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EXTREMAL QUASICONFORMAL POLYGON MAPPINGS FOR ARBITRARY SUBDOMAINS OF COMPACT RIEMANN SURFACES

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Abstract. By a polygon we originally mean a plane disk Δ with a finite number P of distinguished boundary points ω_i. Let (Δ, P), $P = \{\omega_i\}$ and (Δ', P'), $P' = \{\omega_i'\}$ be two polygons. The extremal quasiconformal mapping f_n of (Δ, P) onto (Δ', P'), $f_n(\omega_i) = \omega_i'$ for all i, is a Teichmüller mapping, associated with a quadratic differential φ_n, called polygon differential. To a given extremal qc mapping $f\colon \Delta \to \Delta'$ we can associate extremal polygon mappings f_n, $f_n(\omega_i) = f(\omega_i)$ for all vertices ω_i. This construction has applications in the theory of extremal qc mappings. It is the purpose of this article to generalize the notion of polygon and polygon mappings and prove their properties for arbitrary subdomains of compact Riemann surfaces.

I. Introduction: the disk case

Let $\Delta : |z| < 1$ and $\Delta' : |w| < 1$ be unit disks in the z- and w-plane respectively and let $w = f(z)$ be a quasiconformal mapping of Δ onto Δ' with maximal dilatation K and complex dilatation μ, $\|\mu\|_\infty = (K-1)/(K+1)$. Let P be a finite set of points ω_i, $i = 1, \dots, n$, $n \ge 4$ on $\partial\Delta$. The disk Δ together with the set $P = \{\omega_i\}$ is called a polygon, the points ω_i are its vertices. The mapping f takes the polygon (Δ, P) with vertices ω_i into a polygon (Δ', P') with vertices $\omega_i' = f(\omega_i)$, $i = 1, \dots, n$. Let f_n be the extremal qc mapping of Δ onto Δ' which takes the vertices ω_i into the vertices ω_i'. This mapping determines a pair of quadratic differentials φ_n on Δ and ψ_n on Δ'. The trajectory structures of φ_n and ψ_n partition the disks into finitely many horizontal strips R_{jn} in Δ and R_{jn}' in Δ'. They are mapped by the integrals $\Phi_n(z) = \int \sqrt{\varphi_n(z)}\,dz$ and $\Psi_n(w) = \int \sqrt{\psi_n(w)}\,dw$ onto Euclidean horizontal rectangles

$$\begin{aligned} \Phi_n(R_{jn}) &: 0 < \xi_n < a_{jn}, \qquad 0 < \eta_n < b_{jn}, \\ \Psi_n(R_{jn}') &: 0 < \xi_n' < K_n a_{jn}, \quad 0 < \eta_n' < b_{jn}. \end{aligned}$$

The extremal mapping f_n satisfies the relation

$$\Psi_n \circ f_n \circ \Phi_n^{-1}(\zeta_n) = K_n \xi_n + i\eta_n, \quad \zeta_n = \xi_n + i\eta_n,$$

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with $K_n \geq 1$ the constant dilatation of f_n. This is a consequence of Teichmüller's theorem; the extremal f_n is uniquely determined.

Next we introduce the metrics induced by the two quadratic differentials. They are nothing else than the Euclidean length elements

$$|d\Phi_n(z)| = |\varphi_n(z)|^{1/2}|dz|, \qquad |d\Psi_n(w)| = |\psi_n(w)|^{1/2}|dw|$$

in the Φ_n- and Ψ_n-plane respectively. The length inequality is usually expressed for closed trajectories. In our case the trajectories are cross cuts of Δ and Δ', but they become closed curves by reflection on $\partial\Delta$ and $\partial\Delta'$ respectively. The length inequality says that every closed trajectory is shortest in its free homotopy class with respect to the planes punctured at the points P_n and P'_n respectively. Here we have actually trajectory intervals connecting vertical sides of the disks, with lengths one half of the lengths of the closed trajectories.

