Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 27, 2002, 151–161

A REMARK ON THE AHLFORS–LEHTO UNIVALENCE CRITERION

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Abstract. In this note, we will prove the Ahlfors–Lehto univalence criterion in a general form. This enables us to deduce lower estimates of the inner radius of univalence for an arbitrary quasidisk in terms of a given quasiconformal reflection.

1. Introduction

Let D be a domain in the Riemann sphere $\widehat{\mathbf{C}}$ with hyperbolic metric $\varrho_D(z)|dz|$ of constant negative curvature -4. For a holomorphic function φ on D, we define the hyperbolic sup-norm of φ by

$$\|\varphi\|_D = \sup_{z\in D} \varrho_D(z)^{-2}|\varphi(z)|.$$

We denote by $B_2(D)$ the complex Banach space consisting of all holomorphic functions of finite hyperbolic sup-norm. For a holomorphic map $g\colon D_1 \to D_2$, the pullback $g_2^*\colon \varphi \mapsto \varphi\circ g\cdot (g')^2$ is a linear contraction from $B_2(D_2)$ to $B_2(D_1)$. In particular, if g is biholomorphic, $g_2^*\colon B_2(D_2)\to B_2(D_1)$ becomes an isometric isomorphism. As is well known, the Schwarzian derivative $S_f = (f''/f')' - (f''/f')^2/2$ of a univalent function f on D satisfies $\|S_f\|_D \le 12$ (see [3]). This result is classical for the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, actually, the better estimate $\|S_f\|_{\mathbf{D}} \le 6$ is known. On the other hand, Nehari's theorem [13] asserts that if a locally univalent function f on $\mathbf{D}$ satisfies $\|S_f\|_{\mathbf{D}} \le 2$, then f is necessarily univalent. Hille's example [7] shows that the number 2 is best possible. We now define the quantity $\sigma(D)$, which is called the *inner radius of univalence* of D, as the infimum of the norm $\|S_f\|_D$ of those locally univalent meromorphic functions f on D which are not globally univalent in D. In other words, $\sigma(D)$ is the possible largest number $\sigma\ge 0$ with the property that the condition $\|S_f\|_D\le\sigma$ implies univalence of f in D. Note that the inner radius of univalence is Möbius invariant, namely, $\sigma\bigl(L(D)\bigr) = \sigma(D)$ for a Möbius transformation L. In the case $D = \mathbf{D}$, we already know $\sigma(\mathbf{D}) = 2$. For a comprehensive exposition of these notions and some background, we refer the reader to the book [9] of O. Lehto.

2000 Mathematics Subject Classification: Primary 30C62; Secondary 30C55, 30F60.

Ahlfors [1] showed that every quasidisk has positive inner radius of univalence. Conversely, Gehring [6] proved that if a simply connected domain has positive inner radius of univalence then it must be a quasidisk. Later, Lehto [8] pointed out the inner radius of univalence of a quasidisk can be estimated by using Ahlfors' method as

$$\sigma(D) \ge 2 \inf_{z\in D'} \frac{|\bar\partial\lambda(z)| - |\partial\lambda(z)|}{|\lambda(z)-z|^2\varrho_D(z)^2}, \tag{1}$$

where λ is a quasiconformal reflection in ∂D which is continuously differentiable off ∂D and $D' = D\setminus\{\infty,\lambda(\infty)\}$. This result may be called the Ahlfors–Lehto univalence criterion. However, in order to obtain estimate (1) rigorously, a kind of approximation procedure must work, so an additional assumption was needed. For example, Lehto [9, Lemma III.5.1] assumed the quasidisk D to be exhausted by domains of the form $\{rz : z\in D\}$ for $0<r<1$. More recently, Betker [5] gave a similar result for general quasidisks under the assumption that the quasiconformal reflections λ are of a special form associated with the Löwner chains. For another additional condition, see a remark at the end of the next section.

We remark that if we content ourselves with an estimate of the form $\sigma(D)\ge c(K)$ for a K-quasidisk D, where $c(K)$ is a positive constant depending only on K, the original idea of Ahlfors [1] is sufficient. (See Section 2. See also [2, Chapter VI] and [9, Theorem II.4.1] for slightly different approaches.)

Our main result is to show (1) without any additional assumption, even the continuous differentiability of λ. This might be known as a kind of folklore.

Theorem 1. *Let D be a quasidisk with a quasiconformal reflection λ in ∂D. Then the inequality $\sigma(D)\ge\varepsilon(\lambda,D)$ holds for D, where*

$$\varepsilon(\lambda,D) = 2\operatorname*{ess.inf}_{z\in D} \frac{|\bar\partial\lambda(z)| - |\partial\lambda(z)|}{|\lambda(z)-z|^2\varrho_D(z)^2}. \tag{2}$$

Actually, this estimate is known to give often sharp results for several concrete examples (see [9]). The author, however, does not know if the equality $\sigma(D) = \sup_\lambda \varepsilon(\lambda,D)$ always holds or not, where the supremum is taken over all possible quasiconformal reflections λ in ∂D.

