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ON ZEROS OF NORMAL FUNCTIONS

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Abstract. We give necessary conditions for zero sets of normal functions, little normal functions and functions of uniformly bounded characteristic.

1. Introduction

A function f meromorphic in the unit disc $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ is normal if

$$\|f\| = \sup_{z\in\mathbf{D}} (1-|z|^2) f^\sharp(z) < \infty, \tag{1.1}$$

where $f^\sharp(z) = |f'(z)|/\bigl(1+|f(z)|^2\bigr)$. If

$$\lim_{|z|\to 1} (1-|z|^2) f^\sharp(z) = 0, \tag{1.2}$$

then f is called a little normal function. The class of normal and little normal fuctions will be denoted by N and N_0, respectively.

The Bloch space $\mathscr{B}$ consists of those functions f analytic on $\mathbf{D}$ for which

$$\sup_{z\in\mathbf{D}} |f'(z)|(1-|z|^2) < \infty,$$

and the little Bloch space $\mathscr{B}_0$ consists of those functions $f \in \mathscr{B}$ for which

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0.$$

Since $|f^\sharp(z)| \le |f'(z)|$, it is clear that every Bloch function is a normal function. It was also observed by Tse [T] that if $f \in \mathscr{B}$, then $g = e^f \in N$. There are extensive results about normal and Bloch functions, see, e.g., [L], [ACP] and references given there.

In 1988 D. Ulrich [U] used random series to show that zero sets of elements of $\mathscr{B}_0$ are different from the zero sets of elements of $\mathscr{B}$. More exactly, he proved that there is a function in $\mathscr{B}$ whose zeros cannot be zeros of any function in $\mathscr{B}_o$.

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If $\{z_n\}$ is the sequence of zeros of a normal function f and $|z_1| \leq |z_2| \leq \cdots < 1$, then we call $\{z_n\}$ the ordered zeros of f.

Recently it has been proved in [GNW] that if $\{z_n\}$ are the ordered zeros of a Bloch function nonvanishing at zero, then

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = O\big((\log N)^{1/2}\big), \qquad \text{as } N \to \infty, \tag{1.3}$$

and if $\{z_n\}$ ($z_n \neq 0$) are the ordered zeros of a little Bloch function, then

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = o\big((\log N)^{1/2}\big), \qquad \text{as } N \to \infty.$$

This result has been motivated by Ch. Horowitz' paper [Hor] on zeros of functions in Bergman space A^p. Horowitz proved that if $\{z_n\}$ are ordered zeros of a function in A^p, $0 < p < \infty$, nonvanishing at zero, then

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = O(N^{1/p}), \qquad \text{as } N \to \infty.$$

Here we apply the above-mentioned result in [U], to show that (1.3) is sharp for the Bloch space in the sense that $O\big((\log N)^{1/2}\big)$ cannot be replaced by $o\big((\log N)^{1/2}\big)$.

In 1972 Anderson, Clunie and Pommerenke [ACP] showed that if f is normal, $\{z_n\}$ is the sequence of zeros of f and D_1 is a disc that touches $\partial\mathbf{D}$ from inside, then

$$\sum_{z_n \in D_1} (1 - |z_n|) < \infty.$$

Here we obtain the following

Theorem 2. *If f is a normal function, $f(0) \neq 0$, and $\{z_n\}$ are ordered zeros of f, then*

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = O\big(N^{\|f\|^2/2}\big), \qquad \textit{as } N \to \infty. \tag{1.4}$$

We also obtain a similar result for little normal functions. In the last section we consider the class of functions of uniformly bounded characeristic (UBC) introduced by Yamashita in [Y].