Let α_{jn} be a trajectory interval of φ_n in Δ. It is stretched by K_n onto the corresponding trajectory interval α'_{jn} of ψ_n in the w-plane. Let

$$dw = p(z)\,dz + q(z)\,d\bar{z}$$

be the differential of the qc mapping f. Its length in terms of the ψ_n-metric is

$$|d\widetilde{w}| = |\psi_n(w)|^{1/2}|dw| = |\psi_n(w)|^{1/2}|p(z)\,dz + q(z)\,d\bar{z}|.$$

Now the length inequality reads

$$K_n a_{jn} \leq \int\limits_{f(\alpha_{jn})} |d\widetilde{w}| = \int\limits_{f(\alpha_{jn})} |\psi_n(w)|^{1/2}|p(z)\,dz + q(z)\,d\bar{z}|.$$

Integration over the disk Δ gives the polygon inequality

$$K_n \leq \iint\limits_{|z|<1} |\varphi_n(z)| \frac{\left|1 + \mu(z)\dfrac{\varphi_n(z)}{|\varphi_n(z)|}\right|^2}{1 - |\mu(z)|^2}\,dx\,dy, \tag{1}$$

$z = x + iy$, $\mu(z) = q(z)/p(z)$ the complex dilatation of f. Developing the integrand gives the equivalent form

$$\begin{aligned} \frac{k_n}{1-k_n} &\leq \operatorname{Re} \iint\limits_{|z|<1} \frac{\mu(z)\varphi_n(z)}{1 - |\mu(z)|^2}\,dx\,dy + \iint\limits_{|z|<1} |\varphi_n(z)| \frac{|\mu(z)|^2}{1 - |\mu(z)|^2}\,dx\,dy, \\ k_n &= \frac{K_n - 1}{K_n + 1}, \end{aligned} \tag{1'}$$

which is easier to explore. This is true for an arbitrary qc self-mapping of Δ and all inscribed polygons (for details, see [5], pp. 383–386).

Next, let f be extremal for its boundary values, with maximal dilatation K. Then, $K_n \leq K$ for all polygons. Let the point sets P_n become dense on $\partial\Delta$ for $n \to \infty$. By general principles of qc mappings $K_n \to K$. An easy consequence is the so called Hamilton–Krushkal relation

$$\operatorname{Re} \iint_{\Delta} \mu(z)\varphi_n(z)\, dx\, dy \to \|\mu\| = \|\mu\|_\infty = \operatorname{ess\,sup} |\mu(z)| \tag{2}$$

for $\|\varphi_n\| = \iint_\Delta |\varphi_n(z)|\, dx\, dy = 1$, $n \to \infty$, as a necessary condition for f to be extremal. (The computations are immediate if $|\mu(z)| = \|\mu\|$ a.e. In the general case we first realize that the integral tends to zero for every subset $E \subset \Delta$ with $|\mu(z)| < \|\mu\| - \varepsilon$, $\varepsilon > 0$. For a detailed discussion see [5].)

Another application of the polygon inequality is the following. Let K be the maximal dilatation of an extremal mapping f. Can one have

$$\sup K_n = K, \qquad n \leq N < \infty?$$

Can e.g. the above relation hold for quadrilaterals? Unless the boundary mapping has a so called substantial (essential) boundary point, this is only possible if f itself is a polygon mapping (see [7]).

II. The general case

The Hamilton–Krushkal relation (2) is an important ingredient in the theory of extremal qc mappings. The polygon inequality (1) provides us with Hamilton sequences which have an intuitive geometric meaning.

However, the inequality has so far only been proved for the disk (see [5], inequality 3.2.6). It is the purpose of this paper to prove it for arbitrary plane domains Ω and, slightly more generally, for arbitrary subdomains of compact Riemann surfaces.

Let Ω be a subdomain of a compact Riemann surface R the boundary $\partial\Omega$ of which contains infinitely many points. Let μ be an extremal Beltrami coefficient in Ω. We set $\mu_0 = \mu$ in Ω and $\mu_0 = 0$ in $R \setminus \Omega$. (This extension of μ was used in [2] to investigate different kinds of local boundary dilatations.) Then, μ_0 is a Beltrami differential in R. For the norms we have $\|\mu\| = \|\mu_0\|$.

