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NEW POLYNOMIALS P FOR WHICH $f'' + P(z)f = 0$ HAS A SOLUTION WITH ALMOST ALL REAL ZEROS

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Abstract. Let $a, b, c \in \mathbf{R}$, $a \neq 0$ with $ac \leq 0$. We prove that there exists a sequence of positive real numbers $\mu_k \to \infty$ such that for each k, the equation $f''(z)+(az^3+bz^2+cz-\mu_k)f(z) = 0$ admits a solution with infinitely many real zeros and at most finitely many non-real zeros. This gives a new class of cubic polynomials P for which $f'' + P(z)f = 0$ has a solution with almost all real zeros.

We also find new quartic examples: for each $a > 0$ and $b \in \mathbf{R}$, there exists a sequence of real numbers $\mu_k \to \infty$ such that for each k, $f''(z) + (az^4 + bz^2 - \mu_k)f(z) = 0$ has a solution with almost all real zeros. The case $a > 0$ and $b > 0$ was discovered earlier by Gundersen.

1. Introduction

Consider the following ordinary differential equation

$$f''(z) + P(z)f(z) = 0, \tag{1}$$

where P is a non-constant polynomial. It is well known that any solution $f(z)$ of (1) is an entire function. Hellerstein and Rossi posed the question of characterizing the polynomials P for which the equation (1) admits a solution having only real zeros and infinitely many of them [1, Problem 2.71]. (Steven Bank posed the same question in 1983, but not in print.)

Examples of such polynomials are known when P is linear and real, the Airy differential equation (see [7, pp. 413–415]), and when $P(z) = z^4 - \lambda$ for certain $\lambda > 0$, a result of Titchmarsh, see [10, pp. 172–173].

Gundersen [4] weakened the condition "only real zeros" in the question to "at most finitely many non-real zeros". He introduced a quadratic term to the potential, proving that for each $a > 0$ and $b \geq 0$, there exists a sequence of real numbers $\mu_k \to \infty$ such that for each k, the equation (1) with the quartic $P(z) = az^4 + bz^2 - \mu_k$ admits a solution with almost all real zeros. When $b = 0$, the polynomial reduces to that of Titchmarsh.

In Section 2, we will prove a similar result, Theorem 1, for cubic P. Throughout this paper, "almost all real zeros" means "infinitely many real zeros and at most finitely many non-real zeros".

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Theorem 1. *Suppose that* $a, b, c \in \mathbf{R}$, $a \neq 0$ *with* $ac \leq 0$. *Then there exists a sequence of positive real numbers* $\mu_k \to \infty$ *such that for each* k, *the equation* (1) *with* $P(z) = az^3 + bz^2 + cz - \mu_k$ *admits a solution with almost all real zeros.*

Then in Section 3 we extend Gundersen's examples $P(z) = az^4 + bz^2 - \mu_k$ for $a > 0$ and $b < 0$. More precisely, we will prove the following theorem.

Theorem 2. *Suppose that* $a > 0$, $b \in \mathbf{R}$. *Then there exists a sequence of real numbers* $\mu_k \to \infty$ *such that for each* k, *the equation* (1) *with* $P(z) = az^4 + bz^2 - \mu_k$ *admits a solution with almost all real zeros.*

Recently, Eremenko and Merenkov [2] showed that for each non-negative integer m, there exists a polynomial P of degree m for which (1) has a solution with real zeros only. Moreover, if m is odd, then the solution has infinitely many real zeros. And if $m \equiv 0 \bmod 4$, then there exists a polynomial P of degree m for which (1) has a solution with real zeros only and infinitely many of them.

Now we provide some context regarding the problem. First, we recall the necessity of P being real. That is, if the equation (1) has a solution with infinitely many real zeros, then P is a real polynomial by [3, Theorem 3]. Second, Hellerstein and Rossi [5], and Gundersen [3], showed that if linearly independent solutions f_1 and f_2 have only finitely many non-real zeros, then P must be a constant. See also the paper by Gundersen [4] for a brief survey of other known facts.

2. Background

It is also known that the solutions f of (1) have rather simple asymptotic behavior near infinity. To describe this behavior, which we will use in various places, we need to establish some terminology.

