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A DISTORTION THEOREM FOR BOUNDED UNIVALENT FUNCTIONS

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Abstract. We prove a distortion theorem for bounded univalent functions. Our result includes and refines distortion theorems due to Koebe, Pick, Blatter, Kim and Minda, Ma and Minda, and Jenkins.

1. Introduction

Recently, using the general coefficient theorem, Jenkins [6] proved the sharp estimate

$$|f(z_1) - f(z_2)| \geq \frac{\sinh 2\varrho}{2(2\cosh 2p\varrho)^{1/p}} \bigl(|D_1 f(z_1)|^p + |D_1 f(z_2)|^p\bigr)^{1/p} \tag{1.1}$$

for any function f analytic and univalent in the unit disk $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$ and any $p \geq 1$, where ϱ denotes the hyperbolic distance $\mathrm{d}_{\mathbf{D}}(z_1, z_2)$ between z_1 and z_2 obtained from the line element $|dz|/(1-|z|^2)$, and $D_1 f(z) = (1-|z|^2) f'(z)$ is the "hyperbolic" derivative of f. Jenkins also showed that inequality (1.1) is not true for $0 < p < 1$.

The case $p = 2$ was established earlier by Blatter [2] using a mixture of coefficient inequalities for normalized univalent functions and a comparison theorem for solutions of certain linear differential inequalities. Blatter's method was later extended by Kim and Minda [7]. They showed that inequality (1.1) holds for all $p \geq \frac{3}{2}$ for any univalent function and is not only necessary but also sufficient for univalence (for nonconstant analytic functions). They also observed that the choice $p = \infty$ in (1.1) is just an invariant version of the classical Koebe distortion theorem and that the right-hand side of (1.1) is a decreasing function of p on $[1, \infty)$. Thus, the case $p = 1$, proved by Jenkins by means of his general coefficient theorem, is the sharpest result in the one-parameter family of distortion theorems (1.1). The best possible p that can be obtained by Blatter's method is

$$\frac{1}{2}\frac{2e^3 + 1}{e^3 - 1} \doteq 1.07859, \qquad \text{cf. [4].}$$

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Ma and Minda [8] applied Blatter's method to *bounded* univalent functions $f\colon \mathbf{D} \to \mathbf{D}$ and established sharp distortion theorems similar to (1.1), i.e., bounds on the hyperbolic distance $\mathrm{d}_{\mathbf{D}}\big(f(z_1), f(z_2)\big)$ in terms of $\mathrm{d}_{\mathbf{D}}(z_1, z_2)$ and the value of the hyperbolic derivative Df of f,

$$Df(z) = \frac{1-|z|^2}{1-|f(z)|^2} f'(z),$$

at z_1 and z_2.

In this paper we prove an extension of the Ma–Minda distortion theorem by a modification of a method due to Robinson [10]. We prefer to state our result in the following form in order to point out the analogy with (1.1).

Theorem 1. *If $f\colon \mathbf{D} \to \mathbf{D}$ is univalent, and z_1, z_2 are two distinct points in $\mathbf{D}$, then we have, for $p \ge 1$,*

(1.2)

$$\left(\left(\frac{|Df(z_1)|}{1-|Df(z_1)|}\right)^p + \left(\frac{|Df(z_2)|}{1-|Df(z_2)|}\right)^p\right)^{1/p} \le \big(2\cosh(2p\varrho)\big)^{1/p} \frac{\sinh(2\varrho')}{\sinh\big(2(\varrho-\varrho')\big)},$$

where ϱ is the hyperbolic distance between z_1 and z_2 and ϱ' is the hyperbolic distance between $f(z_1)$ and $f(z_2)$. Equality occurs if f maps $\mathbf{D}$ onto $\mathbf{D}$ slit along a hyperbolic ray on the hyperbolic geodesic determined by $f(z_1)$ and $f(z_2)$. The inequality (1.2) is not true for $0 < p < 1$.

For $p \ge \frac{3}{2}$, Theorem 1 was proved by Ma and Minda [8]. The classical distortion theorem for bounded univalent functions due to Pick [9] is obtained for $p = \infty$. This is the weakest case of Theorem 1. As in the case of unbounded univalent functions, Blatter's method seems not to be capable to prove Theorem 1 for $p = 1$.

A simple rescaling argument shows that all distortion theorems for unbounded univalent functions mentioned above are limiting cases of Theorem 1, which therefore includes and refines the distortion theorems of Koebe, Pick, Blatter, Kim and Minda, Ma and Minda, and Jenkins. Our proof of Theorem 1 is methodologically remarkably simple. We shall only employ Löwner's theory combined with an elementary variational argument. In particular, a new proof of (a refinement of) Jenkins' distortion theorem is obtained without making use of the general coefficient theorem.

An inequality in the opposite sense is easier to establish. We shall prove the following result which provides a counterpart to Theorem 2 in [6] for bounded univalent functions and includes it as a limiting case. Ma and Minda (cf. Theorem 2(ii) in [8]) also found an estimate for $\mathrm{d}_{\mathbf{D}}\big(f(z_1), f(z_2)\big)$ from above in terms of $|Df(z_1)|$, $|Df(z_2)|$ and $\mathrm{d}_{\mathbf{D}}(z_1, z_2)$ which, however, is of a different nature.

Theorem 2. *If $f\colon \mathbf{D} \to \mathbf{D}$ is univalent, and z_1, z_2 are two distinct points in $\mathbf{D}$, then we have, for $p > 0$,*

$$\left(\left(\frac{|Df(z_1)|}{1-|Df(z_1)|}\right)^p + \left(\frac{|Df(z_2)|}{1-|Df(z_2)|}\right)^p\right)^{1/p} \ge 2^{1/p}\frac{\sinh(2\varrho')}{\sinh(2\varrho)-\sinh(2\varrho')}, \tag{1.3}$$

where ϱ is the hyperbolic distance between z_1 and z_2 and ϱ' is the hyperbolic distance between $f(z_1)$ and $f(z_2)$. Equality occurs if f maps $\mathbf{D}$ onto $\mathbf{D}$ slit along two hyperbolic rays on the hyperbolic geodesic determined by $f(z_1)$ and $f(z_2)$ such that $f(z_1)$ and $f(z_2)$ have the same hyperbolic distance to the boundary of $f(\mathbf{D})$.