Our main theorem has applications to lower estimates of the inner radius of univalence for a strongly starlike domain (see [15]) and for a round annulus (see [14]). Indeed, the quasiconformal reflections used in those papers are not necessarily of class C^1 off the boundary.

Finally, the author would like to express his sincere thanks to Professor F.W. Gehring, whose suggestion improved the statement of our main theorem.

2. Proof of the main result

Basically, we shall go along the same line as in [1]. Let a quasiconformal reflection λ in ∂D be given, i.e., λ is an orientation-reversing homeomorphic involution of $\widehat{\mathbf{C}}$ keeping each boundary point of D fixed and satisfying that $\lambda(\bar{z})$ is quasiconformal. We note that $|\partial\lambda| \le k_0|\bar\partial\lambda|$ a.e. for some constant $0 \le k_0 < 1$.

Noting that the quantity $\varepsilon(\lambda, D)$ is invariant under the Möbius transformations (see [9, Section II 4.1]), we assume that a quasidisk D is contained in $\mathbf{C}$ for a moment. We take a non-constant meromorphic function f on D with $\|S_f\|_D < \varepsilon_0 = \varepsilon(\lambda, D)$. We wish to show that f is univalent in D. Set $\varphi = S_f$.

Let η_0 and η_1 be linearly independent solutions of the linear differential equation

$$2y'' + \varphi y = 0 \tag{3}$$

in D. Note that the Wronskian $\eta_0\eta_1' - \eta_0'\eta_1$ is a non-zero constant and, as is well known, η_1/η_0 satisfies the Schwarzian differential equation $S_{\eta_1/\eta_0} = \varphi = S_f$ in D. In particular, there exists a Möbius transformation L satisfying $\eta_1/\eta_0 = L \circ f$. Therefore, when we try to show the univalence of f, we can assume, and always do so in the sequel, $f = \eta_1/\eta_0$ and $\eta_0\eta_1' - \eta_0'\eta_1 \equiv 1$.

For instance, if η_0 and η_1 are taken by the solutions of (3) satisfying the initial conditions $\eta_0 = 1$, $\eta_0' = 0$ and $\eta_1 = 0$, $\eta_1' = 1$, respectively, at a reference point z_0 in D, then $f = \eta_1/\eta_0$ is strongly normalized at z_0: $f(z_0) = f'(z_0) - 1 = f''(z_0) = 0$.

To extend f to the whole sphere, we consider the map

$$F(z) = \frac{\eta_1(z) + \big(\lambda(z) - z\big)\eta_1'(z)}{\eta_0(z) + \big(\lambda(z) - z\big)\eta_0'(z)}.$$

We first note the Möbius invariance of the above construction. For an $A = \binom{a\,b}{c\,d} \in \mathrm{SL}(2, \mathbf{C})$ let L_A be the Möbius transformation induced by the matrix A. We set $A_2^*\varphi = (\varphi\circ L_A)(L_A')^2$ and $A_{-1/2}^*\eta = (\eta\circ L_A)(L_A')^{-1/2}$, where $(L_A')^{-1/2}(z) = cz + d$. A straightforward calculation shows that $A_{-1/2}^*\eta$ is a solution of the differential equation $2y'' + A_2^*\varphi y = 0$ in $A^{-1}(D)$ if η is a solution of (3) in D. In particular, we can see that differential equation (3) always admits two linearly independent (single-valued) solutions in D even if $\infty \in D$. Setting $\tilde\lambda = L_A^{-1}\circ\lambda\circ L_A$, we consider the map

$$\widetilde{F}(z) = \frac{A_{-1/2}^*\eta_1(z) + \big(\tilde\lambda(z) - z\big)(A_{-1/2}^*\eta_1)'(z)}{A_{-1/2}^*\eta_0(z) + \big(\tilde\lambda(z) - z\big)(A_{-1/2}^*\eta_0)'(z)}.$$

Then we have the relation $F \circ L_A = \widetilde{F}$. In fact, using the relation $L_A(w) - L_A(z) = (w-z)/(cw+d)(cz+d)$, we obtain

$$\begin{aligned}
&\eta \circ L_A(z) + \big(\lambda \circ L_A(z) - L_A(z)\big)\eta' \circ L_A(z) \\
&= \frac{A^*_{-1/2}\eta(z)}{cz+d} + \big(L_A\big(\tilde\lambda(z)\big) - L_A(z)\big)(cz+d)\big((A^*_{-1/2}\eta)'(z) - c\eta \circ L_A(z)\big) \\
&= \frac{A^*_{-1/2}\eta(z)}{cz+d} + \frac{\tilde\lambda(z) - z}{c\tilde\lambda(z)+d}\left((A^*_{-1/2}\eta)'(z) - \frac{cA^*_{-1/2}\eta(z)}{cz+d}\right) \\
&= \frac{A^*_{-1/2}\eta(z) + (\tilde\lambda(z) - z)(A^*_{-1/2}\eta)'(z)}{c\tilde\lambda(z)+d}.
\end{aligned}$$

Taking η_1 and η_0 as the above η, we see the desired relation.

