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INFINITELY MANY ASYMPTOTIC VALUES OF LOCALLY UNIVALENT MEROMORPHIC FUNCTIONS

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Abstract. McMillan and Pommerenke showed that a locally univalent meromorphic function f in the disc, without Koebe arcs, has at least three asymptotic values in each boundary arc. The modular function shows that the number three is best possible. We show that if f satisfies certain further conditions, each of which narrowly excludes the modular function, then the number of asymptotic values in each boundary arc must be infinite.

1. Introduction

Let D denote the unit disc $\{z : |z| < 1\}$, C the unit circle $\{z : |z| = 1\}$, and $\widehat{\mathscr{C}}$ the extended complex plane. Let the function f be meromorphic in D. We say that f is *locally univalent* if it has at most simple poles and $f' \neq 0$, and that f has *Koebe arcs* if there are curves $J_n \subset D$ such that, for some $\alpha < \beta < \alpha + 2\pi$ and some $a \in \widehat{\mathscr{C}}$,

(a) J_n meets the radii $\arg z = \alpha$ and $\arg z = \beta$, for $n = 1, 2, \dots$;
(b) $|z| \to 1$, for $z \in J_n$, $n \to \infty$;
(c) $f(z) \to a$, for $z \in J_n$, $n \to \infty$.

A curve $\gamma : z(t)$, $0 \le t < 1$, in D is a *boundary path* if $|z(t)| \to 1$ as $t \to 1$. The set $\overline{\gamma} \cap C$ is called the *end* of γ. We say that f has the *asymptotic value* $a \in \widehat{\mathscr{C}}$ if there is a boundary path $\gamma : z(t)$, $0 \le t < 1$, such that

$$f\bigl(z(t)\bigr) \to a \qquad \text{as } t \to 1.$$

If f maps γ one-to-one onto a line segment, then the asymptotic value a is said to be *linearly accessible* along γ. Whenever the end of γ is contained in a subset E of C, we say that f has the asymptotic value a *in* E; if the end of γ is a singleton $\{\zeta\}$, then we say that f has the (*point*) asymptotic value a *at* ζ. It is clear that if f has no Koebe arcs, then all the asymptotic values of f are point asymptotic values.

In [17], McMillan and Pommerenke proved the following elegant result.

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Theorem A. *If f is meromorphic and locally univalent in D, and has no Koebe arcs, then f has at least three distinct point asymptotic values in each non-trivial arc of C.*

The modular function is locally univalent and its only asymptotic values are 0, 1 and ∞; see [14, p. 56] for a summary of facts about the modular function. Thus the number three is best possible in Theorem A. In this paper we show that if f satisfies certain further conditions, each of which narrowly excludes the modular function, then $\Gamma_P(f,\gamma)$, which denotes the set of point asymptotic values of f in any non-trivial arc γ of C, is infinite.

First we give a result for locally univalent analytic functions for which the maximum modulus $M(r,f)=\max\{|f(z)|:|z|=r\}$ does not grow too quickly.

Theorem 1. *Let f be analytic and locally univalent in D, with*

$$(1.1)\qquad (1-r)\log^+ M(r,f)=o(1)\qquad \text{as } r\to 1.$$

Then for each non-trivial arc γ in C, the set $\Gamma_P(f,\gamma)$ is infinite.

The modular function satisfies $(1-r)\log^+ M(r,f) = O(1)$, because it is normal; see below. Thus the condition $o(1)$ in (1.1) cannot be replaced by $O(1)$. Also, the assumption that f is locally univalent cannot be dropped. Indeed, there are unbounded analytic functions in D which grow 'arbitrarily slowly' but have just one asymptotic value, namely ∞; see [13] and the references therein.

We give an example in Section 4 to show that the conclusion of Theorem 1 cannot be strengthened to 'uncountable' no matter what restriction is placed on the growth of an unbounded $M(r,f)$. We remark that Bagemihl [3] gave several conditions on unbounded analytic functions f in D under which the set $\Gamma_P(f,\gamma)$ is always of positive linear measure.

