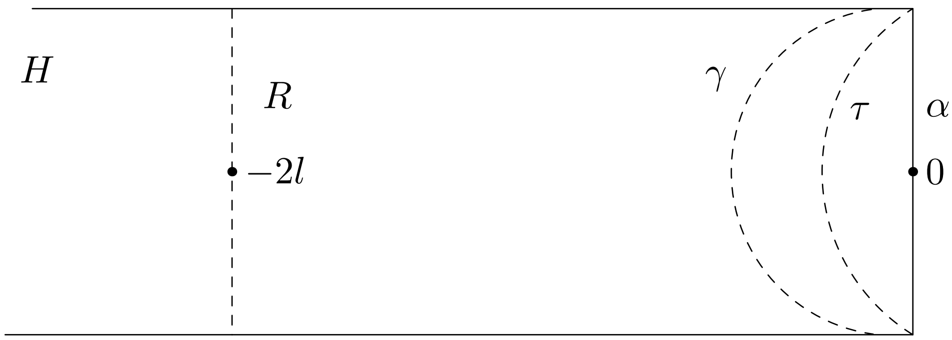

Figure 3.

Let α be the right-hand side of both H and $R(l)$, and let ω_H and ω_R be the harmonic measures of α with respect to H and $R(l)$, respectively. As $\omega_R \le \omega_H$ in $\partial R(l)$, with a strict inequality holding on the left-hand edge, we see immediately that $\omega_R < \omega_H$ in $R(l)$. This means that the geodesic τ separates γ from the right-hand edge α (as illustrated in Figure 3). Standard techniques show that ω_R is increasing with l, and that $\omega_R \to \omega_H$ locally uniformly on H as $l \to +\infty$. If we now note that τ and γ are given by $\omega_R = \frac{1}{2}$ and $\omega_H = \frac{1}{2}$, respectively, this completes the proof of (2.2), and hence of Theorem 1.1(a). □

3. Elliptic integrals

A quadrilateral $Q(z_1, z_2, z_3, z_4)$ is a Jordan domain Q with four given points z_1, z_2, z_3, z_4 lying in this order clockwise around the boundary of Q. The modulus of this quadrilateral is given by

$$\operatorname{mod} Q(z_1, z_2, z_3, z_4) = b/a,$$

where a and b are determined by any conformal map f of Q onto a rectangle R' such that the $f(z_j)$ are the vertices of R', $a = |f(z_1) - f(z_2)|$ and $b = |f(z_2) - f(z_3)|$. Now let

$$M(k) = \operatorname{mod} \mathbf{H}(-k^{-1}, -1, 1, k^{-1}) = \frac{2K(k)}{K'(k)}.$$

The values of $K(k)$ and $K'(k)$ can be found in standard numerical tables for, say, $0 < k < 0.9$ so that, for k in this range, $M(k)$ can be found numerically. For example, if $k = \frac{4}{5}$ then $K = 1.9953\ldots$, $K' = 1.7508\ldots$ and $M(k) = 2.2793\ldots$ (see [6, pp. 278–279]). Tables do not seem to be available for values of k close to 1, but the asymptotic behaviours of K and K' are known. For example, it is well known that as $k \to 0$, $K(k) \to \pi/2$ and $K'(k) - \log 4/k \to 0$ ([8, p. 522]). Thus, as $K'(k) = K(k')$, we see that

$$M(k) = \frac{2K(k)}{K'(k)} = \frac{2K'(k')}{K(k')} \sim \frac{4}{\pi}\log\frac{4}{\sqrt{1-k^2}}$$

as $k \to 1$ (and $k' \to 0$). In fact, we have the following stronger result.

Lemma 3.1. *As* $k \to 1$,

$$M(k) = \frac{4}{\pi}\log\frac{4}{\sqrt{1-k^2}} + o(1).$$

Proof. Chapter 5 of [1] is devoted to a study of the function $\mu(r)$, where

$$\mu(k') = \frac{\pi K'(k')}{2K(k')} = \frac{\pi K(k)}{2K'(k)} = \frac{\pi M(k)}{4}.$$

With this, Lemma 3.1 follows immediately from the inequality

$$k\log\frac{4}{k'} < \mu(k') < \log\frac{4}{k'}$$

which is (5.3) in [1, p. 80]. □

We remark that Chapter 5 of [1] contains additional information about the function $\mu(k)$, and hence about the function $M(k)$; for example, $\mu(k)\mu(k') = \pi^2/4$.