Let f_0 be a μ_0-quasiconformal mapping of R onto a compact surface R', and let $\Omega' = f_0(\Omega)$. The complex dilatation of f_0 in Ω is μ. Therefore any qc mapping f of Ω with complex dilatation μ differs from the restriction $f_0 \mid \Omega$ by a conformal mapping. However, f_0 is qc in all of R, and in particular homeomorphic on $\partial\Omega$. If μ is extremal on Ω, $f_0 \mid \Omega$ is an extremal qc mapping of Ω onto $f_0(\Omega)$.

Choose a finite set P_n of points ω_i on $\partial\Omega$ and let G_n be the surface R punctured at the set P_n, $G_n = R \setminus P_n$. Let f_n be the extremal qc mapping of G_n onto $G'_n = R' \setminus P'_n$ which takes ω_i into $\omega'_i = f_0(\omega_i)$ for all indices i and is homotopic to f_0 modulo the boundary $P'_n = \partial G'_n$. It is a Teichmüller mapping associated with a rational quadratic differential φ_n of finite norm $\|\varphi_n\| < \infty$, which is holomorphic in R except for possible poles of the first order at the points ω_i. Changing the earlier nomenclature we call G_n a polygon, f_n a polygon mapping and φ_n a polygon differential.

We are trying to establish a polygon inequality similar to (1). In the disk case we used reflection on the boundary of the disk and thus had actually to deal with quadratic differentials with closed trajectories. The polygon differentials in this more general case have no symmetry. Therefore we have to make use of the general trajectory structure. This gives rise to a length inequality between homotopic closed curves which can then be integrated over the whole surface R. The fact that $\mu_0 = 0$ outside Ω finally reduces the domain of integration to Ω, and we arrive at the polygon inequality (1), but now for general domains $\Omega \subset R$.

III. Trajectory structure

This chapter is based on reference [6], where the definitions, statements and their proofs can be found.

Let φ be a rational quadratic differential of finite norm on the extended plane or, more generally, on an arbitrary compact Riemann surface R. Its critical points are finitely many zeroes and, possibly, finitely many first order poles. The critical trajectories, i.e. those which tend in at least one direction to a critical point of φ, are finite in number ([6], §7.1). All the other trajectories are either closed Jordan curves or spirals. A spiral is a trajectory which is not closed but both ends of which diverge, i.e. have a limit set which consists of at least two points ([6], §10.2).

Every closed trajectory α of an arbitrary quadratic differential φ on an arbitrary Riemann surface R is embedded in a ring domain swept out by closed trajectories homotopic to α and of the same φ-length ([6], §9.4).

The behaviour of non closed trajectories is more complicated. But on a compact Riemann surface and for a rational quadratic differential of finite norm every divergent trajectory ray α^+ is recurrent ([6], §11, Theorem 11.1 and Corollary). The limit set A of a recurrent ray α^+ is the closure of a domain ([6], §11, Theorem 11.2 and Corollaries). The boundary of A consists of critical trajectories of finite length connecting zeroes of φ. Both subrays α^+ and α^- of a spiral have the same limit set, which we call for short a spiral set. There are only finitely many spiral sets, and different ones are non-overlapping. Therefore the entire surface R is decomposed into finitely many ring domains and spiral sets.

Let A be a spiral set and choose a closed vertical interval β in the interior A° of A. Then, A is covered by a finite set of closed, horizontal, non-overlapping strips S_i corresponding to horizontal rectangles in the $\Phi = \int \sqrt{\varphi}$ -plane with both their vertical sides on β but no other points in common with β. Strips of the first kind have their vertical intervals on different sides of β, strips of the second kind on the same side, either both on the right-hand side or both on the left-hand side (for the construction see ([6], §11.3).

The finitely many rays α^+ and α^- with initial point on β^+ or β^- which end up in a critical point or an end point of β before they meet β again, together with the endpoints of β, determine the strip decomposition of A.

IV. Length inequality

Let α be a closed trajectory of φ. If γ is any closed curve which is freely homotopic to α, then its φ-length $\int_\gamma |\varphi(z)|^{1/2}\,|dz|$ is at least as big as the φ-length of α, with equality only if γ is a closed trajectory in the free homotopy class of α ([6], §17.1). The length inequality, in combination with quasiconformal mappings, will be integrated over the annulus swept out by the closed trajectories of φ parallel to α.