Next, we need the following fundamental property of F.

Lemma 2. *The map $F\colon D \to \widehat{\mathbf{C}}$ constructed above is K-anti-quasiregular, where $K = (1+k)/(1-k)$, $k = 1-(1-k_0)(1-k_1) < 1$, and $k_1 = \|\varphi\|_D/\varepsilon_0 < 1$.*

Proof. By the Möbius invariance of the construction of F, we may assume here that $\infty \in \partial D$. We note that the numerator and the denominator in the definition of the map F can never vanish simultaneously because of the relation $\eta_0\eta_1' - \eta_0'\eta_1 \equiv 1$. Since K-anti-quasiregularity is a local property, it is enough to show that F is K-anti-quasiregular in a neighbourhood of an arbitrary point, say a, in D. We may assume that $\eta_0(a) + \big(\lambda(a) - a\big)\eta_0'(a) \neq 0$. (If not, consider $1/F$ instead.) By continuity, we can take an open neighbourhood V of a in D such that $\eta_0 + (\lambda - z)\eta_0'$ does not vanish at any point of V.

Here, we recall that a non-constant continuous function $h\colon V \to \mathbf{C}$ is K-anti-quasiregular if and only if h is ACL and has locally square integrable partial derivatives satisfying $|\partial h| \le k|\bar\partial h|$ a.e. in V (see [10, Chapter VI], where the authors used the term "quasiconformal function" instead of "quasiregular mapping").

Now we show that F is ACL in V, precisely, for any closed rectangle $\{x+iy : x_0 \le x \le x_1,\ y_0 \le y \le y_1\}$ contained in V, $F(x+iy)$ is absolutely continuous in $x \in [x_0, x_1]$ for a.e. $y \in [y_0, y_1]$ and in $y \in [y_0, y_1]$ for a.e. $x \in [x_0, x_1]$. Since $\eta_j + (\lambda - z)\eta_j'$ is absolutely continuous in $x \in [x_0, x_1]$ for a.e. y and for $j = 0, 1$, and since $\eta_0 + (\lambda - z)\eta_0'$ does not vanish there, we can conclude that the quotient $F(x+iy)$ is also absolutely continuous in $x \in [x_0, x_1]$ for a.e. $y \in [y_0, y_1]$ and in $y \in [y_0, y_1]$ for a.e. $x \in [x_0, x_1]$ (see, for example, [12, p. 50]). Hence, F is ACL in V.

Next, we investigate the partial derivatives of F. A formal calculation gives us

$$\partial F = \frac{\partial\lambda + (\lambda - z)^2\varphi/2}{\big(\eta_0 + (\lambda - z)\eta_0'\big)^2} \quad \text{and} \quad \bar\partial F = \frac{\bar\partial\lambda}{\big(\eta_0 + (\lambda - z)\eta_0'\big)^2}.$$

Since λ has locally square integrable partial derivatives in D and since the denominator is locally bounded away from 0 in V, we can observe that ∂F and $\bar\partial F$ are both locally square integrable in V. Furthermore, we have

$$\frac{\partial F(z)}{\bar\partial F(z)} = \frac{\partial\lambda(z) + \bigl(\lambda(z)-z\bigr)^2\varphi(z)/2}{\bar\partial\lambda(z)}.$$

Hence the assumption $\|\varphi\|_D = \varepsilon_0 k_1$ implies

$$\|\partial F/\bar\partial F\|_\infty \le 1-(1-k_0)(1-k_1) = k.$$

Hence, we have shown that F is K-anti-quasiregular in V. □

Now we consider the map $\hat f\colon D\cup D^* \to \widehat{\mathbf{C}}$ defined by

$$\hat f(z) = \begin{cases} f(z) & \text{for } z\in D, \\ F\bigl(\lambda(z)\bigr) & \text{for } z\in D^*, \end{cases} \tag{4}$$

where $D^* = \widehat{\mathbf{C}}\setminus\overline{D}$. It is not so clear that $\hat f$ can be extended to ∂D continuously because φ cannot be extended to ∂D or beyond it in general. In order to overcome this difficulty, we approximate φ by better quadratic differentials. In fact, for a general $\varphi\in B_2(D)$, we have the following result, which is essentially due to Bers [4, Lemma 1].