4. Proof of Theorems 1.1 and 1.2

We begin with our proof of Theorem 1.2. Because μ: $(0,1) \to (0,+\infty)$ is a homeomorphism, given any positive l we can choose a unique value of k in $(0,1)$ such that

$$l = \frac{\pi K(k)}{K'(k)}, \tag{4.1}$$

and we recall that $M(k) = 2l/\pi$. Then the map

$$z \mapsto \frac{\pi}{K'(k)}F(z) - \frac{i\pi}{2}$$

is a conformal map of $\mathbf{H}$ onto $R(l)$. As the hyperbolic metric is invariant under conformal maps it is clear that

$$2d_R(0, iv) = d_R(-iv, iv) = d_{\mathbf{H}}(i, i/k) = \log 1/k,$$

where $d_{\mathbf{H}}$ is the hyperbolic metric in $\mathbf{H}$, because $\log 1/k$ is the hyperbolic distance between the geodesics given by $|z| = 1$ and $|z| = 1/k$ in $\mathbf{H}$. Now

$$d_R(0, iv)\, e^l = \left(\frac{1}{2} \log \frac{1}{k}\right) \exp\left(\frac{\pi M(k)}{2}\right),$$

and as (from Lemma 3.1)

$$\frac{\pi M(k)}{2} = \log \frac{16}{1 - k^2} + o(1),$$

we see that $d_R(0, iv)\, e^l \to 4$ as $k \to 1$ and this proves Theorem 1.2. □

We can now complete the proof of Theorem 1.1. As we have already proved (a), and as (c) follows trivially from (a) and (b), we need only prove (b). This can be proved in a similar way by computing the hyperbolic distance in $\mathbf{H}$ between the geodesic given by $x = 0$ and the geodesic whose Euclidean diameter is $[1, 1/k]$ (and using the formula given in [2, p. 145]). Alternatively, we can argue as follows. The geodesics σ and τ, and the hyperbolic segments $[0, u]$ and $[0, iv]$, in Figure 1 bound a Saccheri quadrilateral (a hyperbolic quadrilateral with three right-angles and one angle zero). Now the trigonometry of Saccheri quadrilaterals is known (see [2, p. 156]), so we see immediately that

$$\sinh d_R(0, u) \sinh d_R(0, iv) = 1.$$

This with Theorem 1.2 leads easily to Theorem 1.1(b). □

5. Proof of Corollary 1.3

We start with a simply connected domain D that contains the channel $R(l)$. Let β be the 'upper' side of $R(l)$ so that $\beta \subset \partial D$. Now let ω_R and ω_D be the harmonic measures of β with respect to R and D, respectively. It is clear that $\omega_D > \omega_R$ on R, and hence that $\omega_D > \frac{1}{2}$ on the region bounded by β and σ. The corresponding statement holds if we replace β by the 'lower' side of R, and this implies that the geodesics relative to D that join the ends of the upper side, and the ends of the lower side, of R form a narrower channel than the corresponding geodesics for R. Corollary 1.4 follows directly from this and Theorem 1.3.

6. Proof of Theorem 1.4

We begin by establishing the formula for $d_R(0,x)$. Let

$$g(z) = (z^2-1)/(z^2+1);$$

then g maps Σ conformally onto Ω, where $\Sigma = \{x+iy : x > 0\}$ and Ω is the complex plane $\mathbf{C}$ cut from $-\infty$ to -1, and from 1 to ∞. Next, the Reflection Principle implies that F maps Ω conformally onto $\mathscr{R}$, where

$$\mathscr{R} = \{x+iy : |x| < K,\ |y| < K'\}.$$

Now let $h(z) = \pi z/(2K')$; then h maps $\mathscr{R}$ conformally onto R. Finally, let $\Phi = hFg$; then Φ maps Σ conformally onto R, where now,

$$l = \frac{\pi K(k)}{2K'(k)} \tag{6.1}$$

(note that this differs from the relationship (4.1) between l and k given earlier). Note also that Φ maps $(0,+\infty)$ in Σ onto $(-l,l)$ in R, and $\Phi(1) = 0$.

As Φ is a hyperbolic isometry of Σ onto R, if $x > 1$ then

$$d_R\big(0,\Phi(x)\big) = d_\Sigma(1,x) = \log x. \tag{6.2}$$

We also know that $\lambda_R(z)$ is given by

$$1/\operatorname{Re}[z] = \lambda_\Sigma(z) = \lambda_R\big(\Phi(z)\big)|\Phi'(z)|.$$

Thus, from the Chain Rule applied to $\Phi = hFg$, we obtain

$$|\Phi'(x)| = \frac{\pi}{K'\sqrt{(x^2+1)^2 - k^2(x^2-1)^2}}\,,$$

and hence

$$\lambda_R\big(\Phi(x)\big) = \frac{2K'}{\pi}\sqrt{\frac{1}{4}\left(x+\frac{1}{x}\right)^2 - \frac{k^2}{4}\left(x-\frac{1}{x}\right)^2}\,. \tag{6.3}$$