Next recall that a meromorphic function f is said to be *normal* if the functions

$$f\big(\phi(z)\big),\qquad \text{where } \phi(z)=e^{i\theta}\left(\frac{z+a}{1+\bar a z}\right),\ |a|<1,\ \theta\in\mathscr{R},$$

form a normal family or, equivalently, if

$$(1.2)\qquad \sup_{z\in D}(1-|z|^2)f^{\#}(z)<\infty,\qquad \text{where } f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2};$$

see [10]. For example, the modular function is normal because it omits the three values 0, 1 and ∞. By a theorem of Bagemihl and Seidel [4], normal meromorphic functions do not have Koebe arcs, so all their asymptotic values are point asymptotic values, and even angular limits by a theorem of Lehto and Virtanen [12]. Thus Theorem A applies to normal meromorphic functions that are locally univalent, and the modular function again shows that the number three is best possible.

Note that there exist normal meromorphic functions in D with no asymptotic values. See [12] for an example based on a modification of the modular function; alternatively, certain other Schwarz triangle functions with multiple points give an example directly (those described in [18, p. 294, case 3]).

The class $\mathscr{N}_0$ consists of functions meromorphic in D such that

$$(1 - |z|^2) f^{\#}(z) \to 0 \qquad \text{as } |z| \to 1. \tag{1.3}$$

Such 'little normal' functions have been characterised in various ways; see [2], and also [7] where they were called 'strongly normal'. As noted in [1, p. 31], the hypothesis (1.3) means that the spherical radius of the largest schlicht disc around $f(z)$ on the Riemann image surface of f tends to 0 as $|z| \to 1$. In particular, every univalent function is in $\mathscr{N}_0$. The 'little Bloch' class $\mathscr{B}_0$ consists of analytic functions in D which satisfy

$$(1 - |z|^2)|f'(z)| \to 0 \qquad \text{as } |z| \to 1,$$

and these evidently lie in $\mathscr{N}_0$. There are functions in $\mathscr{B}_0$ which have finite angular limits almost nowhere on C, but all such functions must have finite angular limits on a set of Hausdorff dimension 1, by a result of Makarov; see [20, Chapters 8 and 11].

For locally univalent functions in $\mathscr{N}_0$, the conclusion of Theorem A can be greatly strengthened.

Theorem 2. *Let f be meromorphic in D, locally univalent and in $\mathscr{N}_0$. Then for each non-trivial arc γ in C, the set $\Gamma_P(f, \gamma)$ is of positive linear measure.*

We remark that the set of asymptotic values of a meromorphic function is known to be analytic and hence linearly measurable; see [16]. The conclusion of Theorem 2 is best possible, and the modular function shows that the hypothesis in Theorem 2 that $f \in \mathscr{N}_0$ cannot be replaced by the hypothesis that f is normal.

Finally, recall that a function f analytic in D is said to be in the MacLane class $\mathscr{A}$ if f is non-constant and has point asymptotic values at a dense set of points in C; see [14]. It is known that if

$$\int_0^1 \log^+ \log^+ M(r, f)\, dr < \infty, \tag{1.4}$$

then $f \in \mathscr{A}$; see [11]. In particular, normal analytic functions are in $\mathscr{A}$ because they satisfy $(1 - r) \log^+ M(r, f) = O(1)$ as $r \to 1$; see [10, p. 165].

The class $\mathscr{A}_m$, introduced in [5], is the extension of the class $\mathscr{A}$ to meromorphic functions. Theorem A shows that locally univalent normal meromorphic functions belong to $\mathscr{A}_m$. Theorem 2 shows that a stronger result holds if 'normal' is replaced by $\mathscr{N}_0$. We say that a set is *uncountably dense* in C if the set meets each non-trivial arc of C in an uncountable set.

Corollary 1. *Let f be meromorphic in D, locally univalent and in $\mathcal{N}_0$. Then f has angular limits at an uncountably dense set of points of C.*

It is natural to ask about the asymptotic values of a meromorphic function f in $\mathcal{N}_0$ without the hypothesis of local univalence. If

$$(1-|z|^2)f^{\#}(z) = O(1)(1-|z|)^{\varepsilon} \qquad \text{as } |z|\to 1,$$

where $\varepsilon > 0$, then it follows from a result of Carleson [6, p. 61] that f has angular limits at all points of C apart from a set of (an appropriate) capacity zero. It is plausible that if f is meromorphic in D and in $\mathcal{N}_0$, then f belongs to $\mathcal{A}_m$; we intend to return to this matter in another paper.

2. Proof of Theorem 1

It is sufficient to consider the case when the function f is unbounded near each interior point of the arc γ. Indeed, if f is bounded near some interior point ζ of γ, then $\Gamma_P(f,\gamma)$ has positive linear measure by [8, p. 120].