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2. Representation and variational lemmas

It suffices to consider appropriately normalized univalent functions $f\colon \mathbf{D} \to \mathbf{D}$ since if (1.2) or (1.3) is proved for some f, then it follows for $S \circ f \circ T$, where S and T are conformal automorphisms of $\mathbf{D}$. We shall use the standard normalization and denote by $\mathscr{S}_0$ the set of univalent functions $f\colon \mathbf{D} \to \mathbf{D}$ with $f(0) = 0$.

We consider, for fixed $0 < v < r < 1$, the set

$$\mathscr{D}(v,r) := \bigl\{ \bigl(|Df(0)|, |Df(z_0)|\bigr) \bigm| f \in \mathscr{S}_0,\ |z_0| = r,\ |f(z_0)| = v \bigr\}.$$

In order to prove (1.2) and (1.3) we have to find the maximum and the minimum of the function

$$F(a,e) = \left(\frac{a}{1-a}\right)^p + \left(\frac{e}{1-e}\right)^p$$

for $(a,e) \in \mathscr{D}(v,r)$.

The following lemma shows that $\mathscr{D}(v,r)$ admits a very simple description. It is the key to the proof of Theorem 1 and is maybe interesting in its own.

Lemma 1. *For any $0 < v < r < 1$*

$$\mathscr{D}(v,r) = \left\{ \left(\exp\left[-\int_v^r \frac{u(x)}{x}\,dx\right], \exp\left[-\int_v^r \frac{dx}{xu(x)}\right]\right) \Bigm| u \in \mathscr{U}(v,r) \right\}, \tag{2.1}$$

where

$$\mathscr{U}(v,r) = \left\{ u : [v,r] \to \mathbf{R} \textit{ measurable} \Bigm| \frac{1-x}{1+x} \le u(x) \le \frac{1+x}{1-x} \textit{ for a.e. } x \in [v,r] \right\}.$$

Moreover, $\mathscr{D}(v,r)$ is convex in logarithmic coordinates, that is, the set $\{(\log x, \log y) \mid (x,y) \in \mathscr{D}(v,r)\}$ is convex.

Remark. The lemma is a consequence of the Löwner theory which gives a parametric representation of $\mathscr{S}_0$ in conjunction with a version of Liapunov's convexity theorem on vector measures due to Aumann [1]. Parts of the following argument can be found in Robinson's paper [10].

Proof. (i) We denote the set on the right-hand side of (2.1) by $\mathscr{E}(v,r)$. Note that both sets, $\mathscr{D}(v,r)$ and $\mathscr{E}(v,r)$, are compact. This is obvious for $\mathscr{D}(v,r)$. To prove the compactness of $\mathscr{E}(v,r)$, we first observe that

$$\begin{aligned}(2.2)\qquad &\left\{\left(\int_v^r \frac{u(x)}{x}\,dx, \int_v^r \frac{dx}{xu(x)}\right) \,\Big|\, u \in \mathscr{U}(v,r)\right\}\\ &= \left\{\int_v^r \alpha(x)\,dx \,\Big|\, \alpha\colon [v,r] \to \mathbf{R}^2 \text{ measurable}, \ \alpha(x) \in A(x),\ x \in [v,r]\right\},\end{aligned}$$

where

$$A(x) = \left\{\left(\frac{u}{x}, \frac{1}{xu}\right) \,\Big|\, \frac{1-x}{1+x} \le u \le \frac{1+x}{1-x}\right\}, \qquad v \le x \le r.$$

The sets $A(x)$ are nonempty compact subsets of $\mathbf{R}^2$ and the set-valued function $x \mapsto A(x)$ is continuous in the Hausdorff topology of compact subsets of $\mathbf{R}^2$. Thus a version of Liapunov's convexity theorem due to Aumann [1] (cf. also [5, p. 29]) applies and shows that the set in (2.2) is convex and compact. Hence $\mathscr{E}(v,r)$ is compact and convex in logarithmic coordinates, i.e., $\mathscr{E}(v,r)$ is compact (but not necessarily convex, cf. Figure 1).

(ii) We are now going to prove $\mathscr{D}(v,r) \subseteq \mathscr{E}(v,r)$. Since both sets are compact it suffices to show that the dense subset of $\mathscr{D}(v,r)$ which corresponds to one-slit mappings in $\mathscr{S}_0$ is contained in $\mathscr{E}(v,r)$.

Let $f(z) = az + \cdots\colon \mathbf{D} \to \mathbf{D}$ be such a univalent function, i.e., $\mathbf{D} \setminus f(\mathbf{D})$ is a Jordan arc. Then $f(z) = aw(z,T)/|a|$, $T = -\log|f'(0)|$, where $w(z,t)$ is the solution of the Löwner differential equation

$$\begin{aligned}(2.3)\qquad &\frac{d}{dt}w(z,t) = -w(z,t)\frac{1+\kappa(t)w(z,t)}{1-\kappa(t)w(z,t)}, \qquad t \in [0,T],\\ &w(z,0) = z,\end{aligned}$$

for some measurable function $\kappa(t)\colon [0,T] \to \partial\mathbf{D}$. See, for instance, [3].

Fix $z_0 \in \mathbf{D}$, let $x(t) = |w(z_0,t)|$ and $r = |z_0|$. We deduce from (2.3)

$$\begin{aligned}(2.4)\qquad &\frac{d}{dt}x(t) = -\frac{x(t)}{u\big(x(t)\big)},\\ &x(0) = r,\end{aligned}$$

with

$$u\big(x(t)\big) = \frac{1}{\operatorname{Re}\left(\dfrac{1+\kappa(t)w(z_0,t)}{1-\kappa(t)w(z_0,t)}\right)} = \frac{|1-\kappa(t)w(z_0,t)|^2}{1-|w(z_0,t)|^2}.$$

Taking into account that $t \mapsto x(t)$ is monotonically decreasing from $x(0) = r$ to $x(T) = v = |f(z_0)|$, the latter identity defines a function $u\colon [v,r] \to \mathbf{R}$ which satisfies

$$\frac{1-x}{1+x} \leq u(x) \leq \frac{1+x}{1-x}.$$

Thus $u \in \mathscr{U}(v,r)$ for $v = |f(z_0)|$, and

$$|Df(0)| = |f'(0)| = e^{-T} = \exp\left(-\int_0^T dt\right) = \exp\left(-\int_v^r \frac{u(x)}{x}\,dx\right)$$

by (2.4). A straightforward calculation using the Löwner ODE (2.3) and (2.4) shows

$$\frac{d}{dt}\log\left(\frac{|w'(z_0,t)|}{1-|w(z_0,t)|^2}\right) = \frac{x'(t)}{x(t)u\big(x(t)\big)},$$

that is,

$$\begin{aligned}|Df(z_0)| &= (1-|z_0|^2)\frac{|w'(z_0,T)|}{1-|w(z_0,T)|^2} = \exp\left(\int_0^T \frac{d}{dt}\log\left(\frac{|w'(z_0,t)|}{1-|w(z_0,t)|^2}\right)dt\right)\\ &= \exp\left(\int_0^T \frac{x'(t)}{x(t)u\big(x(t)\big)}\,dt\right) = \exp\left(-\int_v^r \frac{dx}{xu(x)}\right).\end{aligned}$$

This proves $\mathscr{D}(v,r) \subseteq \mathscr{E}(v,r)$.