Proposition 3. *Let D be a Jordan domain in $\widehat{\mathbf{C}}$. For any $\varphi\in B_2(D)$ there exists a sequence $(\varphi_j)_j$ of holomorphic functions in $\overline{D}$ such that $\|\varphi_j\|_D \le \|\varphi\|_D$ and φ_j tends to φ uniformly on each compact subset of D as $j\to\infty$.*

Proof. We denote by $g\colon \mathbf{D}\to D$ the Riemann mapping function of D with $g(0)=z_0$ and $g'(0)>0$. Let D_j, $j=1,2,\dots$, be Jordan domains with $\overline{D}_{j+1}\subset D_j$ and with $\bigcap_j D_j = D$. Then the Carathéodory kernel theorem implies that the Riemann mapping functions g_j of D_j with $g_j(0)=z_0$ and $g_j'(0)>0$ converge to g uniformly on each compact subset of the unit disk as j tends to ∞. Now we set $\varphi_j = (g\circ g_j^{-1})_2^*\varphi$. We then have $\|\varphi_j\|_D \le \|\varphi_j\|_{D_j} = \|\varphi\|_D$ by the Schwarz–Pick lemma: $\varrho_D \ge \varrho_{D_j}$. We also have $\varphi_j\to\varphi$ locally uniformly as $j\to\infty$. □

With this result in mind, we can deduce our main result from the following lemma.

Lemma 4. *Suppose that $\varphi\in B_2(D)$ with $\|\varphi\|_D \le k_1\varepsilon_0$ is holomorphic in $\overline{D}$, where $0\le k_1<1$ and $\varepsilon_0=\varepsilon(\lambda,D)$, which is given by (2). Then the function $\hat f$ defined by (4) extends to a K-quasiconformal homeomorphism of the Riemann sphere, where $K=(1+k)/(1-k)$ and $k=1-(1-k_0)(1-k_1)$.*

Actually, we can prove our main theorem as follows. Let $\varphi \in B_2(D)$ satisfy $\|\varphi\|_D < \varepsilon_0$ and set $k_1 = \|\varphi\|_D/\varepsilon_0$. We take a sequence $(\varphi_j)_j$ as in Proposition 3. Let $\hat{f}$ and $\hat{f}_j$ be the functions in $\widehat{\mathbf{C}}\setminus\partial D$ defined by (4) for φ and φ_j, respectively, so that both are strongly normalized at $z_0 \in D$. Then, by the above lemma, each $\hat{f}_j$ can be continued to a K-quasiconformal homeomorphism of $\widehat{\mathbf{C}}$. Since those K-quasiconformal mappings which are conformal in D and strongly normalized at z_0 form a normal family, $\hat{f}_j$ has a subsequence converging to a K-quasiconformal mapping uniformly in $\widehat{\mathbf{C}}$. By construction, the limit mapping coincides with $\hat{f}$ in $\widehat{\mathbf{C}} \setminus \partial D$. This implies that $\hat{f}$ has a K-quasiconformal extension to the whole sphere. Now the proof of our main theorem is complete up to the above lemma.

Remark. Under the assumption that λ is of class C^1 off the boundary ∂D and that φ is holomorphic in $\overline{D}$ with $\|\varphi\|_D < \varepsilon_0$, a direct calculation shows

$$\partial\hat{f}(z) = -\frac{1+\big(z-\lambda(z)\big)^2\varphi\big(\lambda(z)\big)\partial\lambda(z)/2}{\Big(\eta_0\big(\lambda(z)\big)+\big(z-\lambda(z)\big)\eta_0'\big(\lambda(z)\big)\Big)^2} \qquad \text{and}$$

$$\bar{\partial}\hat{f}(z) = -\frac{\big(z-\lambda(z)\big)^2\varphi\big(\lambda(z)\big)\bar{\partial}\lambda(z)/2}{\Big(\eta_0\big(\lambda(z)\big)+\big(z-\lambda(z)\big)\eta_0'\big(\lambda(z)\big)\Big)^2}$$

at every $z \in D^*\setminus\{\infty, \lambda(\infty)\}$. Therefore, if $\big(\lambda(z)-z\big)^2\bar{\partial}\lambda(z)$ vanishes at the boundary, then we would obtain continuous extensions of $\partial\hat{f}$ and $\bar{\partial}\hat{f}$ to $\widehat{\mathbf{C}}$. Moreover, the limits of

$$\frac{\hat{f}(z+t)-\hat{f}(z)}{t} \quad \text{and} \quad \frac{\hat{f}(z+it)-\hat{f}(z)}{t},$$

when t tends to 0 along the real axis, both exist and are equal to $f'(z)$ and $if'(z)$, respectively, for each $z \in \partial D$. In fact, when $z+t$ or $z+it$ approaches to z in D^*, the above quotients tend to the desired values by (7) below. This implies that our $\hat{f}$ has continuous partial derivatives everywhere in $\widehat{\mathbf{C}}$. Hence, we can conclude that $\hat{f}$ is a local C^1-diffeomorphism of $\widehat{\mathbf{C}}$, and hence, a global C^1-diffeomorphism of it. Thus, if we restrict ourselves to this case, the proof would become much simpler than ours.