Definition. Consider the equation (1) with $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_k \in \mathbf{C}$ for $0 \leq k \leq m$ with $a_m \neq 0$. Let

$$\theta_j = \frac{2j\pi - \arg a_m}{m+2} \qquad \text{for } j \in \mathbf{Z}.$$

For $j \in \mathbf{Z}$ we call the open sectors

$$S_j = \{z \in \mathbf{C} : \theta_j < \arg z < \theta_{j+1}\}$$

the *Stokes sectors* of (1). Also we call the rays $\{\arg z = \theta_j\}$ the *critical rays.*

In particular, when $a_m > 0$ the Stokes sectors of (1) are

$$S_j = \left\{ z \in \mathbf{C} : \frac{2j\pi}{m+2} < \arg z < \frac{2(j+1)\pi}{m+2} \right\} \quad \text{for } j \in \mathbf{Z}.$$

We are now ready to introduce the asymptotic results of Hille [6].

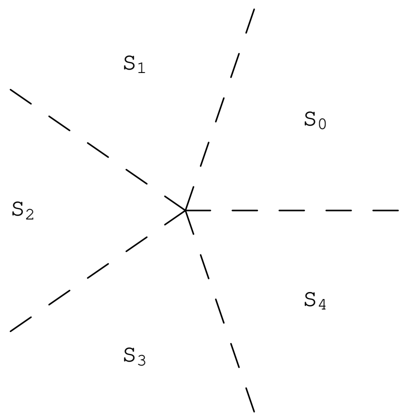

Figure 1. The Stokes sectors for $P(z) = z^3 + bz^2 + cz + d$. The dashed rays are the critical rays; $\arg z = 0, \pm\frac{2}{5}\pi, \pm\frac{4}{5}\pi$.

Proposition 1 ([6, Section 7.4]). *Let f be a non-constant solution of* (1) *with $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_k \in \mathbf{C}$ for $0 \le k \le m$ with $a_m \neq 0$. Then the following hold.*

(i) *In each Stokes sector S_j, f either blows up or decays to zero exponentially in S_j (or more precisely, as z tends to infinity in any closed subsector within S_j), and f has at most finitely many zeros in any closed subsector of S_j.*
(ii) *If f decays to zero in S_j, for some j, then it must blow up in S_{j-1} and S_{j+1}. (However, it is possible for f to blow up in many adjacent Stokes sectors.)*
(iii) *If f decays to zero in S_j, then f has at most finitely many zeros in any closed subsector within $S_{j-1} \cup \overline{S_j} \cup S_{j+1}$.*
(iv) *If f blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, f has infinitely many zeros in each sector $\theta_j - \varepsilon \le \arg z \le \theta_j + \varepsilon$.*

The following proposition on existence of solutions is also known.

Proposition 2 ([6, Section 7.4]). *Consider the equation* (1) *with $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, where $a_k \in \mathbf{C}$ for $0 \le k \le m$ with $a_m \neq 0$. For each $j \in \mathbf{Z}$, this equation*

(I) *has a solution that decays in S_j, and*
(II) *has a solution that blows up in S_j.*

3. Necessary and sufficient conditions

In this section, we will provide necessary and sufficient conditions on the polynomial P and on asymptotic behavior of a solution f of (1), for the solution f to have almost all real zeros. And we will use this sufficient condition in proving Theorems 1 and 2.

3.1. Necessary conditions. Suppose that a solution f of (1) has almost all real zeros. (That is, f has infinitely many real zeros and at most finitely many non-real zeros.) Then as we mentioned in the introduction, the polynomial P must be real on the real line [3, Theorem 3]. Moreover, when the degree m of the real polynomial $P(z) = a_m z^m + \cdots + a_0$ is even, we see that $a_m > 0$ because otherwise, every critical ray is away from the real axis, and hence f would not have infinitely many real zeros (see Proposition 2(i)). When the degree of the real polynomial P is odd, then exactly one critical ray is on the real axis. If $a_m > 0$ then the positive real axis is a critical ray, and if $a_m < 0$ then the negative real axis is a critical ray.

Also since f has at most finitely many non-real zeros, by Proposition 2(iv), for each critical ray that is away from the real axis, the solution f is decaying in one of two adjacent Stokes sectors, and the solution f blows up in the other Stokes sector. Otherwise, the solution f had to blow up in these two adjacent Stokes sectors since a non-constant solution f cannot decays to zero in any two adjacent Stokes sectors (see Proposition 1(ii)). Then Proposition 1(iv) implies that f would have infinitely many non-real zeros.