(iii) In order to prove the converse inclusion, we fix $u \in \mathscr{U}(v,r)$ and define measurable functions $\zeta\colon [v,r] \to [-1,1]$ and $\varphi\colon [v,r] \to \partial\mathbf{D}$ by

$$\zeta(x) = \frac{(1+x^2)-(1-x^2)u(x)}{2x}, \qquad \varphi(x) = \zeta(x) + i\sqrt{1-\zeta(x)^2}\,,$$

so that

$$u(x) = \frac{|1-\varphi(x)x|^2}{1-x^2}.$$

Let $x(t)$ be the uniquely determined absolutely continuous solution of the separable ODE (2.4) corresponding to our choice of u, i.e., $x(t) = R^{-1}(t)$, where

$$R(x) = -\int_r^x \frac{u(\eta)}{\eta}\,d\eta, \qquad x \in [v,r].$$

Note that $x(t)$ is absolutely continuous because $c \leq R'(x) \leq 1/c$ for a.e. $x \in [v,r]$ for some constant $c < 0$. Moreover, $x(t)$ is monotonically decreasing, satisfies $x(t) \leq r + t/c$ and exists as long as $x(t) \geq v$, i.e., $x(T) = v$ for some $T > 0$. Finally, let

$$\varrho(t) = -\int_0^t \frac{2\,\mathrm{Im}\big(\varphi\big(x(t)\big)\big)x(t)}{|1-\varphi\big(x(t)\big)x(t)|^2}\,dt.$$

A calculation shows $x(t)e^{i\varrho(t)} = w(r,t)$, where $w(z,t)$ is the solution of the Löwner ODE (2.3) for $\kappa(t) = \varphi(x(t))e^{-i\varrho(t)}$. Therefore, $w(\,\cdot\,,t) \in \mathscr{S}_0$ for every $t \in [0,T]$ and $|w(r,t)| = x(t)$. Moreover,

$$\frac{d}{dt}\log\left(\frac{|w'(r,t)|}{1-|w(r,t)|^2}\right) = \frac{x'(t)}{x(t)u\big(x(t)\big)}.$$

Thus

$$\begin{aligned} \exp\left[-\int_v^r \frac{u(x)}{x}\,dx\right] &= \exp\left[-\int_0^T dt\right] = e^{-T} = |w'(0,T)| = |Dw(0,T)|, \\ \exp\left[-\int_v^r \frac{dx}{xu(x)}\right] &= (1-r^2)\frac{|w'(r,T)|}{1-|w(r,T)|^2} = |Dw(r,T)|, \end{aligned}$$

and we have proved $\mathscr{E}(v,r) \subseteq \mathscr{D}(v,r)$. □

We now combine the representation Lemma 1 with an elementary variational argument which is in fact a particularly simple case of Pontryagin's maximum principle in optimal control theory. It essentially reduces extremal problems over $\mathscr{D}(v,r)$ to a family of extremal problems over the intervals

$$U(x) = \left[\frac{1-x}{1+x}, \frac{1+x}{1-x}\right], \qquad x \in [v,r].$$

Lemma 2. *Let a: $\mathscr{U}(v,r) \to \mathbf{R}$ and e: $\mathscr{U}(v,r) \to \mathbf{R}$ be defined by*

$$\begin{aligned} a(u) &:= \exp\left[-\int_v^r \frac{u(x)}{x}\,dx\right], \\ e(u) &:= \exp\left[-\int_v^r \frac{dx}{xu(x)}\right]. \end{aligned} \tag{2.5}$$

Let F: $[0,1]\times[0,1] \to \mathbf{R}$ be differentiable and $\hat{u} \in \mathscr{U}(v,r)$ such that

$$\sup_{u\in\mathscr{U}(v,r)} F\big(a(u),e(u)\big) = F\big(a(\hat{u}),e(\hat{u})\big).$$

Then

$$H\big(\hat{u}(x),x\big) = \min_{u\in U(x)} H(u,x) \qquad \text{for a.e. } x \in [v,r], \tag{2.6}$$

where the hamiltonian H: $(0,\infty)\times[v,r] \to \mathbf{R}$ is given by

$$H(u,x) = \frac{\partial F}{\partial a}\big(a(\hat{u}),e(\hat{u})\big)\frac{a(\hat{u})}{x}u + \frac{\partial F}{\partial e}\big(a(\hat{u}),e(\hat{u})\big)\frac{e(\hat{u})}{x}\frac{1}{u}. \tag{2.7}$$

Proof. Fix a Lebesgue point $x_0 \in [v,r]$ of the functions $x \mapsto \hat{u}(x)/x$ and $x \mapsto 1/\big(x\hat{u}(x)\big)$, that is, suppose

$$\lim_{\delta\to 0+} \frac{1}{\delta}\int_{x_0}^{x_0+\delta} \frac{\hat{u}(x)}{x}\,dx = \frac{\hat{u}(x_0)}{x_0}, \qquad \lim_{\delta\to 0+} \frac{1}{\delta}\int_{x_0}^{x_0+\delta} \frac{dx}{x\hat{u}(x)} = \frac{1}{x_0\hat{u}(x_0)},$$

and let u be an arbitrary point in $U(x_0)$. For $\delta > 0$ consider the needle variation

$$u_\delta(x) = \begin{cases} u & \text{for } x \in [x_0, x_0+\delta], \\ \hat{u}(x) & \text{for } x \in [v,r]\setminus[x_0,x_0+\delta]. \end{cases}$$