Now (6.2) and (6.3) yield

$$\lambda_R\big(\Phi(x)\big) = \frac{2K'}{\pi}\sqrt{1+(1-k^2)\sinh^2 d_R\big(0,\Phi(x)\big)}\,,$$

and as $\Phi(x)$ is any point of $(0,l)$, this with (6.1) proves that if $0 < x < l$ then

$$\lambda_R(x) = \frac{K}{l}\sqrt{1+(1-k^2)\sinh^2 d_R(0,x)}\,. \tag{6.4}$$

For brevity, we now write $\varrho(t) = d_R(0,t)$ for $0 \le t < l$. Then (6.4) implies that ϱ satisfies the differential equation

$$\frac{d\varrho}{dt} = \frac{K}{l}\sqrt{1 + k'^2 \sinh^2 \varrho},$$

and if we make the substitution $m(t) = \sinh \varrho(t)$ this gives

$$\frac{dm}{dt} = \frac{dm}{d\varrho}\,\frac{d\varrho}{dt} = \frac{K}{l}\sqrt{1+m^2}\,\sqrt{1+k'^2m^2}\,.$$

However, directly from (1.4), we have

$$\frac{d}{dx}\,F\!\left(\frac{x}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}\,\sqrt{1+k'^2x^2}}$$

so that if $x > 0$ then

$$F\!\left(\frac{x}{\sqrt{1+x^2}}\right) = \int_0^x \frac{dm}{\sqrt{1+m^2}\sqrt{1+k'^2m^2}} = \frac{K}{l}\int_0^\tau dt = \frac{K\tau}{l},$$

where $x = \sinh \varrho(\tau) = \sinh\, d_R(0,\tau)$. As $\mathrm{sn} = F^{-1}$, this shows that

$$\tanh\, d_R(0,\tau) = \mathrm{sn}\big(K\tau/l\big),$$

and as $\tanh^{-1}(y) = \frac{1}{2}\log\big[(1+y)/(1-y)\big]$, this gives (1.5).

Finally, we need to show that $d_R(0,x)$ is a convex function of x on $[0,l]$. The elliptic functions sn, cn and dn satisfy

$$\mathrm{cn}^2 + \mathrm{sn}^2 = 1, \qquad \mathrm{dn}^2 + k^2\mathrm{sn}^2 = 1, \qquad \mathrm{sn}'(y) = \mathrm{cn}(y)\mathrm{dn}(y),$$

and a straightforward calculation using (1.5) shows that

$$\frac{d}{dx}\big[d_R(0,x)\big] = \frac{K}{l}\sqrt{\frac{1-k^2\mathrm{sn}(y)^2}{1-\mathrm{sn}(y)^2}}\,,$$

where $y = Kx/l$. As $t \mapsto (1-k^2t)/(1-t)$ is an increasing function of t, it is immediate that $d_R(0,x)$ is a convex function of x as required. $\square$

We remark that Theorem 1.4 shows that the derivative of $d_R(0,x) - x$ is increasing, and as

$$\frac{d}{dx}d_R(0,x)\,\Big|_{x=0} = \lambda_R(0) > \lambda_S(0) = 1,$$

we see that $d_R(0,x)-x$ is increasing on $(0,l)$ and this proves (1.3). The same argument shows that when $0 \le x \le l - \frac{1}{2}\pi$ we have the sharp inequality

$$x \le d_R(0,x) \le x + d_R(0, l-\pi/2) - l + \pi/2, \tag{6.5}$$

and the constant on the right-hand side of this inequality can be evaluated using numerical tables for the values of sn. This technique allows us to see how accurate (1.1) and (1.2) are, and we give one example to illustrate these ideas. First, (6.1) implies that $(2K'/\pi)(l-\pi/2) = K - K'$, and this with (1.5) shows that

$$d_R\left(0, l - \frac{\pi}{2}\right) = \frac{1}{2}\log\left(\frac{1+\mathrm{sn}(K-K')}{1-\mathrm{sn}(K-K')}\right). \tag{6.6}$$

For example, if $k = 0.9$ then $K = 2.28055\ldots$ and $K' = 1.65462\ldots$, so that $l = 2.165\ldots$. Thus if $k = 0.9$ the lengths of the sides of R are approximately 3.14 and 4.33, and R is almost a square. Also, $K - K' = 0.62593\ldots$, so that $\mathrm{sn}(K-K')$ is between $\mathrm{sn}(0.6)$ and $\mathrm{sn}(0.7)$. Using the tables given in, for example, [6, pp. 278–284], and (6.6), we find that in this case, and with $x = l - \pi/2$, we have

$$0.01317\ldots \le d_R(0,x) - x \le 0.11786\ldots.$$

In any event, we now have the sharp inequality

$$x \le d_R(0,x) \le x + \frac{1}{2}\log\left(\frac{1+\mathrm{sn}(K-K')}{1-\mathrm{sn}(K-K')}\right) - l + \frac{\pi}{2},$$

where k is chosen so that $l = \pi K(k)/2K'(k)$, and this inequality is valid for all x in $(0, l-\pi/2)$.

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