Finally, it is clear that f must have infinitely many zeros.

3.2. Sufficient conditions. In the following theorem, we provide a sufficient condition for having a solution of (1) with almost all real zeros. We will use this theorem in proving Theorems 1 and 2.

Theorem 3. *Let* $P(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ *with* $m \geq 1$, $a_j \in \mathbf{R}$ *for* $0 \leq j \leq m-1$. *Suppose that* f *is a solution of*

$$f''(z) + P(z)f(z) = 0 \tag{2}$$

such that

(i) *for each critical ray, away from real axis, the solution* f *is decaying in one of the two adjacent Stokes sectors, and the solution* f *blows up in the other Stokes sector, and*

(ii) *the solution* f *has infinitely many zeros.*

Then f *has almost all real zeros.*

Remarks. (1) Earlier we explained that the conditions (i) and (ii) are necessary. And we know that for some polynomial P, if (1) has a solution f with infinitely many real zeros, then P must be real on the real line [3, Theorem 3]. In order to apply Theorem 3 for real polynomials of the form $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$, one can use the change of variables $z \mapsto |a_m|^{-1/(m+2)} z$, and $z \mapsto -z$ if necessary.

(2) By the work of Eremenko and Merenkov [2], one can see that for each $m \in \mathbf{N}$, if $m \not\equiv 2 \bmod 4$ then there exists a real polynomial P of degree m, with which (2) has a solution f satisfying the hypotheses in Theorem 3.

(3) Gundersen [3, Theorem 2], and Hellerstein and Rossi [5, Theorem 3] showed that if $m \equiv 2 \bmod 4$, then there exists no real polynomial P of degree

m for which a solution of $f'' + P(z)f = 0$ satisfies the conditions (i) and (ii) in Theorem 3.

Proof of Theorem 3. First note that the positive real axis is a critical ray for equation (2). The condition (i) along with Proposition 1(iii) implies that f has at most finitely many zeros in $\pi/3m < |\arg z| < \pi - \pi/3m$. So f has infinitely many zeros either in the sector $|\arg z| \le \pi/3m$, or in the sector $|\arg z - \pi| \le \pi/3m$. We will first prove that if f has infinitely many zeros in $|\arg z| \le \pi/3m$, then the zeros of f in this sector must be all real except finitely many. To this end we will show that f has only finitely many zeros in the sector $0 < |\arg z| \le \pi/3m$.

Suppose that f has infinitely many zeros in $|\arg z| \le \pi/3m$. Then by Proposition 1(ii) and (iii), we see that $f(z)$ blows up in $0 < |\arg z| < 2\pi/(m+2)$. Since the critical rays $\arg z = \pm 2\pi/(m+2)$ are not on the real axis, we know, by the condition (i), that $f(z)$ must decay to zero in $2\pi/(m+2) < |\arg z| < 4\pi/(m+2)$. Since P is real on the real line, we know that $\overline{f(\overline{z})}$ and $f(z)$ solve the same equation (2). Moreover, $\overline{f(\overline{z})}$ and $f(z)$ both decay in the Stokes sector $S_1 = \{2\pi/(m+2) < \arg z < 4\pi/(m+2)\}$. Thus we know one is a constant multiple of the other. Otherwise, any solution of (2) would be a linear combination of these two. And hence there would be no solution of (2) that blows up in S_1. This contradicts Proposition 2(II). So one is a constant multiple of the other. Since $|\overline{f(\overline{z})}| = |f(z)|$ on the real axis, we see that $\overline{f(\overline{z})} = e^{i\theta_0} f(z)$ for some $\theta_0 \in \mathbf{R}$, and hence $|f(x+iy)|$ is an even function of y. Thus

$$0 = \frac{\partial}{\partial y}|f(x+iy)|^2\Big|_{y=0} = -2\,\mathrm{Im}\big(f'(x)\overline{f(x)}\big) \qquad \text{for all } x \in \mathbf{R}. \tag{3}$$

Fix $x \in \mathbf{R}$ and $y > 0$, and let $g(t) := f(x+it)$. Then we have $g'(t) = if'(x+it)$ and $g''(t) = -f''(x+it)$. We then substitute f in (2) by g, multiply this by $\overline{g(t)}$, and integrate the resulting equation over $0 \le t \le y$ to have

$$-\int_0^y g''(t)\overline{g(t)}\,dt + \int_0^y \big[(x+it)^m + a_{m-1}(x+it)^{m-1} + \cdots + a_0\big]|g(t)|^2\,dt = 0.$$

This equation is called the Green's transform [6, Section 11.3].