Since $U(x_0) \subseteq U(x_1)$ for $x_1 \ge x_0$, we have $u_\delta \in \mathscr{U}(v,r)$. A calculation yields

$$\begin{aligned} \lim_{\delta\to 0+} \frac{a(u_\delta)-a(\hat{u})}{\delta} &= \frac{a(\hat{u})}{x_0}\big(\hat{u}(x_0)-u\big), \\ \lim_{\delta\to 0+} \frac{e(u_\delta)-e(\hat{u})}{\delta} &= \frac{e(\hat{u})}{x_0}\left(\frac{1}{\hat{u}(x_0)}-\frac{1}{u}\right), \end{aligned}$$

and thus

$$\lim_{\delta\to 0+} \frac{F\big(a(u_\delta),e(u_\delta)\big)-F\big(a(\hat{u}),e(\hat{u})\big)}{\delta} = H\big(\hat{u}(x_0),x_0\big) - H(u,x_0).$$

By hypothesis,

$$\frac{F\big(a(u_\delta),e(u_\delta)\big)-F\big(a(\hat{u}),e(\hat{u})\big)}{\delta} \le 0, \qquad \delta > 0,$$

so that (2.6) follows since almost every point in $[v,r]$ is a Lebesgue point of the L^1-functions $x \mapsto \hat{u}(x)/x$ and $x \mapsto 1/\big(x\,\hat{u}(x)\big)$. □

3. An extremal problem for the set $\mathscr{U}(v,r)$

We next apply Lemma 2 to $F(a,e) = \big(a/(1-a)\big)^p + \big(e/(1-e)\big)^p$, $p > 0$.

Lemma 3. *Let $\hat{u} \in \mathscr{U}(v,r)$ such that*

$$\left(\frac{a(u)}{1-a(u)}\right)^p + \left(\frac{e(u)}{1-e(u)}\right)^p \le \left(\frac{a(\hat{u})}{1-a(\hat{u})}\right)^p + \left(\frac{e(\hat{u})}{1-e(\hat{u})}\right)^p$$

for every $u \in \mathscr{U}(v,r)$. Then there exists a constant $\gamma \in [v,r]$ such that either

(a) $$\hat{u}(x) = u_{1,\gamma}(x) := \begin{cases} \dfrac{1-x}{1+x} & \text{for a.e. } x \in [v,\gamma], \\ \dfrac{1-\gamma}{1+\gamma} & \text{for a.e. } x \in [\gamma,r]; \end{cases}$$

or

$$\text{(b)} \qquad \hat{u}(x) = u_{2,\gamma}(x) := \begin{cases} \dfrac{1+x}{1-x} & \text{for a.e. } x \in [v,\gamma], \\ \dfrac{1+\gamma}{1-\gamma} & \text{for a.e. } x \in [\gamma,r]; \end{cases}$$

or

$$\text{(c)} \qquad \hat{u}(x) = u_{3,\alpha}(x) = \alpha \qquad \text{for a.e. } x \in [v,r],$$

for some $\alpha \in \big[(1-v)/(1+v), (1+v)/(1-v)\big]$.

Proof. By Lemma 3, $H(u,x) \geq H\big(\hat{u}(x),x\big)$ for a.e. $x \in [v,r]$ where

$$H(u,x) = p\left(\frac{a(\hat{u})}{1-a(\hat{u})}\right)^{p-1} \frac{a(\hat{u})}{\big(1-a(\hat{u})\big)^2}\frac{1}{x}u + p\left(\frac{e(\hat{u})}{1-e(\hat{u})}\right)^{p-1} \frac{e(\hat{u})}{\big(1-e(\hat{u})\big)^2}\frac{1}{x}\frac{1}{u}.$$

Since $u \mapsto H(u,v)$ is a non-constant convex function, it attains its minimum on the interval $U(v) = \big[(1-v)/(1+v), (1+v)/(1-v)\big]$ at $(1-v)/(1+v)$, at $(1+v)/(1-v)$ or at some unique point α in between. In the latter case, $u \mapsto H(u,x)$ attains its minimum on $U(x)$ only at α for *every* $x \in [v,r]$, so that $\hat{u}(x) = \alpha$ for a.e. $x \in [v,r]$. If $u \mapsto H(u,v)$ takes its minimum on $U(v)$ at $(1-v)/(1+v)$ but not at $(1+v)/(1-v)$, then obviously case (a) holds. □

Remark. In the proof of Lemma 3 we only used the convexity of $u \mapsto H(u,x)$ and the fact that in this case $u \mapsto H(u,x)$ attains its minimum always at the same point (independent of x).

Lemma 4. *The functional*

$$\eta(u) := \frac{a(u)}{1-a(u)} + \frac{e(u)}{1-e(u)}$$

attains its maximum on $\mathscr{U}(v,r)$ *only at* $u = u_{1,r}$ *and* $u = u_{2,r}$.

Proof. In view of Lemma 3, the only candidates for maximizing $\eta(u)$ over $\mathscr{U}(v,r)$ are the functions $u_{1,\gamma}$ and $u_{2,\gamma}$ for $\gamma \in [v,r]$ and the functions $u_{3,\alpha}$ for $\alpha \in \big[(1-v)/(1+v), (1+v)/(1-v)\big]$. To decide which of these functions actually maximizes $\eta(u)$ is now a pure calculus problem. The basic steps are as follows.

Integrating (2.5) with $u = u_{1,\gamma}$ yields

$$\begin{aligned} \hat{a}(\gamma) &:= a(u_{1,\gamma}) = \frac{(1+\gamma)^2}{\gamma}\frac{v}{(1+v)^2}\exp\left[-\frac{1-\gamma}{1+\gamma}\log\frac{r}{\gamma}\right], \\ \hat{e}(\gamma) &:= e(u_{1,\gamma}) = \frac{(1-\gamma)^2}{\gamma}\frac{v}{(1-v)^2}\exp\left[-\frac{1+\gamma}{1-\gamma}\log\frac{r}{\gamma}\right]. \end{aligned} \tag{3.1}$$

It is easy to see that $\hat{a}'(\gamma) > 0$ and $\hat{e}'(\gamma) < 0$. Thus, $\hat{e}(\gamma) < \hat{a}(\gamma)$, since $\hat{e}(v) < \hat{a}(v)$. Next, for $\gamma \in [v, r]$, we have

$$\begin{aligned} &\frac{d}{d\gamma}\left(\frac{\hat{a}(\gamma)}{1-\hat{a}(\gamma)} + \frac{\hat{e}(\gamma)}{1-\hat{e}(\gamma)}\right) \\ &= 2\left(\log\frac{r}{\gamma}\right)\left[\frac{1}{(1+\gamma)^2}\frac{\hat{a}(\gamma)}{(1-\hat{a}(\gamma))^2} - \frac{1}{(1-\gamma)^2}\frac{\hat{e}(\gamma)}{\big(1-\hat{e}(\gamma)\big)^2}\right] \geq 0, \end{aligned} \tag{3.2}$$

with equality if and only if $\gamma = r$, since

$$\left[\frac{1}{(1+\gamma)^2}\frac{\hat{a}(\gamma)}{\big(1-\hat{a}(\gamma)\big)^2} - \frac{1}{(1-\gamma)^2}\frac{\hat{e}(\gamma)}{\big(1-\hat{e}(\gamma)\big)^2}\right] > 0.$$