We assume that the set $\Gamma_P(f,\gamma)\setminus\{\infty\}$ of finite asymptotic values of f at points of γ is finite, and obtain a contradiction. It follows from (1.1), (1.4) and Cauchy's estimate that f and f' belong to the class $\mathcal{A}$. Since $f'\neq 0$, we deduce from [15, Theorem 9] that f has at least three distinct point asymptotic values in each non-trivial arc of C. Hence there is a cross-cut γ' of D ending at distinct points of γ on which f is bounded. Let $D(\gamma)$ be the component of $D\setminus\gamma'$ such that $\partial D(\gamma)\cap C\subset\gamma$. We can choose $R>0$ so that

$$\Gamma_P(f,\gamma)\setminus\{\infty\}\subset\{|w|\le R\}, \tag{2.1}$$

and

$$|f(z)|\le R, \qquad \text{for } z\in\gamma'. \tag{2.2}$$

Since f is unbounded near each interior point of γ, there is at least one component, Ω say, of $\{z:|f(z)|>R+1\}$ lying entirely in $D(\gamma)$, by (2.2). Pick $z_0\in\Omega$, let $w_0=f(z_0)$, and let g denote the branch of f^{-1} such that $g(w_0)=z_0$. By (2.1), (2.2) and the local univalence of f, the branch g can be analytically continued along any path from w_0 in $\{|w|>R\}$ without meeting any singularity. Indeed, since the Riemann surface of f has no branch points, such a singularity would give rise to an asymptotic value α of f, with $|\alpha|>R$, at a point of γ.

Thus if $H=\{t:\mathrm{Re}(t)>\log R\}$ and t_0 satisfies $e^{t_0}=w_0$, then $h(t)=g(e^t)$ can be analytically continued from t_0 along any path in H to give a single-valued analytic function, by the monodromy theorem. Two cases can then arise [18, p. 283], as follows.

(a) The function h is univalent in H.
(b) The function h is periodic with period $2\pi i m$, where m is a minimal positive integer.

In the present situation case (b) cannot occur. Indeed, if case (b) holds, then the function $G(w) = g(w^m)$ must be univalent in $\Delta_0 = \{|w| > R^{1/m}\}$, so it has a Laurent expansion there of the form

$$G(w) = a_1 w + a_0 + a_{-1} w^{-1} + \cdots .$$

Since g takes values in D, we have $a_1 = 0$. Hence G has a univalent extension to $\Delta_0 \cup \{\infty\}$ and so $a_{-1} \neq 0$. Then $G(\infty) = a_0$ is a pole of f in D, which is impossible.

Thus only case (a) can occur. Since the values taken by the function h in H all lie in the component of $\{z : |f(z)| > R\}$ which contains Ω, we deduce that Ω is conformally equivalent to $\{t : \mathrm{Re}(t) > \log(R+1)\}$ and that $\partial\Omega$ consists of a simple analytic curve in $D(\gamma)$.

Now we use the fact that f belongs to the MacLane class $\mathscr{A}$. Then by [14, Theorem 1], both ends of the simple curve $\partial\Omega$ must approach points of γ. Moreover, by the Carathéodory boundary correspondence theorem [20, p. 24], both ends of $\partial\Omega$ must approach the same point, ζ_0 say, of γ. It follows that the function

$$u(z) = \begin{cases} \log\bigl(|f(z)|/(R+1)\bigr), & \text{for } z \in \Omega, \\ 0, & \text{for } z \in D \setminus \Omega, \end{cases}$$

is subharmonic in D and tends to zero at each point of $C \setminus \{\zeta_0\}$. The hypothesis (1.1) implies that $\max\{u(z) : |z| = r\} = o(1)/(1-r)$ as $r \to 1$ so, by a special case of Dahlberg's radial maximum theorem [9, Theorem 2], u must be identically zero, a contradiction. This completes the proof of Theorem 1.

Remark. By a result of MacLane [15, Theorem 8], it follows that under the hypotheses of Theorem 1 the set of linearly accessible finite asymptotic values of f is unbounded and so infinite.

3. Proof of Theorem 2

First we consider the case when there exists $\alpha \in \widehat{\mathscr{C}}$ such that f is bounded away from α, in the spherical metric, near some interior point ζ of γ. Then f or $g(z) = 1/\bigl(f(z) - \alpha\bigr)$ is bounded near ζ, and it follows from [8, p. 120] that $\Gamma_P(f, \gamma)$ is of positive linear measure.