Next we integrate the first term by parts and take the real part of the resulting equation. Then using (3) in the form of $\mathrm{Re}\big(g'(0)\overline{g(0)}\big) = 0$, we have

$$\begin{aligned}\mathrm{Re}\, g'(y)\overline{g(y)} = &\int_0^y |g'(t)|^2\,dt \\ &+ \int_0^y \mathrm{Re}\big[(x+it)^m + a_{m-1}(x+it)^{m-1} + \cdots + a_0\big]|g(t)|^2\,dt.\end{aligned} \tag{4}$$

Suppose that $0 < \arg(x+iy) \le \pi/3m$, and let $x + it = re^{i\theta}$ for $0 < t \le y$.

Then $0 < \theta \leq \pi/3m$ and hence $\cos(m\theta) \geq \frac{1}{2}$. Thus

$$\begin{aligned} &\operatorname{Re}\left[(x+it)^m + a_{m-1}(x+it)^{m-1} + \cdots + a_0\right] \\ &= r^m \cos(m\theta) + a_{m-1} r^{m-1} \cos((m-1)\theta) + \cdots + a_0 \\ &\geq \tfrac{1}{2} r^m \big(1 + O(r^{-1})\big) \quad \text{as } r \to \infty \\ &> 0 \quad \text{for all large } r > 0, \text{ say } r \geq r_0. \end{aligned} \tag{5}$$

Then the formula (4) with (5) says that if $x \geq r_0$ and $0 < \arg(x+iy) \leq \pi/3m$, then

$$-\operatorname{Im} f'(x+iy)\overline{f(x+iy)} = \operatorname{Re} g'(y)\overline{g(y)} > 0,$$

and so $f(x+iy) \neq 0$. And hence the entire function f has at most finitely many zeros in $0 < \arg z \leq \pi/3m$. Then evenness of $y \mapsto |f(x+iy)|$ shows that f has at most finitely many non-real zeros in the sector $0 < |\arg z| \leq \pi/3m$. Thus f has infinitely many real zeros.

When $m = \deg P$ is odd, f cannot have infinitely many zeros near the negative real axis since the negative real axis is the center of a Stokes sector. However, when m is even, we need to consider the case that f has infinitely many zeros in $|\arg z - \pi| \leq \pi/3m$ since the negative real axis is a critical ray. In fact, if $f(z)$ has infinitely many zeros in $|\arg z| \leq \pi/3m$, then since $\overline{f(\overline{z})} = e^{i\theta_0} f(z)$ for some $\theta_0 \in \mathbf{R}$, $f(z)$ must blow up in the two adjacent Stokes sectors of the negative real axis because $f(z)$ cannot decay to zero in these two adjacent Stokes sectors.

In order to show that f has at most finitely many non-real zeros near the negative real axis, we use the change of the variable $z \mapsto -z$, and follow the argument that we used for proving that f has at most finitely many non-real zeros near the positive real axis.

So far we have proved the theorem when f has infinitely many zeros in $|\arg z| \leq \pi/3m$. Now we suppose that f has infinitely many zeros in $|\arg z - \pi| \leq \pi/3m$. (So m must be even.) Then we use the change of the variable $z \mapsto -z$ and follow the above argument to complete the proof.

4. The cubic case: Proof of Theorem 1

In proving Theorem 1, we will further use the following proposition on existence of certain polynomials for which the equation (1) has solutions decaying in both ends of the real axis.

Proposition 3. *We consider the following Schrödinger eigenvalue problem with non-real potential*:

$$-u''(z) - \left[(iz)^3 + \beta(iz)^2 + \gamma(iz)\right]u(z) = \lambda u(z), \qquad u(\pm\infty + 0i) = 0, \tag{6}$$

where $\beta, \gamma \in \mathbf{C}$. *Then*:

(i) *There are infinitely many eigenvalues* $\lambda_k \in \mathbf{C}$; *moreover,* $|\lambda_k| \sim Ck^{6/5}$ *as* $k \to \infty$.