The latter inequality is equivalent to

$$(1+v)\big(1-\hat{a}(\gamma)\big) - \big(1-\hat{e}(\gamma)\big)(1-v)\exp\left[\frac{2\gamma}{1-\gamma^2}\log\frac{r}{\gamma}\right] < 0, \quad 0 < v \leq \gamma \leq r < 1, \tag{3.3}$$

which in turn follows from the easily verified fact that the left-hand side of (3.3) is a strictly convex function of the variable v which is obviously not positive for $v = 0$ and takes the value

$$2\left[(1+\gamma)\sinh\left(\frac{1}{2}\frac{1-\gamma}{1+\gamma}\log\frac{r}{\gamma}\right) - (1-\gamma)\sinh\left(\frac{1}{2}\frac{1+\gamma}{1-\gamma}\log\frac{r}{\gamma}\right)\right]\exp\left[-\frac{1}{2}\frac{1-\gamma}{1+\gamma}\log\frac{r}{\gamma}\right] \tag{3.4}$$

at $v = \gamma$. The first factor in (3.4) is strictly decreasing on $[\gamma, 1]$ as a function of variable r and vanishes at $r = \gamma$. This proves (3.3) and hence (3.2).

It follows that $\eta(u_{1,\gamma}) < \eta(u_{1,r})$ for all $\gamma \in [v, r]$, so that among $u_{1,\gamma}$ only $u_{1,r}$ maximizes $\eta(u)$. Similarly, $\eta(u)$ is maximized among $u_{2,\gamma}$ only by $u_{2,r}$. Moreover, $\eta(u_{1,r}) = \eta(u_{2,r})$.

Finally, on $\big[(1-v)/(1+v), (1+v)/(1-v)\big]$, the function $\alpha \mapsto \eta(u_{3,\alpha})$ is a convex function since the second derivatives

$$\begin{aligned} \frac{d^2}{d\alpha^2}\frac{a(u_{3,\alpha})}{1-a(u_{3,a\alpha})} &= \left(\log\frac{r}{v}\right)^2 \frac{1+a(u_{3,\alpha})}{\big(1-a(u_{3,\alpha})\big)^3}a(u_{3,\alpha}), \\ \frac{d^2}{d\alpha^2}\frac{e(u_{3,\alpha})}{1-e(u_{3,a\alpha})} &= \left(\log\frac{r}{v}\right)\frac{2}{\alpha^3}\frac{1+e(u_{3,\alpha})}{\big(1-e(u_{3,\alpha})\big)^3}e(u_{3,\alpha}) \\ &\quad \times\left[\frac{\log(r/v)}{2\alpha} - \tanh\left(\frac{\log(r/v)}{2\alpha}\right)\right] \end{aligned} \tag{3.5}$$

are obviously positive. Thus, $\alpha \mapsto \eta(u_{3,\alpha})$ is maximized at $\alpha^- = (1-v)/(1+v)$ or at $\alpha^+ = (1+v)/(1-v)$. However, $\eta(u_{3,\alpha^-}) = \eta(u_{1,v}) < \eta(u_{1,r})$ by the first part of the proof. Similarly, $\eta(u_{3,\alpha^+}) = \eta(u_{2,v}) < \eta(u_{2,r})$. Therefore, none of the functions $u_{3,\alpha}$ maximizes η over $\mathscr{U}(v, r)$. □

Lemma 5. *For any $p \geq 1$ the functional*

$$\eta_p(u) := \left(\frac{a(u)}{1-a(u)}\right)^p + \left(\frac{e(u)}{1-e(u)}\right)^p \tag{3.6}$$

attains its maximum on $\mathscr{U}(v,r)$ only at $u = u_{1,r}$ and $u = u_{2,r}$. For $0 < p < 1$ the functional (3.6) *is not maximized by $u_{1,r}$ and $u_{2,r}$.*

Proof. Employing the notation used in the proof of Lemma 4, we obtain

$$\begin{aligned}
\frac{d}{d\gamma}\left[\left(\frac{\hat a(\gamma)}{1-\hat a(\gamma)}\right)^p + \left(\frac{\hat e(\gamma)}{1-\hat e(\gamma)}\right)^p\right] &= p\left(\frac{\hat a(\gamma)}{1-\hat a(\gamma)}\right)^{p-1}\frac{\hat a'(\gamma)}{(1-\hat a(\gamma))^2} \\
&\quad + p\left(\frac{\hat e(\gamma)}{1-\hat e(\gamma)}\right)^{p-1}\frac{\hat e'(\gamma)}{\big(1-\hat e(\gamma)\big)^2} \\
&\overset{(3.2)}{\geq} p\left[-\left(\frac{\hat a(\gamma)}{1-\hat a(\gamma)}\right)^{p-1} + \left(\frac{\hat e(\gamma)}{1-\hat e(\gamma)}\right)^{p-1}\right] \\
&\quad \times \frac{\hat e'(\gamma)}{\big(1-\hat e(\gamma)\big)^2} > 0,
\end{aligned} \tag{3.7}$$

if $p \geq 1$, since $\hat e'(\gamma) < 0$ and $\hat e(\gamma) < \hat a(\gamma)$. Therefore, $\eta_p(u)$ is maximal among $u_{1,\gamma}$ only for $u_{1,r}$. Similarly, $\eta_p(u)$ is maximal among $u_{2,\gamma}$ only for $u_{2,r}$ and $\eta_p(u_{1,r}) = \eta_p(u_{2,r})$. Finally, $\alpha \mapsto \eta_p(u_{3,\alpha})$ is strictly convex for $p \geq 1$ since its second derivative is found to be