Since we cannot restrict ourselves to quadratic differentials with closed trajectories, we need a length inequality for spiral domains A°.

Let β be a compact vertical interval in A° and let S be a horizontal strip of the first kind based on β. Let α be a trajectory interval in S with initial point P^+ and end point P^- on the two different sides of β. Let δ be the subinterval of β with initial point P^- and end point P^+. The horizontal length of the closed Jordan curve $\alpha+\delta$ is equal to $a = \int_\alpha |\varphi(z)|^{1/2}\,|dz|$ ([6], §24, Theorem 24.1, where the property is expressed for the vertical trajectories and vertical length = height rather than for the horizontal length).

Let γ be any closed curve in the free homotopy class of $\alpha+\delta$. Then, since its φ-length is at least equal to its horizontal length with respect to φ and this is minimized (see [6], Definition 24.1), in its free homotopy class, by a, we have $\int_\gamma |\varphi(z)|^{1/2}\,|dz| \geq a$.

The process is slightly more complicated for strips of the second kind, since in this case the closed curve $\alpha+\delta$ evidently would not minimize the horizontal length in its free homotopy class. In order to get a minimizing closed curve we must combine a trajectory interval α_1 of the second kind on one side of β with a trajectory interval α_2 of the second kind on the other side of β. The two connecting subintervals δ_1 and δ_2 on β can be chosen in such a way that the curve $\alpha_1+\delta_1+\alpha_2+\delta_2$ is a Jordan curve, possibly after a slight shift of δ_1, say, away from β, (see [6], Fig. 68, p. 156, where again the reasoning is carried out for heights instead of horizontal lengths). Then, again by ([6], Theorem 24.1), the φ-length of any closed curve γ in the free homotopy class of the above step curve is at least equal to the sum a_1+a_2 of the lengths of α_1 and α_2.

V. Polygon inequality

We are now ready to prove the Polygon Inequality (1) and, equivalently (1′), for the general case of Section II.

First, let α be a closed trajectory of φ_n. In the maximal annulus S determined by α on $G_n = R - \{\omega_i\}$ we introduce the parameter $\zeta = \xi + i\eta = \Phi_n(z) = \int \sqrt{\varphi_n(z)}\, dz$, z a local parameter on R. The Φ_n-image of S cut along a vertical trajectory β is a rectangle $0 \le \xi \le a$, $0 \le \eta \le b$. The trajectory α is mapped by f_n onto a closed trajectory $\alpha' = f_n(\alpha)$ of the image differential $\psi_n = f_n(\varphi_n)$ and the ring domain S on G_n onto a ring domain $S' = f_n(S)$ on R' swept out by closed trajectories of ψ_n. The mapping f_0 takes α into a closed curve $\widetilde{\alpha} = f_0(\alpha)$ of ψ_n-length at least equal to $K_n a$, because it is freely homotopic on G'_n to the closed trajectory α' with ψ_n-length $K_n a$.

Let us introduce the notations, in terms of local parameters z and w on R and R' respectively, $w = f_0(z)$, $dw = p(z)\, dz + q(z)\, d\bar{z}$,

$$dz = \frac{dz}{d\zeta}\, d\zeta = \frac{1}{\Phi'_n(z)}\, d\zeta.$$

This gives, for the ψ_n-length of $\widetilde{\alpha}$,

$$K_n a \le \int_{\tilde{\alpha}} |d\Psi_n| = \int_{\tilde{\alpha}} |\psi_n(w)|^{1/2}\, |dw| = \int_{\alpha} |\psi_n(w)|^{1/2} |p(z)\, dz + q(z)\, d\bar{z}|. \tag{3}$$

Along the horizontals in the $\zeta = \xi + i\eta$-plane we have $dz = d\xi/\Phi'_n(z)$, and thus

$$K_n a \le \int \frac{|\psi_n(w)|^{1/2}}{|\varphi_n(z)|^{1/2}} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| d\xi, \tag{4}$$

and hence by integration over the η-variable

$$\begin{aligned} K_n ab &\le \iint_{\Phi_n(S)} \frac{|\psi_n(w)|^{1/2}}{|\varphi_n(z)|^{1/2}} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| d\xi\, d\eta \\ &= \iint_S |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| dx\, dy. \end{aligned} \tag{5}$$