There remains the case when, for each $\alpha \in \widehat{\mathscr{C}}$, the function f takes values arbitrarily close to α, in the spherical metric, in any neighbourhood of any interior point of γ. To deal with this case, we suppose that the set $\Gamma_P(f, \gamma)$ is of linear measure zero, and obtain a contradiction.

Since normal functions do not have Koebe arcs [4], we deduce from Theorem A that there is a cross-cut γ' of D ending at distinct points of γ on which f is bounded, say

$$|f(z)| \le R, \qquad \text{for } z \in \gamma', \tag{3.1}$$

for some $R > 3$. Let $D(\gamma)$ be the component of $D \setminus \gamma'$ such that $\partial D(\gamma) \cap C \subset \gamma$. Since f takes values arbitrarily close to any $\alpha \in \widehat{\mathscr{C}}$ near each point of γ, there is some $z_0 \in D(\gamma)$ such that

$$R + 2 < |f(z_0)| < 2R, \tag{3.2}$$

and, by (1.3),

$$(1 - |z_0|^2) f^{\#}(z_0) < \frac{1}{1 + 4R^2}. \tag{3.3}$$

Moreover, since f is an open mapping and $\Gamma_P(f, \gamma)$ has linear measure zero, we can assume, by varying z_0 slightly if necessary, that

$$w_0 = f(z_0) \text{ does not belong to } \Gamma_P(f, \gamma). \tag{3.4}$$

From (3.2) and (3.3), we deduce that

$$(1 - |z_0|^2)|f'(z_0)| < 1. \tag{3.5}$$

For $z \in D$, let $d(z)$ denote the radius of the maximal schlicht disc in the image surface $\mathscr{F}$ of f centred at the point of $\mathscr{F}$ corresponding to $f(z)$. Since f is locally univalent the surface $\mathscr{F}$ has no branch points, so this maximal schlicht disc has a (linearly accessible) asymptotic value of f on its boundary corresponding to an inverse function singularity of f. It follows from Schwarz's lemma (see [21]) that if f is analytic at z, then

$$d(z) \leq (1 - |z|^2)|f'(z)|. \tag{3.6}$$

We claim that there is a path γ_0 in D with one endpoint at z_0 such that

$$\text{there exist distinct } \zeta_n \in \gamma_0 \text{ such that } f(\zeta_n) = w_0, \text{ for } n = 1, 2, \ldots, \tag{3.7}$$

and

$$|f(z) - w_0| \leq 2d(z_0), \qquad \text{for } z \in \gamma_0. \tag{3.8}$$

To prove this claim, let Σ_0 denote the maximal schlicht disc in $\mathscr{F}$ centred at the point of $\mathscr{F}$ corresponding to $w_0 = f(z_0)$ and let w_0' be a singularity of f^{-1} on the boundary of Σ_0. Then $|w_0' - w_0| = d(z_0) < 1$ by (3.5) and (3.6). Also, let Γ_0' denote the radius of Σ_0 from w_0 to w_0'.

Next let g denote the branch of f^{-1} such that $g(w_0) = z_0$, so g is univalent in Σ_0. Let Σ_0' denote the open disc in $\mathscr{C}$ with diameter along the line segment Γ_0'.

Then Σ_0' has radius $\frac{1}{2}d(z_0)$ and centre $\frac{1}{2}(w_0 + w_0') = w_0^*$, say. We consider the analytic continuation of g from near w_0 along paths in $\mathscr{C} \setminus \overline{\Sigma_0'}$.

First suppose that such analytic continuation of g meets no singularities of f^{-1}. Choose t_0 such that

$$w_0^* + \tfrac{1}{2}d(z_0)e^{t_0} = w_0. \tag{3.9}$$

Then the function

$$h(t) = g\big(w_0^* + \tfrac{1}{2}d(z_0)e^{t}\big),$$

can be analytically continued along any path in $H = \{\mathrm{Re}(t) > 0\}$, starting near t_0. Thus h is a single-valued analytic function in H, by the monodromy theorem, and $h(t_0) = z_0$. Two cases can then arise [18, p. 283], as follows.

(a) The function h is univalent in H.
(b) The function h is periodic with period $2\pi i m$, where m is a minimal positive integer.