We note that it is always possible to take such a quasiconformal reflection λ for any quasidisk D (see [1] or [9, Section II.4]).

3. Proof of Lemma 4

Let φ be as in Lemma 4. We assume, for a moment, that D is bounded. Then the solutions η_0 and η_1 of (3) are holomorphic in $\overline{D}$. Thus $\hat{f}$ can be continuously extended to the whole sphere and $\hat{f}(\partial D)$ is the image of the quasicircle ∂D under the locally univalent meromorphic map η_1/η_0. Now we require an extension theorem for quasiregular mappings.

Lemma 5. *Let Ω be a plane domain and C be an open quasiarc (or a quasicircle) in Ω such that $\Omega \setminus C$ is an open set in $\widehat{\mathbf{C}}$. Suppose that $h\colon \Omega \to \widehat{\mathbf{C}}$ is a continuous map such that $h|_{\Omega\setminus C}$ is a K-quasiregular map and that, for each $x \in C$, h maps $C \cap U$ injectively onto a quasiarc for some open neighbourhood U of x in Ω. Then h is K-quasiregular in Ω.*

Proof. If we know that h is quasiregular in Ω, we can conclude that h is K-quasiregular because $|\bar\partial h/\partial h| \le (K-1)/(K+1)$ a.e. by assumption. Since quasiregularity is a local property, it suffices to show that h is quasiregular in an open neighbourhood U of each $x \in C$. The assumption allows us to take an open neighbourhood U of x so that h maps $U \cap C$ injectively onto a quasiarc. Then, by composing suitable quasiconformal mappings, we may further assume that U is an open disk centered at $x = 0$ with $U \cap C = U \cap \mathbf{R}$ and that $h(U \cap \mathbf{R}) \subset \mathbf{R}$. Set $U_\pm = \{z \in U : \pm \operatorname{Im} z \ge 0\}$. By the reflection principle for quasiregular mappings [11], the mapping $h|_{U_\pm}$ extends to a quasiregular one in U for each signature. This means that h is ACL and has locally square integrable partial derivatives in U, and hence h is quasiregular there. □

By this lemma, our mapping $\hat f$ turns out to be a K-quasiregular mapping on $\widehat{\mathbf{C}}$, and hence, $\hat f$ can be decomposed to the form $g \circ \omega$ for a K-quasiconformal mapping $\omega\colon \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ and a rational function g (see [10, Chapter VI]). Suppose that the degree of g is greater than one. Then there exists a branch point, say b^*, of g. Set $a^* = \omega^{-1}(b^*)$ and $a = \lambda(a^*)$. At this time, by Möbius conjugation, we assume that $0, \infty \in \partial D$ and the branch points of $\hat f$ and their reflections under λ are all finite.

Since $\hat f$ is locally injective in D, the point a^* must lie in $\widehat{\mathbf{C}} \setminus D$, thus $a \in \overline{D}$.

First, we show that $a \notin D$. Suppose that $a \in D$. By assumption, note that $\eta_0(a) + \bigl(\lambda(a) - a\bigr)\eta_0'(a) \ne 0$. We now investigate the local behaviour of F near the point a. Since a^* is a branch point of $F \circ \lambda$, the image of the positively oriented loop $|z - a| = r$ under F would have winding number N with $N < -1$ around $F(a)$ for a sufficiently small $r > 0$. Setting $\delta = z - a$ and $\delta^* = \lambda(z) - \lambda(a)$, we have

$$\begin{aligned}\eta_j(z) + \bigl(\lambda(z) - z\bigr)\eta_j'(z) &= \eta_j(a) + \eta_j'(a)\delta \\ &\quad + \bigl(\lambda(a) - a + \delta^* - \delta\bigr)\bigl(\eta_j'(a) + \eta_j''(a)\delta\bigr) + O(\delta^2) \\ &= \eta_j(a) + \bigl(\lambda(a) - a + \delta^*\bigr)\eta_j'(a) + \bigl(\lambda(a) - a\bigr)\eta_j''(a)\delta + o(\delta)\end{aligned}$$

as $\delta \to 0$ for $j = 0, 1$. Using the relations $\eta_0\eta_1' - \eta_0'\eta_1 = 1$ and $\eta_j'' = -\varphi\eta_j/2$ also, we calculate