There is the variable $w = f_0(z)$ which has no place in S but in $\widetilde{S} = f_0(S)$. By means of the Jacobian $du\, dv = J(w/z)\, dx\, dy = \big(|p(z)|^2 - |q(z)|^2\big)\, dx\, dy$ we go back to $S \subset R$ with the integral

$$K_n ab \le \iint_S |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} J^{1/2}\, \frac{\left| p(z) + q(z) \dfrac{\varphi_n(z)}{|\varphi_n(z)|} \right|}{J^{1/2}}\, dx\, dy. \tag{6}$$

Secondly, let A be a spiral set of the trajectory structure of φ_n. Choose a closed, regular vertical interval β in the interior A° of A. β is the basis of a finite set of non overlapping horizontal φ_n-strips which cover A. They go over into horizontal rectangles in the $\zeta = \xi + i\eta$-plane by the conformal mapping Φ_n. For given $\varepsilon > 0$ we can choose β so short that the ψ_n-length of $\tilde{\beta} = f_0(\beta)$ is smaller than ε.

Let S be a strip of the first kind. It connects an interval on the positive side of β with one on the negative side. Let α be a trajectory interval of φ_n in S and δ the interval on β which connects the two end points of α. It can be considered as a shift vector on β pointing from the end point of α on the negative side of β to its initial point on the positive side. It is the same for all trajectory intervals α in S (for details see [6], p. 68).

The curve $\alpha + \delta$ is mapped by f_n onto a curve $\alpha' + \delta'$ on R', composed of a horizontal interval α' and a vertical interval δ' of the image differential $\psi_n = f_n(\varphi_n)$. The curve $f_0(\alpha+\delta) = f_0(\alpha) + f_0(\delta) = \widetilde{\alpha} + \tilde{\delta}$ has ψ_n-length at least equal to the ψ_n-length of α', which is equal to $|\alpha'|_{\psi_n} = K_n \cdot |\alpha|_{\varphi_n} = K_n \cdot a$. For the lengths of the vertical intervals δ, δ', we have $|\delta|_{\varphi_n} = |\delta'|_{\psi_n} = d$. We use the rectangle $0 \le \xi \le a$, $0 \le \eta \le b$, b the φ_n-height of the strip S, in the $\zeta = \xi + i\eta$-plane, $\zeta = \Phi_n(z)$, as parameter domain. From $|\widetilde{\alpha} + \tilde{\delta}|_{\psi_n} = |\widetilde{\alpha}|_{\psi_n} + |\tilde{\delta}|_{\psi_n} \ge K_n a$ and $|\tilde{\delta}|_{\psi_n} < \varepsilon$ we get $|\widetilde{\alpha}|_{\psi_n} \ge K_n a - \varepsilon$, and hence

$$
\begin{aligned}
K_n a - \varepsilon &\le \int_{\alpha} |\psi_n(w)|^{1/2} |p(z)\, dz + q(z)\, d\bar{z}| \\
&= \int_{\Phi_n(\alpha)} |\psi_n(w)| \frac{1}{|\varphi_n(z)|^{1/2}} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| d\xi .
\end{aligned} \tag{7}
$$

Integration over η yields

$$
\begin{aligned}
K_n ab - \varepsilon b &\le \iint |\psi_n(w)|^{1/2} \frac{1}{|\varphi_n(z)|^{1/2}} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| d\xi\, d\eta \\
&= \iint_S |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| dx\, dy .
\end{aligned} \tag{8}
$$

Here, $z = x + iy$ is an arbitrary parameter on R. Since $\psi_n(w)$ depends on $w = f_0(z)$, we introduce the Jacobian $J(w/z) = |p(z)|^2 - |q(z)|^2$. Then, the integral reads

$$
K_n ab - \varepsilon b \le \iint_S |\psi_n(w)|^{1/2} J^{1/2} |\varphi_n(z)|^{1/2} \frac{\left| p + q \dfrac{\varphi_n}{|\varphi_n|} \right|}{J^{1/2}}\, dx\, dy . \tag{9}
$$