Once again, case (b) cannot occur in the present situation. Indeed, if case (b) holds, then the function $G(w) = g\big(w_0^* + (w - w_0^*)^m\big)$ must be univalent in $\Delta_0 = \big\{|w - w_0^*| > \big(\frac{1}{2}d(z_0)\big)^{1/m}\big\}$, so it has a Laurent expansion there of the form

$$G(w) = a_1(w - w_0^*) + a_0 + a_{-1}(w - w_0^*)^{-1} + \cdots.$$

Since g takes values in D, we have $a_1 = 0$. Hence G has a univalent extension to $\Delta_0 \cup \{\infty\}$ and so $a_{-1} \neq 0$. Then $G(\infty) = a_0$ is a pole of f which must be simple. Writing $z = G(w)$, we obtain $f(z) = w_0^* + (w - w_0^*)^m$ and

$$z - a_0 = \frac{a_{-1}}{w - w_0^*}\big(1 + o(1)\big) \qquad \text{as } w \to \infty,$$

so

$$f(z) = w_0^* + \frac{(a_{-1})^m}{(z - a_0)^m}\big(1 + o(1)\big) \qquad \text{as } z \to a_0.$$

Thus $m = 1$, so g is univalent in $\Delta_0 \cup \{\infty\}$. Because $\partial\Delta_0 \setminus \{w_0'\} \subset \Sigma_0$, it follows that g is analytic in $\Sigma_0 \cup (\Delta_0 \cup \{\infty\}) = \widehat{\mathscr{C}} \setminus \{w_0'\}$. Since g takes values in D, this contradicts Liouville's theorem.

Thus case (a) occurs, so h is univalent in H. Moreover, we can find open discs D_n with centres $t_n = t_0 + 2\pi i n$, $n \in \mathscr{Z}$, such that h is univalent in the larger set

$$H' = H \cup \bigcup_{n \in \mathscr{Z}} D_n.$$

This follows from (3.9) and the fact that the analytic continuation of g along any path in $\mathscr{C} \setminus \overline{\Sigma_0'}$ which ends at w_0 can be extended to a neighbourhood of w_0; for otherwise $w_0 \in \Gamma_P(f, \gamma)$ contrary to (3.4).

We now take a simple path β_0 in $H' \cap \{\mathrm{Re}(t) < 1\}$ with one endpoint at t_0, passing through every $t_n, n = 0, 1, 2, \ldots$, and tending to ∞. Then $\gamma_0 = h(\beta_0)$ is a simple path in D with one endpoint at z_0. Since $f(h(t)) = w_0^* + \frac{1}{2}d(z_0)e^t$, we deduce that (3.7) holds, with $\zeta_n = h(t_n)$, and also (3.8) holds, as required.

Suppose, on the other hand, that the analytic continuation of g from w_0 along some path Γ_0'' in $\mathscr{C} \setminus \overline{\Sigma_0'}$ *does* encounter a singularity of f^{-1}, at w_0'' say. Choose $r_0 > 0$ such that $r_0 < \min\{d(z_0), \operatorname{dist}(w_0', \Gamma_0'')\}$. Then let Γ_0 be a Jordan curve in $\mathscr{C}$ which surrounds w_0' and consists of two chords through w_0 of the disc $\overline{\Sigma_0'}$ together with an arc of a circle $\{|w - w_0'| = r\}$, where $0 < r < r_0$. Since $\Gamma_P(f, \gamma)$ has linear measure zero, it fails to meet almost all such chords and almost all such circular arcs, so we can assume by (3.2) that $\Gamma_0 \subset \{|w| > R\} \setminus \Gamma_P(f, \gamma)$. In particular, the branch g can be analytically continued from w_0 around Γ_0 arbitrarily far (in either direction). Note that by construction Γ_0, Γ_0', Γ_0'' meet only at w_0.

Let γ_0, γ_0', γ_0'' denote the paths in D obtained by analytic continuation of g along Γ_0, Γ_0', Γ_0'', respectively. Then γ_0' and γ_0'' are asymptotic paths meeting only at z_0, so the path $\gamma_0' \cup \gamma_0''$ forms a cross-cut of D. The path γ_0 can meet $\gamma_0' \cup \gamma_0''$ only at z_0. Thus, by the local univalence of f, the path γ_0 must pass through z_0 exactly once, as it crosses $\gamma_0' \cup \gamma_0''$. Also, if the part of γ_0 on one side of the cross-cut were to intersect itself, then at the first point of intersection there would again be a contradiction to local univalence. It follows that γ_0 is a simple path passing through z_0 such that (3.7) and (3.8) hold. Thus we have established the claim.