$$\begin{aligned} F(z)-F(a) &= \frac{\eta_1(z)+\big(\lambda(z)-z\big)\eta_1'(z)}{\eta_0(z)+\big(\lambda(z)-z\big)\eta_0'(z)} - \frac{\eta_1(a)+\big(\lambda(a)-a\big)\eta_1'(a)}{\eta_0(a)+\big(\lambda(a)-a\big)\eta_0'(a)} \\ &= \frac{\varphi(a)\big(\lambda(a)-a\big)^2\delta/2+\delta^*+o(\delta)}{\big(\eta_0(a)+\big(\lambda(a)-a\big)\eta_0'(a)\big)^2+o(1)} \\ &= \frac{\lambda(z)-\lambda(a)+c(z-a)+o(z-a)}{\big(\eta_0(a)+\big(\lambda(a)-a\big)\eta_0'(a)\big)^2+o(1)} \end{aligned}$$

as $z\to a$, where $c=\varphi(a)\big(\lambda(a)-a\big)^2/2$. From $\|\varphi\|_D\le k_1\varepsilon_0$, we deduce

$$|c|\le k_1\bigg(\frac{|\lambda(a)-a|\varrho_D(a)}{|\lambda(z)-z|\varrho_D(z)}\bigg)^2\big(|\bar\partial\lambda(z)|-|\partial\lambda(z)|\big)$$

for almost all $z\in D$. So, if we are given a number s with $k_1<s<1$, then we can find a sufficiently small number r so that

$$|c|<s\cdot\operatorname*{ess.inf}_{\mathbf{D}(a,r)}\big(|\bar\partial\lambda|-|\partial\lambda|\big) \tag{5}$$

holds, where $\mathbf{D}(a,r)=\{z\in\mathbf{C}:|z-a|<r\}$. We now need the following fact about the local behaviour of quasiconformal maps, which might be interesting in itself.

Lemma 6. *Let $h\colon\mathbf{C}\to\mathbf{C}$ be a quasiconformal homeomorphism. For a point $a\in\mathbf{C}$ and a radius $r>0$, suppose that $E=\operatorname{ess.inf}_{\mathbf{D}(a,r)}\big(|\partial h|-|\bar\partial h|\big)>0$. Then, the map $h_t(z)=h(z)+t\bar z$ is quasiconformal on the disk $\mathbf{D}\big(a,(1-s)r\big)$ for any $t\in\mathbf{C}$ with $|t|<sE$. Furthermore, we have*

$$|h_t(z)-h_t(a)|\ge(sE-|t|)|z-a|\quad \text{for } z\in\mathbf{D}\big(a,(1-s)r\big). \tag{6}$$

We postpone the proof to Section 4 because we require a trick to show this.

Set $H(z)=\big(\lambda(z)+cz\big)/\big(\eta_0(a)+\big(\lambda(a)-a\big)\eta_0'(a)\big)^2$. Using (5), we now apply the above lemma to the case $h=\bar\lambda$ and $t=\bar c$ and see that H is anti-quasiconformal near the point a. In particular, the image of the positively oriented loop $l_r:|z-a|=r$ under H has winding number -1 around the point $H(a)$ for r small enough. With the help of the estimate in (6), we now conclude that $F(z)-F(a)=\big(H(z)-H(a)\big)\big(1+o(1)\big)$ as $z\to a$, which implies that the winding number of the image of l_r under F around the point $F(a)=H(a)$ is equal to that of H for sufficiently small r. This contradicts the fact that a is a branch point of F. We now conclude that $a\notin D$.

Therefore, the point a^* must lie in ∂D if $\hat f$ has a branch point a^*. Since $a=\lambda(a^*)=a^*$ in this case, we may use the letter a instead of a^*.

We will use the following important fact on quasiconformal reflections to deduce a contradiction.

Lemma 7 [9, Lemma I.6.3]. *Let λ be a K-quasiconformal reflection in C with $\infty \in C$. Then*

$$\frac{1}{M(K)}|z-\zeta| \le |\lambda(z)-\zeta| \le M(K)|z-\zeta|$$

for any $z \in \mathbf{C}$ and $\zeta \in C$, where $M(K) > 1$ is a constant depending only on K.

The map $\hat{f}$ is never injective near a while $\hat{f}|_D = f = \eta_1/\eta_0$ is injective near a, so we can select sequences of pairs of points z_n and w_n in D and closed arcs α_n connecting z_n and w_n in D such that $F(z_n) = F(w_n)$ and $F(\alpha_n)$ has winding number ± 1 around $F(a) = f(a)$, and that $z_n \to a$, $w_n \to a$ and $\operatorname{diam} \alpha_n \to 0$ as $n \to \infty$, where diam stands for the Euclidean diameter. Here and hereafter, we always understand that curves are parametrized by the standard interval $[0,1]$.

Now we consider the asymptotic behaviour of $F(z)$ as z tends to a in D. Keeping Lemma 7 in mind, in the same way as above, we can show that

$$\begin{aligned} F(z) - F(a) &= \frac{\eta_1(z) + \big(\lambda(z)-z\big)\eta_1'(z)}{\eta_0(z) + \big(\lambda(z)-z\big)\eta_0'(z)} - \frac{\eta_1(a)}{\eta_0(a)} \\ &= \eta_0(a)^{-2}\big(\lambda(z)-a\big) + O\big((z-a)^2\big) \end{aligned} \tag{7}$$

as $z \to a$ in D.