$$\begin{aligned}
p(p-1)\Bigg[&\left(\frac{a(u_{3,\alpha})}{1-a(u_{3,\alpha})}\right)^{p-2}\left(\frac{d}{d\alpha}\frac{a(u_{3,\alpha})}{1-a(u_{3,\alpha})}\right)^2 \\
&+ \left(\frac{e(u_{3,\alpha})}{1-e(u_{3,\alpha})}\right)^{p-2}\left(\frac{d}{d\alpha}\frac{e(u_{3,\alpha})}{1-e(u_{3,\alpha})}\right)^2\Bigg] \\
&+ p\left(\frac{a(u_{3,\alpha})}{1-a(u_{3,\alpha})}\right)^{p-1}\frac{d^2}{d\alpha^2}\left(\frac{a(u_{3,\alpha})}{1-a(u_{3,\alpha})}\right) \\
&+ p\left(\frac{e(u_{3,\alpha})}{1-e(u_{3,\alpha})}\right)^{p-1}\frac{d^2}{d\alpha^2}\left(\frac{e(u_{3,\alpha})}{1-e(u_{3,\alpha})}\right) > 0
\end{aligned}$$

by (3.5). Thus $\eta_p(u)$ is maximal among $u_{3,\alpha}$ only for $\alpha = (1-v)/(1+v)$ or for $\alpha = (1+v)/(1-v)$ where its value is less than for $u = u_{1,r}$.

It remains to consider the case $0 < p < 1$. Since

$$\frac{d}{d\gamma}\left(\frac{\hat a(\gamma)}{1-\hat a(\gamma)} + \frac{\hat e(\gamma)}{1-\hat e(\gamma)}\right)_{\gamma=r} = 0,$$

$\hat{e}(\gamma) < \hat{a}(\gamma)$ and $\hat{e}'(\gamma) < 0 < \hat{a}'(\gamma)$, (3.7) shows

$$\frac{d}{d\gamma}\left[\left(\frac{\hat{a}(\gamma)}{1-\hat{a}(\gamma)}\right)^p + \left(\frac{\hat{e}(\gamma)}{1-\hat{e}(\gamma)}\right)^p\right] < 0$$

for all $\gamma \in [v, r]$ sufficiently close to r. Therefore, $\eta_p(u)$ is not maximized at $u = u_{1,r}$. □

4. Proof of Theorem 1

Using Lemma 1 we deduce from Lemma 5

Lemma 6. *For any $p \geq 1$ and any $0 < v < r < 1$, we have*

(4.1)
$$\left(\left(\frac{|Df(0)|}{1-|Df(0)|}\right)^p + \left(\frac{|Df(z_0)|}{1-|Df(z_0)|}\right)^p\right)^{1/p} \leq \Big(2\cosh(2p\varrho)\Big)^{1/p} \frac{\sinh(2\varrho')}{\sinh\big(2(\varrho-\varrho')\big)}$$

for $\big(|Df(0)|, |Df(z_0)|\big) \in \mathscr{D}(v, r)$, where ϱ and ϱ' denotes the hyperbolic distance between 0 and z_0, and 0 and $f(z_0)$, respectively. Equality occurs if f maps $\mathbf{D}$ univalently onto $\mathbf{D}$ slit along a segment on the line determined by 0 and $f(z_0)$. The inequality (4.1) is not valid for $0 < p < 1$.

Proof. Since the hyperbolic distance between 0 and $r = |z_0|$, and between 0 and $v = |f(z_0)|$ is given by

$$\varrho = \frac{1}{2}\log\frac{1+|z_0|}{1-|z_0|} \quad \text{and} \quad \varrho' = \frac{1}{2}\log\frac{1+|f(z_0)|}{1-|f(z_0)|},$$

respectively, we obtain from (3.1) and (3.6)

$$\begin{aligned}\eta_p(u_{1,r})^{1/p} &= \left(\left(\frac{a(u_{1,r})}{1-a(u_{1,r})}\right)^p + \left(\frac{e(u_{1,r})}{1-e(u_{1,r})}\right)^p\right)^{1/p} \\ &= \big(2\cosh(2p\varrho)\big)^{1/p}\frac{\sinh(2\varrho')}{\sinh\big(2(\varrho-\varrho')\big)}.\end{aligned}$$

In view of Lemma 1 and Lemma 5 this proves (4.1) for $p \geq 1$ and shows that this inequality does not hold for $0 < p < 1$. It remains to show that equality occurs in (4.1) if f maps $\mathbf{D}$ univalently onto $\mathbf{D}$ slit along a segment on the line determined by 0 and $f(z_0)$. In this case $g(z) := |f(z_0)|/f(z_0)f(z_0z/|z_0|)$ maps $\mathbf{D}$ univalently onto $\mathbf{D} \setminus [\beta, 1)$ for some $\beta \in (0,1)$ and $g'(0) > 0$. Thus $g(z) = w(z, T)$, $T = -\log g'(0)$, where $w(z,t)$ is the solution of the Löwner equation (2.3) corresponding to $\kappa(t) \equiv 1$. Note that $w(r,t) > 0$. If we define the function $u \in U(v, r)$ by

$$u\big(w(r,t)\big) = \frac{|1-w(r,t)|^2}{1-|w(r,t)|^2}$$

as in part (ii) of the proof of Lemma 1, we see that $u = u_{1,r}$ by construction and $|Df(0)| = |Dw(0,T)| = a(u_{1,r})$ and $|Df(z_0)| = |Dw(r,T)| = e(u_{1,r})$. Thus equality holds in (4.1) in this case. □

Utilizing the invariance property of the problem, we have proved Theorem 1. It remains to deduce Jenkins' distortion theorem from Theorem 1.

Corollary 1. *If $f\colon \mathbf{D} \to \mathbf{C}$ is univalent, and z_1, z_2 are two distinct points in $\mathbf{D}$, then we have, for $p \geq 1$,*

$$|f(z_1) - f(z_2)| \geq \frac{\sinh 2\varrho}{2(2\cosh 2p\varrho)^{1/p}} \big(|D_1 f(z_1)|^p + |D_1 f(z_2)|^p\big)^{1/p}, \tag{4.2}$$

where ϱ denotes the hyperbolic distance between z_1 and z_2.