It follows from (3.7) that $|\zeta_n| \to 1$ as $n \to \infty$. Hence $d(\zeta_n) \to 0$ as $n \to \infty$ by (1.3) and (3.6), because f is bounded on γ_0 by (3.8). Thus we can truncate γ_0 at a point, z_1 say, chosen from the ζ_n, such that

$$f(z_1) = w_0, \quad 1 - |z_1| < \tfrac{1}{2} \quad \text{and} \quad d(z_1) \le \tfrac{1}{2}d(z_0) < \tfrac{1}{2}.$$

Now we repeat the above argument with z_0 replaced by z_1 to obtain a path γ_1 in D with one endpoint at z_1 such that

$$\text{there exist distinct } \zeta_n \in \gamma_1 \text{ such that } f(\zeta_n) = w_0, \text{ for } n = 1, 2, \ldots,$$

and

$$|f(z) - w_0| \le 2d(z_1), \quad \text{for } z \in \gamma_1.$$

We then truncate γ_1 at a point z_2 such that

$$f(z_2) = w_0, \quad 1 - |z_2| < \tfrac{1}{4} \quad \text{and} \quad d(z_2) \le \tfrac{1}{2}d(z_1) < \tfrac{1}{4}.$$

By repeating this process, we obtain a sequence of such truncated paths γ_n, $n = 0, 1, 2, \ldots$, such that $f(z)$ tends to w_0 along the path

$$\gamma_\infty = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \cdots,$$

and, by (3.2), $|f(z)| > R$ for $z \in \gamma_\infty$. By (3.1) the end of γ_∞ must lie in γ, so γ_∞ must tend to a point of γ because f has no Koebe arcs. Hence $w_0 \in \Gamma_P(f, \gamma)$, which contradicts (3.4). The proof of Theorem 2 is complete.

Remarks. 1. The proof can be adapted to show that if, in any neighbourhood of any point of γ, the locally univalent function f in $\mathscr{N}_0$ takes values arbitrarily close to each $\alpha \in \widehat{\mathscr{C}}$, in the spherical metric, then $\Gamma_P(f,\gamma)$ has positive linear measure in any neighbourhood of each point of $\widehat{\mathscr{C}}$. This holds near ∞ by taking R arbitrarily large in (3.1) and, for other points α, we consider the function in $\mathscr{N}_0$ obtained by composing f with a rotation of the Riemann sphere taking α to ∞. It seems possible that under these circumstances we must actually have $\Gamma_P(f,\gamma) = \widehat{\mathscr{C}}$.

2. An earlier version of this proof, which was more like the proof of Theorem 1, examined the components of $\{z : |f(z)| > R\}$ in $D(\gamma)$. This raised the question of whether the level sets $L(R) = \{z : |f(z)| = R\}$, $R > 0$, end at points of C, in the sense of MacLane [14, p. 8]. In fact, the proof of Theorem A in [17] shows that the 'three asymptotic values' obtained in each arc are linearly accessible. It follows that if f satisfies the hypotheses of Theorem A, then every level set $L(R)$, $R > 0$, of f must end at points.

4. An example

We show here that the conclusion in Theorem 1 cannot be strengthened to 'uncountable'.

Example 1. *Let μ be an increasing function on $[0,1)$ such that $\mu(0) = 1$ and $\mu(r) \to \infty$ as $r \to 1$. Then there exists a function f analytic and locally univalent in D, such that*

$$M(r,f) \le \mu(r), \qquad \text{for } 0 \le r < 1, \tag{4.1}$$

and the set of asymptotic values of f is countable.

Our approach is to follow the geometric construction of MacLane [13] for an analytic function in D of 'arbitrarily slow growth' with level curves C_n, $n = 1,2,\ldots$, which are nested Jordan curves that expand towards C, on which $|f| = \varrho_n$, $n = 1,2,\ldots$, where $\varrho_n \to \infty$ as $n \to \infty$. The method is to construct the image Riemann surface of the function by a process of successive enlargements, and the main difference in our construction is that we replace MacLane's adjoined two-sheeted surfaces by adjoined logarithmic spirals.