We sum it up over all strips S of the first kind based on β. Denoting the area of S in terms of φ_n by $|S|_{\varphi_n}$ we have

$$K_n \sum |S|_{\varphi_n} - \varepsilon \sum b \leq \iint\limits_{\sum S} |\psi_n(w)|^{1/2} J^{1/2} |\varphi_n(z)|^{1/2} \frac{\left| p + q \dfrac{\varphi_n}{|\varphi_n|} \right|}{J^{1/2}} \, dx \, dy. \tag{10}$$

We pass now to the strips of the second kind. If we want to have a length inequality, they have to be combined, each strip on the right-hand side with a strip on the left-hand side of the same height. To that end take the highest strip S_r on the right-hand side and the highest strip S_l on the left-hand side. Let b_r and b_l be the respective heights and let $b_r > b_l$. Then we cut off a horizontal rectangle of height $b_r - b_l$ from S_r to get $\widetilde{S}_r$ with height b_l. We can therefore glue $\widetilde{S}_r$ to S_l. The total number of strips is reduced by one. Therefore the process comes to an end.

Let $\alpha_1 + \delta_1 + \alpha_2 + \delta_2$ be a closed step curve with respect to φ_n. It is mapped by f_n onto a similar step curve with respect to ψ_n, with horizontal ψ_n-length $K_n a_1 + K_n a_2$. The f_0 image $\widetilde{\alpha}_1 + \tilde{\delta}_1 + \widetilde{\alpha}_2 + \tilde{\delta}_2$ has ψ_n-length greater or equal to $K_n(a_1 + a_2)$. On the other hand, it is $|\widetilde{\alpha}_1|_{\psi_n} + |\widetilde{\alpha}_2|_{\psi_n} + |\tilde{\delta}_1|_{\psi_n} + |\tilde{\delta}_2|_{\psi_n}$. Setting $|\tilde{\delta}_1|_{\psi_n} = \tilde{d}_1$, $|\tilde{\delta}_2|_{\psi_n} = \tilde{d}_2$ we find

$$|\widetilde{\alpha}_1|_{\psi_n} + |\widetilde{\alpha}_2|_{\psi_n} \geq K_n(a_1 + a_2) - \tilde{d}_1 - \tilde{d}_2 > K_n(a_1 + a_2) - 2\varepsilon. \tag{11}$$

With the values $|\widetilde{\alpha}_1|_{\psi_n} = \int_{\tilde{\alpha}_1} |d\Psi_n|$, $|\widetilde{\alpha}_2|_{\psi_n} = \int_{\tilde{\alpha}_2} |d\Psi_n|$ we get

$$\begin{aligned} K_n(a_1 + a_2) - 2\varepsilon &< \int\limits_{\tilde{\alpha}_1 + \tilde{\alpha}_2} \left| \frac{d\Psi_n}{dw} \right| |dw| = \int\limits_{\alpha_1 + \alpha_2} |\psi_n(w)|^{1/2} |p \, dz + q \, d\bar{z}| \\ &= \int |\psi_n(w)|^{1/2} \frac{1}{|\varphi_n(z)|^{1/2}} \left| p(z) + q(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right| d\xi. \end{aligned} \tag{12}$$

The integral with respect to ξ is thought to be extended over the rectangles corresponding to the two strips. We integrate over η and get, with the height of both strips S_r and S_l being equal to b

$$\begin{aligned} K_n(a_1 + a_2)b - 2\varepsilon b &\leq \iint |\psi_n(w)|^{1/2} \frac{1}{|\varphi_n(z)|^{1/2}} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| d\xi \, d\eta \\ &= \iint\limits_{S_r + S_l} |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| dx \, dy. \end{aligned} \tag{13}$$