Therefore, combining with Lemma 7, we have

$$F\big(\alpha_n(t)\big) - F(a) - \eta_0(a)^{-2}\big(\alpha_n^*(t) - a\big) = O\big(\big(\alpha_n(t)-a\big)^2\big) = O\big(\big(\alpha_n^*(t)-a\big)^2\big)$$

uniformly in t as $n \to \infty$, where $\alpha_n^*(t) = \lambda\big(\alpha_n(t)\big)$. In particular,

$$\eta_0(a)^2\big(F\big(\alpha_n(t)\big) - F(a)\big)/\big(\alpha_n^*(t)-a\big) = 1 + o(1),$$

hence

$$\big|F\big(\alpha_n(t)\big) - F(a) - \eta_0(a)^{-2}\big(\alpha_n^*(t)-a\big)\big| < \big|F\big(\alpha_n(t)\big) - F(a)\big| \tag{8}$$

holds in $t \in [0,1]$ for sufficiently large n.

Now we may assume $|z_n| \ge |w_n|$ for every n. Since $F(z_n) = F(w_n)$ we have $\delta_n := |z_n^* - w_n^*| = O(|z_n - a|^2)$ as $n \to \infty$ by (7), where we set $z_n^* = \lambda(z_n)$ and $w_n^* = \lambda(w_n)$.

Here, we recall a fundamental property of quasidisks. The linear connectedness of D^* asserts the existence of a constant $M > 1$ such that any pair of points in $D^* \cap \mathbf{D}(c,r)$ can be joined by a curve in $D^* \cap \mathbf{D}(c, Mr)$ for all $c \in \mathbf{C}$ and $r > 0$

(see [6] or [9]). In particular, there exists a sequence of curves β_n^* connecting w_n^* and z_n^* in $D^*\cap\mathbf{D}(z_n^*, M\delta_n)$. Therefore we have

$$\begin{aligned}\big|\big(F(z_n)-F(a)\big)-\eta_0(a)^{-2}\big(\beta_n^*(t)-a\big)\big| &\le M|\eta_0(a)|^{-2}\delta_n + O(|z_n-a|^2)\\ &= O(|z_n-a|^2)=O(|z_n^*-a|^2)=O(|F(z_n)-F(a)|^2)\end{aligned}$$

as $n\to\infty$, and then

$$\big|\big(F(z_n)-F(a)\big)-\eta_0(a)^{-2}\big(\beta_n^*(t)-a\big)\big| < |F(z_n)-F(a)| \tag{9}$$

for n large enough.

Now we conclude from (8) and (9) that the closed curves $F(\alpha_n)-F(a)$ and γ_n^*-a, where $\gamma_n^*:=\alpha_n^*\cdot\beta_n^*$, have the same winding number around 0 for sufficiently large n. By the choice of α_n, we see that γ_n^* has winding number ±1, and hence γ_n^* separates a from ∞ for such an n. Since a and ∞ belong to ∂D^* and since γ_n^* is a curve in D^*, this situation contradicts the fact that D^* is simply connected. This contradiction is caused by the assumption $\deg g>1$. Therefore we can now conclude that g is a Möbius transformation, and hence the proof of Lemma 4 is now complete except for Lemma 6.

4. Proof of Lemma 6

Set $k_0=\|\bar\partial h/\partial h\|_\infty<1$. Without loss of generality, we can assume that $a=0$. We simply write $\mathbf{D}(r)=\mathbf{D}(0,r)$. First note that $h_t(z)=h(z)+t\bar z$ is quasiregular in $\mathbf{D}(r)$ for $|t|<E$. In fact, $|\bar\partial h_t/\partial h_t|\le(|\partial\lambda|+|t|)/|\bar\partial\lambda|\le k_0+k_1-k_0k_1=1-(1-k_0)(1-k_1)<1$ a.e. in $\mathbf{D}(r)$ if $|t|/E=k_1<1$.

Put $\varepsilon=1-s$. Now we use the auxiliary function $\chi\colon\mathbf{C}\to\mathbf{C}$ which is defined by

$$\chi(z)=\begin{cases}\bar z & \text{if } |z|\le\varepsilon r,\\ \varepsilon(1-\varepsilon)^{-1}(r-|z|)\bar z/|z| & \text{if } \varepsilon r\le|z|\le r,\\ 0 & \text{if } r\le|z|.\end{cases}$$