We first fix a positive strictly increasing sequence ϱ_n, with $\varrho_n \to \infty$ as $n \to \infty$, and put $\alpha_n = n\sqrt{2}$, $n = 1,2,\ldots$. Then define

$$a_{n,k} = \varrho_n \exp(i\theta_{n,k}), \qquad \text{where } \theta_{n,k} = 2\pi\alpha_n + \frac{2\pi k}{\nu_n},\ k = 1,2,\ldots,\nu_n. \tag{4.2}$$

For $n = 1,2,\ldots$, the points $a_{n,k}$, $k = 1,2,\ldots,\nu_n$, are equally spaced on the circle $\{|w| = \varrho_n\}$ and no two $a_{n,k}$, $k = 1,2,\ldots,\nu_n$, $n = 1,2,\ldots$, are in the same direction from 0. The sequences ϱ_n and α_n are fixed throughout, but ν_n will be varied to determine the required Riemann surface.

Let

$$L(n,k) = \{|w| \geq \varrho_n,\ \arg w = \theta_{n,k}\}, \qquad k = 1,2,\ldots,\nu_n,\ n = 1,2,\ldots,$$

so the rays $L(n,k)$, $k = 1,2,\ldots,\nu_n$, $n = 1,2,\ldots,$ are mutually disjoint. The required surface $\mathscr{S}$ is built in stages. We begin with the surface $\mathscr{S}_1$ consisting of the w-plane slit along all rays $L(1,k)$, $k = 1,2,\ldots,\nu_1$. Then the surface $\mathscr{S}_2$ is constructed by first adjoining a half-logarithmic spiral to each edge of each slit in $\mathscr{S}_1$, and then making a slit over every ray $L(2,k)$, $k = 1,2,\ldots,\nu_2$, in all the resulting sheets. The surface $\mathscr{S}_3$ is constructed in an analogous manner by adjoining half-logarithmic spirals to edges of existing slits in $\mathscr{S}_2$ wherever possible and then making new slits over the rays $L(3,k)$, $k = 1,2,\ldots,\nu_3$, wherever possible.

Repeating this process indefinitely, we obtain a surface $\mathscr{S}$ without algebraic branch points. The surface $\mathscr{S}$ is simply connected because at each stage the addition of each pair of half logarithmic spirals preserves the property of being simply connected, and $\mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{S}_n$. Now let $\mathscr{T}_n$ denote that component of $\mathscr{S}_n$ over $\{|w| < \varrho_n\}$ which contains the origin in $\mathscr{S}_1$. If $\nu_1, \nu_2, \ldots, \nu_{n-1}$ are fixed and $\nu = \nu_n$ is variable, then as $\nu \to \infty$ the sequence of surfaces $\mathscr{T}_{n+1} = \mathscr{T}_{n+1}(\nu)$ converges to its Carathéodory kernel, the unique maximal Riemann surface all of whose compact subsets K can be embedded in $\mathscr{T}_{n+1}(\nu)$, for $\nu \geq \nu(K)$. This kernel is $\mathscr{T}_n$. We now need the following generalisation of the Carathéodory kernel theorem, due essentially to L.I. Volkovyskii; see [13].

Theorem B. *Let R_n, $n = 1,2,\ldots,$ be a sequence of positive numbers such that $\lim_{n\to\infty} R_n = R$, $0 < R < \infty$. Let $\mathscr{F}_n = \{|z| < R_n\}$ and $\mathscr{F} = \{|z| < R\}$. Let $[\mathscr{G}_n; Q_n]$ be a sequence of Riemann surfaces over the w-plane, each containing a schlicht disc of radius $s_0 > 0$ about Q_n, where Q_n is a point of $\mathscr{G}_n$ over $w = 0$. Let $w = F_n(z)$ be holomorphic in $\mathscr{F}_n$ and map $\mathscr{F}_n$ one-one onto $\mathscr{G}_n$ with $F_n(0) = Q_n$ and $F_n'(0) = 1$. Let Φ_n be the inverse of F_n. Then the following are true.*

(a) *If $F_n \to F$ locally uniformly on $\mathscr{F}$, then $[\mathscr{G}_n; Q_n]$ converges to its kernel $[\mathscr{G}; Q]$ and F maps $\mathscr{F}$ one-one onto $\mathscr{G}$, with $F(0) = Q$ and $F'(0) = 1$. Also, $\Phi_n \to \Phi$ locally uniformly on $\mathscr{G}$ and Φ is the inverse of F.*