(ii) *If* $\beta \in \mathbf{R}$ *and* $\gamma \leq 0$, *then all eigenvalues* λ_k *are positive real.*

Proof. The statement (i) is due to Sibuya [9, Theorem 29.1] while (ii) is due to the author [8, Corollary 3]. In fact, both Sibuya and the author proved more general results for higher degree potentials, but these are enough for proving Theorem 1.

Proof of Theorem 1. By a simple rescaling, we can reduce to the cases $a = \pm 1$. And then by changing $z \mapsto -z$ if necessary, we can take $a = 1$.

Let $a = 1$, $b, c \in \mathbf{R}$ with $c \le 0$. Then Proposition 3 says that there exists a sequence of positive real numbers $\lambda_k \to \infty$ such that the equation

$$-u''(z) - \big[(iz)^3 - b(iz)^2 + c(iz) + \lambda_k\big]u(z) = 0$$

has a non-constant solution u that decays to zero as z tends to infinity along both ends of the real axis. This is where we use $c \le 0$ (or, $ac \le 0$ in the general case).

We then let $f(z) = u(iz)$ and see that f is a solution of

$$f''(z) + \big[z^3 + bz^2 + cz - \lambda_k\big]f(z) = 0, \tag{7}$$

which is the case $a = 1$ that we want to prove. Clearly, f decays to zero as z tends to infinity along both ends of the imaginary axis. This implies that f decays in the Stokes sectors S_1 and S_3, and hence blows up in S_0, S_2 and $S_4 = S_{-1}$, by Proposition 1(iii). Since f blows up in the two adjacent Stokes sectors S_{-1} and S_0, f must have infinitely many zeros, by Proposition 1(iv). Thus the conditions (i) and (ii) in Theorem 3 are satisfied, and hence Theorem 3 completes the proof.

5. The quartic case: Proof of Theorem 2

Gundersen [4] showed that for each $b \ge 0$, there exists a sequence of real constants $\lambda_k \to \infty$ such that the equation

$$f''(z) + (z^4 + bz^2 - \lambda_k)f(z) = 0 \tag{8}$$

has a solution f with almost all real zeros. Below we will prove Theorem 2 that extends Gundersen's examples for $b < 0$.

Proof of Theorem 2. By rescaling, it suffices to show for $a = 1$ and $b \in \mathbf{R}$.

Let us consider the differential equation

$$-g''(z) + (z^4 - bz^2)g(z) = \lambda g(z), \tag{9}$$

where $b, \lambda \in \mathbf{R}$. According to Gundersen [4], one can deduce from Chapters 2 and 5 in [10] that for each $b \in \mathbf{R}$, there exists a sequence of real numbers $\lambda_k \to \infty$ such that equation (9) with these λ_k has a solution g in $L^2(\mathbf{R})$. (Also, one can deduce from Sibuya's book [9] that these λ_k are zeros of some entire function of order $\frac{3}{4}$. And hence by the Hadamard factorization theorem we know that there are infinitely many such λ_k.)

Next we set $f(z) = g(iz)$. Then f solves (8) and decays to zero as $z \to \infty$ along both ends of the imaginary axis. In this case, there are six Stokes sectors. And one can see that f decays to zero in $S_1 \cup S_4$ that contains the imaginary axis. So we know, by Proposition 1(ii), that f blows up in $S_0 \cup S_2 \cup S_3 \cup S_5$. Since f blows up in $S_2 \cup S_3$, we know by Proposition 1(iv) that f has infinitely many zeros. So all hypotheses in Theorem 3 are satisfied. Therefore, for each $b \in \mathbf{R}$ there exists a sequence of real numbers λ_k such that a solution of (8) has infinitely many real zeros and at most finitely many non-real zeros.

Remark. Proposition 3 has a natural extension to higher degree polynomials P. However, the number of Stokes sectors is $2 + \deg P$ [6, Section 7.4]. And an extension [8, Theorem 2] of Proposition 3(ii) ensures existence of real polynomials for which the equation (1) has a solution decaying in *at least* two Stokes sectors. But this still leaves more than one critical ray, away from the real axis, along which the zeros might cluster if the degree of P is greater than 4.

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