Now we sum up over all strips S of the first kind and all pairs of strips $S_r + S_l$ of the second kind in A. We generically denote the height of the strips by b. Thus $K_n ab = K_n \|\varphi_n\|_S$, $K_n(a_1 + a_2)b = K_n \|\varphi_n\|_{S_r+S_l}$. This gives

$$\sum_{(S)} K_n \|\varphi_n\|_S - \varepsilon|\beta|_{\varphi_n} \le \iint\limits_{\sum S} |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| dx\, dy \tag{14}$$

and

$$\sum_{(S_r+S_l)} K_n \|\varphi_n\|_{S_r+S_l} - 2\varepsilon \tfrac{1}{2} |\beta|_{\varphi_n} \le \iint\limits_{\sum S_r+S_l} |\psi_n|^{1/2} |\varphi_n|^{1/2} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| dx\, dy \tag{15}$$

and hence

$$K_n \|\varphi_n\|_A - 2\varepsilon|\beta|_{\varphi_n} \le \iint\limits_{A} |\psi_n|^{1/2} |\varphi_n|^{1/2} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| dx\, dy. \tag{16}$$

The integral over A does not depend on the length of the vertical interval β, so we can let $|\beta|_{\varphi_n} \to 0$. Then, ε becomes arbitrarily small too, which leads to

$$K_n \|\varphi_n\|_A \le \iint\limits_{A} |\psi_n(w)|^{1/2} |\varphi_n(z)|^{1/2} \left| p + q \frac{\varphi_n}{|\varphi_n|} \right| dx\, dy. \tag{17}$$

We add up the inequalities for all ring domains and spiral sets of φ_n and get the inequality (17) for R instead of A. Finally, choosing $\|\varphi_n\| = 1$, we have, inserting the Jacobian $J(w/z)$

$$K_n \le \iint\limits_{R} |\psi_n(w)|^{1/2} J^{1/2} |\varphi_n(z)|^{1/2} \frac{\left| p + q \dfrac{\varphi_n}{|\varphi_n|} \right|}{J^{1/2}} dx\, dy. \tag{18}$$

To this expression we apply Schwarz's inequality which gives

$$K_n^2 \le \iint\limits_{R} |\psi_n(w)| J\, dx\, dy \cdot \iint\limits_{R} |\varphi_n(z)| \frac{\left| p + q \dfrac{\varphi_n}{|\varphi_n|} \right|^2}{|p|^2 - |q|^2} dx\, dy. \tag{19}$$

With $\iint_R |\psi_n(w)| J(w/z)\, dx\, dy = \|\psi_n\|_{R'} = K_n \|\varphi_n\|_R = K_n$ we finally get

$$\begin{aligned} K_n &\le \iint\limits_{R} |\varphi_n(z)| \frac{\left| p(z) + q(z) \dfrac{\varphi_n(z)}{|\varphi_n(z)|} \right|^2}{|p(z)|^2 - |q(z)|^2} dx\, dy \\ &= \iint\limits_{R} |\varphi_n(z)| \frac{\left| 1 + \mu_0 \dfrac{\varphi_n(z)}{|\varphi_n(z)|} \right|^2}{1 - |\mu_0(z)|^2} dx\, dy. \end{aligned} \tag{20}$$

Remember that $\mu_0(z) = 0$ outside of Ω and $\mu_0(z) = \mu(z)$ in Ω. Then, the above integral over R reduces to one over Ω,

$$K_n \leq \iint\limits_{\Omega} |\varphi_n(z)| \frac{\left|1 + \mu(z)\dfrac{\varphi_n(z)}{|\varphi_n(z)|}\right|^2}{1 - |\mu(z)|^2}\, dx\, dy. \tag{21}$$

This is the Polygon Inequality. The polygon mapping f_n is the extremal qc mapping on R which takes the points $\omega_i \in \partial\Omega$ into the points $\omega_i' = f_0(\omega_i) \in \partial\Omega'$ and is homotopic on $G_n = R \setminus \{\omega_i\}$ to f_0 modulo the points $P_n = \{\omega_i\}$. It is a Teichmüller mapping associated with a quadratic differential φ_n of finite norm (norm one, if normalized) which is holomorphic on R except for possible first order poles at the vertices ω_i.