(b) *If $\{F_n\}$ is a normal family on $\mathscr{F}$ and if $[\mathscr{G}_n; Q_n]$ converges to its kernel $[\mathscr{G}; Q]$, then $F_n \to F$ locally uniformly on $\mathscr{F}$.*

Now let f_n map $\{|z| < r_n\}$ onto $\mathscr{T}_n$, with $f_n(0) = 0 \in \mathscr{S}$ and $f_n'(0) = 1$. Since $g_n = f_{n+1}^{-1} \circ f_n$ maps $\{|z| < r_n\}$ into $\{|z| < r_{n+1}\}$ with $g_n(0) = 0$ and $g_n'(0) = 1$, we deduce that

$$(4.3) \qquad r_n < r_{n+1}, \qquad \text{for } n = 1,2,\ldots.$$

Let δ_n be any positive sequence such that

$$\sum_{n=1}^{\infty} \delta_n < \infty. \tag{4.4}$$

We shall show that the sequence ν_n, $n = 1, 2, \ldots$, can be chosen so that

$$r_{n+1} < r_n + \delta_n, \qquad \text{for } n = 1, 2, \ldots . \tag{4.5}$$

To do this, suppose that $\nu_1, \nu_2, \ldots, \nu_{n-1}$ have already been chosen. As above, we indicate dependence on $\nu = \nu_n$ by writing $\mathscr{T}_{n+1} = \mathscr{T}_{n+1}(\nu)$, $f_{n+1}(z) = f_{n+1}(z;\nu)$ and $r_{n+1} = r_{n+1}(\nu)$. Now choose a subsequence of values of ν so that

$$r_{n+1}(\nu) \to r_{n+1}(\infty) \geq r_n .$$

We claim that $r_{n+1}(\infty) = r_n$. Indeed, the subsequence $f_{n+1}(z;\nu)$ is defined eventually on each compact subset of $\{|z| < r_{n+1}(\infty)\}$, is uniformly bounded by ϱ_{n+1} and hence forms a normal family. Also, the sequence $\mathscr{T}_{n+1}(\nu)$ converges to its kernel $\mathscr{T}_n$. Thus, by Theorem B, part (b),

$$f_{n+1}(z;\nu) \to f_{n+1}(z;\infty) \qquad \text{as } \nu \to \infty,$$

locally uniformly and, by Theorem B, part (a), $f_{n+1}(z;\infty)$ maps $\{|z| < r_{n+1}(\infty)\}$ onto $\mathscr{T}_n$ with the same normalization as f_n. Hence $r_{n+1}(\infty) = r_n$. Thus we can indeed choose ν_n so that (4.5) holds.

Now we apply Theorem B to the sequences f_n and $\mathscr{T}_n$. By (4.4), $r_n \to R < \infty$ and by construction $\mathscr{T}_n$ converges to its kernel $\mathscr{S}$. To show that the sequence f_n is normal, let f^* map $\{|z| < R^*\} = D^*$ onto $\mathscr{S}$ with $f^*(0) = 0 \in \mathscr{S}$ and $(f^*)'(0) = 1$; here $0 < R^* \leq \infty$. Since $\mathscr{T}_n \subset \mathscr{S}$, the functions $h_n = (f^*)^{-1} \circ f_n$ are conformal and normalised in the usual way. Thus the family $\{h_n\}$ is normal and it follows that the family $f_n = f^* \circ h_n$ is normal in $\{|z| < R\}$. Hence, by Theorem B, part (b), $f_n \to f$ locally uniformly in $\{|z| < R\}$, where f maps $\{|z| < R\}$ onto $\mathscr{S}$, the kernel of the sequence $\mathscr{T}_n$, with $f(0) = 0 \in \mathscr{S}_1$ and $f'(0) = 1$.

The proof that the sequence ν_n can be chosen in such a way that the condition (4.1) holds, is based on (4.3) and (4.4), as in [13, Section 3]. Finally, it is clear from the construction of $\mathscr{S}$ that the function f mapping $\{|z| < R\}$ onto $\mathscr{S}$ is locally univalent and has countably many finite asymptotic values, namely $a_{n,k}$, $k = 1, 2, \ldots, \nu_n$, $n = 1, 2, \ldots$. This completes the proof of Example 1.

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