Then we can extend $h_t|_{\mathbf{D}(\varepsilon r)}$ to the complex plane, which will still be denoted by the same letter, by the relation $h_t=h+t\chi$. Since

$$\partial\chi(z)=-\frac{\varepsilon r}{2(1-\varepsilon)}\cdot\frac{|z|}{z^2}\quad\text{and}\quad\bar\partial\chi(z)=\frac{\varepsilon r}{2(1-\varepsilon)}\left(\frac{1}{|z|}-\frac{2}{r}\right),$$

we have $|\partial\chi|\le1/2(1-\varepsilon)=1/2s$ and $|\bar\partial\chi|\le\max\{\varepsilon,|1-2\varepsilon|\}/2(1-\varepsilon)<1/2s$ on the annulus $\{\varepsilon r<|z|<r\}$. Setting $k=|t|/sE<1$, we see that

$$\left|\frac{\bar\partial h_t}{\partial h_t}\right|\le\frac{|\bar\partial h|+|t|/2s}{|\partial h|-|t|/2s}\le\frac{2|\bar\partial h|+k(|\partial h|-|\bar\partial h|)}{2|\partial h|-k(|\partial h|-|\bar\partial h|)}\le\frac{m+k_0}{1+mk_0}<1$$

holds a.e. in the above annulus, where $m = k/(2-k) < 1$. Combining this with the fact that h_t is quasiregular in $|z| < \varepsilon r$ and in $|z| > r$, we can see that h_t is quasiregular in $\mathbf{C}$ for each $t \in \mathbf{D}(sE)$. Set $\mu_t = \bar\partial h_t/\partial h_t$ and let ω_t be the quasiconformal automorphism of $\mathbf{C}$ satisfying the partial differential equation $\bar\partial\omega_t = \mu_t\partial\omega_t$ a.e. on $\mathbf{C}$ and the normalization $\omega_t(0) = 0$ and $\omega_t(1) = 1$. Then $Q_t = h_t\circ\omega_t^{-1}$ is an entire function for each $t \in \mathbf{D}(sE)$. Since $h_t = h$ near the point at infinity, Q_t can be holomorphically extended to ∞ so that $Q_t^{-1}(\infty) = \{\infty\}$ and that Q_t is locally biholomorphic near ∞. In particular, Q_t is a polynomial of degree 1, and hence an analytic automorphism of $\mathbf{C}$. Thus we can conclude that $h_t = Q_t^{-1}\circ\omega_t$ is also a quasiconformal map of $\mathbf{C}$. Since $h_t(z) = h(z) + t\bar z$ for $z \in \mathbf{D}(\varepsilon r) = \mathbf{D}\big((1-s)r\big)$, the first assertion in Lemma 6 now follows.

The latter part of Lemma 6 can immediately be deduced from the former one. Indeed, for each fixed $z \in \mathbf{D}\big((1-s)r\big)$ other than 0 and for $t \in \mathbf{D}(sE)$, the fact that $h(z) + (t+u)\bar z = h_t(z) + u\bar z$ never vanishes whenever $|t| + |u| < sE$ implies that $|h_t(z)| \ge (sE - |t|)|\bar z| = (sE - |t|)|z|$.

References

[1] Ahlfors, L.V.: Quasiconformal reflections. - Acta Math. 109, 1963, 291–301.

[2] Ahlfors, L.V.: Lectures on Quasiconformal Mappings. - van Nostrand, 1966.

[3] Beardon, A.F., and F.W. Gehring: Schwarzian derivatives, the Poincaré metric and the kernel function. - Comment. Math. Helv. 55, 1980, 50–64.

[4] Bers, L.: A non-standard integral equation with applications to quasiconformal mappings. - Acta Math. 116, 1966, 113–134.

[5] Betker, Th.: Univalence criteria and Löwner chains. - Bull. London Math. Soc. 23, 1991, 563–567.

[6] Gehring, F.W.: Univalent functions and the Schwarzian derivative. - Comment. Math. Helv. 52, 1977, 561–572.

[7] Hille, E.: Remarks on a paper by Zeev Nehari. - Bull. Amer. Math. Soc. 55, 1949, 552–553.

[8] Lehto, O.: Remarks on Nehari' s theorem about the Schwarzian derivative and schlicht functions. - J. Analyse Math. 36, 1979, 184–190.

[9] Lehto, O.: Univalent Functions and Teichmüller Spaces. - Springer-Verlag, 1987.

[10] Lehto, O., and K.I. Virtanen: Quasiconformal Mappings in the Plane, 2nd ed. - Springer-Verlag, 1973.

[11] Martio, O.: Reflection principle for solutions of elliptic partial differential equations and quasiregular mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1980, 179–187.

[12] McShane, E.J.: Integration. - Princeton University Press, Princeton, 1947.

[13] Nehari, Z.: The Schwarzian derivative and schlicht functions. - Bull. Amer. Math. Soc. 55, 1949, 545–551.

[14] Nonin, S., and T. Sugawa: On the inner radius of univalence for a round annulus. - In preparation.

[15] Sugawa, T.: Inner radius of univalence for a strongly starlike domain. - Preprint.

Received 1 November 2000