VI. Homotopy

Let now the set of vertices P_n become arbitrarily dense on $\partial\Omega$ with $n \to \infty$. By general principles of qc mappings (see e.g. [4], II, §5) there is a subsequence of the sequence (f_n) which converges uniformly in R to a qc mapping with maximal dilatation smaller or equal to $\overline{\lim}\, K_n$. Its values on $\partial\Omega$ are the same as the values of f_0 on $\partial\Omega$. We denote its restriction to Ω by f^*, with maximal dilatation K^*. Clearly, the maximal dilatation of f_0 is the same as the maximal dilatation K of its restriction f to Ω.

An extremal mapping we are looking for has the boundary values of f on $\partial\Omega$ and is homotopic to f in Ω modulo the boundary (this is in fact the Teichmüller class of f in Ω).

Let $G_n = R \setminus P_n$, $G_n' = R \setminus P_n'$. In this notation the mappings $f_n\colon G_n \to G_n'$ are homotopic $(\operatorname{mod} P_n)$ to the mapping f_0. This property has an equivalent in terms of covering surfaces. Let $\widehat{G}_n$, $\widehat{G}_n'$ be the universal covering surfaces of G_n and G_n' respectively. We lift the qc mappings f_n and f_0 to the universal covering surfaces $\widehat{G}_n$, $\widehat{G}_n'$. Lifting is done by corresponding arcs with a fixed initial point $z_0 \in R \setminus P_n$ and its image point by f_n or f_0 in $R' \setminus P_n'$, as the case may be. In principle, the point z_0 can be chosen anywhere in G_n. For our purposes it is however important to choose $z_0 \in \Omega$.

Let $\hat{f}_n$ be the lift of f_n, $(f_0, P_n)\widehat{\ }$ the lift of f_0 to $\widehat{G}_n \to \widehat{G}_n'$.

Besides the multisheeted covering surfaces $\widehat{G}_n$, $\widehat{G}_n'$ over G_n, G_n' (which is easier for the lifting process) we represent $\widehat{G}_n$ and $\widehat{G}_n'$ by the disks Δ, Δ' respectively, without changing the notations of the lifts. The mappings $\hat{f}_n$ and $(f_0, P_n)\widehat{\ }$ are thus quasiconformal mappings of the disks $\Delta \to \Delta'$ which agree on the boundary $\partial\Delta \to \partial\Delta'$. Their limits are quasiconformal mappings g and h of Δ onto Δ' respectively. Since the projection of $(f_0, P_n)\widehat{\ }$ onto G_n is always equal to f_0, the same is true if we restrict f_0 to the original mapping $f\colon \Omega \to \Omega'$. For $\hat{f}_n$

the projection f_n maps z_0 onto $f_n(z_0) = z_n \to f^*(z_0) \in \Omega'$. Therefore $g = \lim \hat{f}_n$ cannot degerate. On the other hand, since Δ is connected (by arcs), the projection is connected. It cannot contain a point of $\partial\Omega$, but contains all of Ω and is thus equal to Ω. We have: The projection of g is f^* and thus $g = \hat{f}^*$. The two lifts $\hat{f}$ and $\hat{f}^*$ to the universal covering surfaces $\widehat{\Omega} \to \widehat{\Omega}'$ (which are represented by Δ and Δ') have the same boundary values. It follows that $f^* = \lim f_n \mid \Omega$ and f are homotopic in Ω modulo the boundary $\partial\Omega$.

We conclude: f^* is in the Teichmüller class of f in Ω. Since f is extremal, we must have $K \le K^*$, but evidently only equality is possible. It follows from the Polygon Inequality that $\lim_{n\to\infty} K_n = K$. We finally have the

Theorem. *Let Ω be a subdomain of a compact Riemann surface R with infinitely many boundary points. Let μ be an extremal Beltrami coefficient in Ω. Let $\{\omega_i\}$ be a sequence of points on $\partial\Omega$ which is dense in $\partial\Omega$. Then, there exists a sequence of polygon differentials φ_n of norm $\|\varphi_n\| = 1$ with vertices ω_i, $i = 1, \dots, n$ $(n \to \infty)$ which is a Hamilton–Krushkal sequence for μ in Ω. The usual computations in fact show that the Teichmüller differentials of the extremal polygon mappings satisfy*

$$\lim_{n\to\infty} \operatorname{Re} \iint_{\Omega} \mu(z)\varphi_n(z)\,dx\,dy = \|\mu\|_\infty.$$

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