Proof. We can approximate any univalent function $f\colon \mathbf{D} \to \mathbf{C}$ by *bounded* univalent functions, if the bound is allowed to vary. Let $f\colon \mathbf{D} \to \mathbf{C}$ be univalent and $f_n\colon \mathbf{D} \to \mathbf{C}$ a sequence of univalent functions bounded by M_n such that $f_n \to f$ locally uniformly in $\mathbf{D}$. We may assume that $M_n \to \infty$. Using the easily verified facts

$$M_n \frac{|Dg_n(z)|}{1 - |Dg_n(z)|} \to |D_1 f(z)|,$$

and

$$M_n \sinh\big(2\mathrm{d}_{\mathbf{D}}\big(g_n(z_1), g_n(z_2)\big)\big) \to 2|f(z_1) - f(z_2)|$$

for $g_n(z) = f_n(z)/M_n$, we see that (4.2) follows from (1.2) applied to g_n. □

5. Proof of Theorem 2

Proof. According to Lemma 1 we have to maximize the function

$$\mu_p(u) := -\left(\frac{a(u)}{1 - a(u)}\right)^p - \left(\frac{e(u)}{1 - e(u)}\right)^p$$

on the set $\mathscr{U}(v,r)$, where $a(u)$ and $e(u)$ are given by (2.5). Let $\hat{u} \in \mathscr{U}(v,r)$ such that $\mu_p(u) \leq \mu_p(\hat{u})$ for every $u \in \mathscr{U}(v,r)$. We first show that $\mu_p(\hat{u}) = \mu_p(u_\sigma)$, where

$$u_\sigma(x) := \begin{cases} \dfrac{1-x}{1+x}, & v \leq x < \sigma, \\ \dfrac{1+x}{1-x}, & \sigma \leq x \leq r, \end{cases}$$

for some suitably chosen constant $\sigma \in [v,r]$.

To see this we note that the hamiltonian $H(u,x)$ in Lemma 2 is now a strictly concave function of u for every fixed $x \in [v,r]$. Thus the function $u \mapsto H(u,x)$ is minimal on the interval $\big[(1-x)/(1+x), (1+x)/(1-x)\big]$ at one of its boundary points, and Lemma 2 implies for every $x \in [v,r]$ that $\hat{u}(x) = (1+x)/(1-x)$ or $\hat{u}(x) = (1-x)/(1+x)$. For $u(x) = \hat{u}(x)$ we construct the function $\kappa(t)$, $0 \leq t \leq T$, as in part (iii) of the proof of Lemma 1. Then $\kappa(t)$ takes only the values ± 1.

Hence the solution $w(z,T)$ of the Löwner equation (2.3) generated by $\kappa(t)$ satisfies $a(\hat{u}) = |Dw(0,T)|$ and $e(\hat{u}) = |Dw(r,T)|$, and maps $\mathbf{D}$ univalently onto $\mathbf{D}$ slit along two segments on the real axis (see [3, p. 116]). Therefore, $w(z,T) = \widehat{w}(z,T)$, where $\widehat{w}(z,t)$ is the solution to the Löwner ODE (2.3) corresponding to $\hat{\kappa}(t) = 1$ for $0 \le t < \tau$ and $\hat{\kappa}(t) = -1$ for $\tau \le t \le T$ for some constant $0 \le \tau \le T$. Note that $\widehat{w}(r,t) > 0$. If we now define $u \in \mathscr{U}(v,r)$ as in part (ii) of the proof of Lemma 1 by

$$u\big(\hat{w}(r,t)\big) = \frac{|1-\hat{\kappa}(t)\widehat{w}(r,t)|^2}{1-|\widehat{w}(r,t)|^2},$$

then, by construction, $u(x) = u_\sigma(x)$ for $\sigma = \widehat{w}(r,\tau)$, and $a(u) = |D\widehat{w}(0,T)| = |Dw(0,T)| = a(\hat{u})$ and $e(u) = |D\widehat{w}(r,T)| = |Dw(r,T)| = e(\hat{u})$, that is, $\mu_p(u_\sigma) = \mu_p(\hat{u})$.

We have to decide which of the functions u_σ actually maximizes $\mu_p(u)$. A calculation yields

$$\begin{aligned} a(u_\sigma) &= \frac{(1-r)^2}{r}\frac{v}{(1+v)^2}\left(\frac{1+\sigma}{1-\sigma}\right)^2, \\ e(u_\sigma) &= \frac{(1+r)^2}{r}\frac{v}{(1-v)^2}\left(\frac{1-\sigma}{1+\sigma}\right)^2. \end{aligned} \tag{5.1}$$

Thus

$$\frac{d}{d\sigma}\mu_p(u_\sigma) = -\frac{4p}{1-\sigma^2}\left[\left(\frac{a(u_\sigma)}{1-a(u_\sigma)}\right)^p\frac{1}{1-a(u_\sigma)} - \left(\frac{e(u_\sigma)}{1-e(u_\sigma)}\right)^p\frac{1}{1-e(u_\sigma)}\right] = 0$$

if and only if $a(u_\sigma) = e(u_\sigma)$, because $x^p/(1-x)^{p+1}$ is monotonically increasing on $(0,1)$ for $p>0$. Since $a(u_\sigma) = e(u_\sigma)$ is equivalent to

$$\frac{1+r}{1-r}\cdot\frac{1+v}{1-v} = \left(\frac{1+\sigma}{1-\sigma}\right)^2, \tag{5.2}$$

we conclude that $\sigma \mapsto \mu_p(u_\sigma)$ has precisely one critical point $\sigma = \sigma_0 \in [v,r]$ which is given by (5.2). A straightforward computation using (5.1) shows that the second derivative of $\mu_p(u_\sigma)$ is negative for $\sigma = \sigma_0$, so that σ_0 is the global maximum for $\mu_p(u_\sigma)$. By (5.1) and (5.2)

$$a(u_{\sigma_0}) = \frac{(1-r)^2}{r}\cdot\frac{v}{(1+v)^2}\cdot\frac{1+r}{1-r}\cdot\frac{1+v}{1-v} = \frac{\sinh(2\varrho')}{\sinh(2\varrho)},$$

i.e.,

$$\mu_p(u) \le \mu_p(u_{\sigma_0}) = -2\left(\frac{a(u_{\sigma_0})}{1-a(u_{\sigma_0})}\right)^p = -2\left[\frac{\sinh(2\varrho')}{\sinh(2\varrho)-\sinh(2\varrho')}\right]^p, \tag{5.3}$$

where $\varrho = \frac{1}{2}\log\big((1+r)/(1-r)\big)$ and $\varrho' = \frac{1}{2}\log\big((1+v)/(1-v)\big)$ is the hyperbolic distance between 0 and r and between 0 and v, respectively. We have proved (1.3).

Let f map $\mathbf{D}$ univalently onto $\mathbf{D}$ slit along two hyperbolic rays on the hyperbolic geodesic determined by $f(z_1)$ and $f(z_2)$ such that $f(z_1)$ and $f(z_2)$ have the same hyperbolic distance to the boundary of $f(\mathbf{D})$. We shall show that equality holds in (1.3). By conformal invariance we may assume $z_1 = 0$, $z_2 = r \in (0,1)$, $f(0) = 0$ and $f(r) = v \in (0,r)$. Thus f maps $\mathbf{D}$ onto $\mathbf{D}$ slit along the segments $(-1,-r_1]$ and $[r_2,1)$. Since $\mathrm{d}_{\mathbf{D}}(0,-r_1) = \mathrm{d}_{\mathbf{D}}(r,r_2)$, we have $r_1 = (r_2 - v)/(1 - r_2 v)$. Hence

$$f(z) = -\frac{f\left(\dfrac{r-z}{1-rz}\right) - v}{1 - vf\left(\dfrac{r-z}{1-rz}\right)},$$

which implies $Df(0) = Df(r)$.

On the other hand, $f(z) = w(z,T)$, $T = -\log f'(0)$, where $w(z,t)$ is the solution of the Löwner equation (2.3) for $\kappa(t) = 1$ for $0 \le t < \tau$ and $\kappa(t) = -1$ for $\tau \le t \le T$ for some constant $\tau \in (0,T)$. Note that $w(r,t) > 0$. If we now define the function $u \in \mathscr{U}(v,r)$ as in part (ii) of the proof of Lemma 1 by

$$u\bigl(w(r,t)\bigr) = \frac{|1 - \kappa(t) w(r,t)|^2}{1 - |w(r,t)|^2},$$

then $u = u_\sigma$ for some $\sigma \in [v,r]$ by construction and $a(u_\sigma) = Df(0) = Df(r) = e(u_\sigma)$. As we have seen, this is only possible if $\sigma = \sigma_0$, where σ is given by (5.2). Hence

$$\begin{aligned}\left(\left(\frac{|Df(z_1)|}{1-|Df(z_1)|}\right)^p + \left(\frac{|Df(z_2)|}{1-|Df(z_2)|}\right)^p\right)^{1/p} &= \Bigl(-\mu_p(u_{\sigma_0})\Bigr)^{1/p} \\ &= 2^{1/p}\frac{\sinh(2\varrho')}{\sinh(2\varrho) - \sinh(2\varrho')}\end{aligned}$$

by (5.3). □

Using the same arguments as in the proof of Corollary 1, we can deduce the following result of Jenkins (Theorem 2 in [6]) from Theorem 2.

Corollary 2. *If $f\colon \mathbf{D} \to \mathbf{C}$ is univalent, and z_1, z_2 are two distinct points in $\mathbf{D}$, then we have, for $p \ge 0$,*

$$|f(z_1) - f(z_2)| \le \frac{\sinh 2\varrho}{2^{1+1/p}}\bigl(|D_1 f(z_1)|^p + |D_1 f(z_2)|^p\bigr)^{1/p}, \tag{5.4}$$

where ϱ denotes the hyperbolic distance between z_1 and z_2.

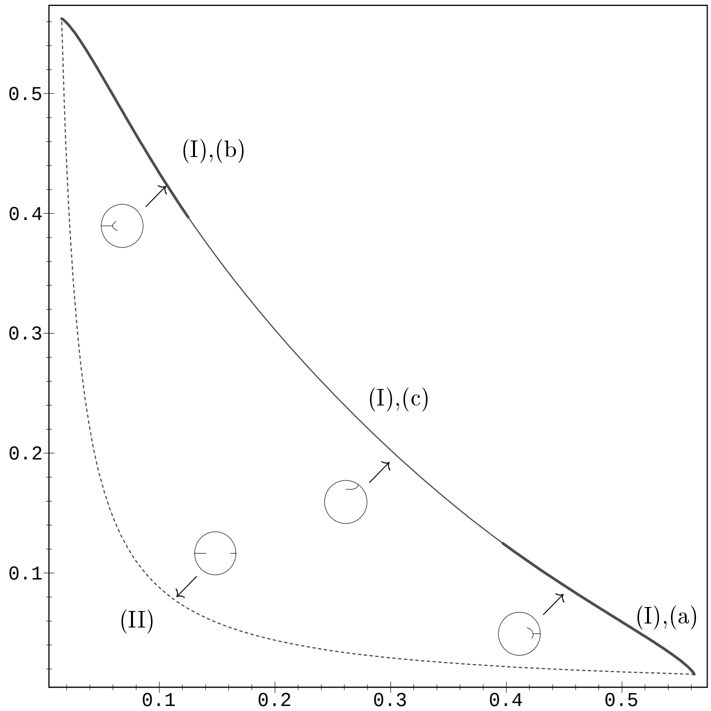

Figure 1. The set $\mathscr{D}(\frac{1}{5}, \frac{4}{5})$.

6. Final remarks

1. In principle, Lemma 2 can be used to solve any extremal problem for the set $\mathscr{D}(v, r)$. Indeed, if we want to solve the extremal problem

$$\max_{(a,e)\in\mathscr{D}(v,r)} F(a, b)$$

for some differentiable function F: $[0, 1] \times [0, 1] \to \mathbf{R}$, then the corresponding hamiltonian $H(u, x)$, cf. (2.7), is either (I) convex or (II) concave as function of u. Case (I) is completely characterized by Lemma 3, whereas case (II) can handled as in the proof of Theorem 2. In either case, the determination of the extremal values of F on $\mathscr{D}(v, r)$ is reduced to a calculus problem, namely to maximize a real-valued function of one real variable.

2. Our method can also be used to describe the set $\mathscr{D}(v, r)$ explicitly. It turns out that $\mathscr{D}(v, r)$ is a simply-connected domain whose boundary consists of two smooth Jordan arcs corresponding to the cases (I) and (II) above, cf. Figure 1. The two common end points of these two Jordan arcs correspond to univalent functions mapping onto $\mathbf{D}$ slit along a radial ray. Points on the Jordan arc (II) correspond to univalent functions mapping onto $\mathbf{D}$ with slits along the positive and the negative real axes or rotations of such functions. Points on the Jordan arc (I) correspond to forked slit mappings in case (a) and (b) of Lemma 3 and to analytic one-slit mappings in case (c) of Lemma 3.

3. The expression

$$|Df(z_1)|^p + |Df(z_2)|^p$$

is not maximized by the extremal functions of Theorem 1. In fact, the extremal functions for this functional belong to the complicated part (I), (c) of the set $\mathscr{D}(v, r)$.

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