2. A remark on zeros of Bloch functions

For a function f analytic on $\mathbf{D}$ and $0<r<1$ set

$$\|f_r\|_0=\exp\biggl(\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta\biggr).$$

It is known (see, e.g., [ACP], [U]) that if f is a Bloch function, then

$$\|f_r\|_0=O\biggl(\Bigl(\log\frac{1}{1-r}\Bigr)^{1/2}\biggr), \tag{2.1}$$

while

$$\|f_r\|_0=o\biggl(\Bigl(\log\frac{1}{1-r}\Bigr)^{1/2}\biggr),$$

if f is in the little Bloch space. Moreover, it has been proved in [U], that there is $f\in\mathscr{B}$ for which

$$\|f_r\|_0\neq o\biggl(\Bigl(\log\frac{1}{1-r}\Bigr)^{1/2}\biggr). \tag{2.2}$$

We will use this result to show

Theorem 1. *There is a function $f\in\mathscr{B}$ with $f(0)\neq 0$ whose ordered zeros $\{z_n\}$ satisfy*

$$\prod_{n=1}^{N}\frac{1}{|z_n|}\neq o\bigl((\log N)^{1/2}\bigr),\qquad \text{as } N\to\infty.$$

Proof. Assume that $f\in\mathscr{B}$ satisfies (2.2) and $f(0)\neq 0$. (One can take $f(z)=f_\omega(z)/z^2$, where f_ω is given by (17) in [U].) Then there is a sequence $\{r_m\}$, $0<r_m<1$, $\lim_{m\to\infty}r_m=1$, and a positive constant c such that

$$\|f_{r_m}\|_0\geq c\Bigl(\log\frac{1}{1-r_m}\Bigr)^{1/2}.$$

This and the Jensen formula give

$$|f(0)|\prod_{|z_k|<r_m}\frac{r_m}{|z_k|}\geq c\Bigl(\log\frac{1}{1-r_m}\Bigr)^{1/2}. \tag{2.3}$$

Let $n(r)$ denote the number of zeros of f in the disc $|z|\leq r$, where each zero is counted according to its multiplicity. Note that (2.3) implies that $n(r_m)\to\infty$ as $m\to\infty$. Moreover, by (2.1),

$$N(r,0)=\int_0^r\frac{n(t)}{t}\,dt\leq C\log\log\frac{1}{1-r},$$

which implies that (see, e.g., [SS, p. 225])

$$n(r) \le \frac{C \log\log \frac{1}{1-r}}{1-r}, \tag{2.4}$$

or, equivalently,

$$\log \frac{1}{1-r}\left(1 + \frac{\log\log\log \frac{1}{1-r} + \log C}{\log \frac{1}{1-r}}\right) \ge \log n(r).$$

So, if $\varepsilon > 0$, then for r sufficiently close to 1,

$$\log \frac{1}{1-r} \ge \frac{1}{1+\varepsilon} \log n(r).$$

Consequently, (2.3) yields

$$\prod_{k=1}^{n(r_m)} \frac{1}{|z_k|} > \prod_{k=1}^{n(r_m)} \frac{r_m}{|z_k|} \ge c_1 \big(\log n(r_m)\big)^{1/2}$$

with some $c_1 > 0$.

3. Proof of Theorem 2

Proof of Theorem 2. For $0 < r < 1$, and for a function f meromorphic in $\mathbf{D}$ set

$$A(r,f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \big(f^\sharp(te^{i\theta})\big)^2 t \, d\theta dt.$$

Then the Ahlfors–Shimizu characteristic $T_0(r,f)$ is given by

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t} \, dt.$$

By (1.1),

$$A(r,f) \le \|f\|^2 \int_0^r \frac{2t}{(1-t^2)^2} \, dt = \|f\|^2 \frac{r^2}{1-r^2}.$$

Consequently,

$$T_0(r,f) \le \|f\|^2 \int_0^r \frac{tdt}{1-t^2} = \frac{\|f\|^2}{2} \log\left(\frac{1}{1-r^2}\right) \le \frac{\|f\|^2}{2} \log\left(\frac{1}{1-r}\right).$$

On the other hand, if $f(0) \neq 0$, then by the Ahlfors–Shimizu theorem [Hay, p. 12]

$$T_0(r, f) = N(r, 0) + m_0(r, 0) - m_0(0, 0),$$

where

$$N(r, 0) = \sum_{|z_n|<r} \log \frac{r}{|z_n|}$$

and

$$m_0(r, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{k\big(f(re^{i\theta}), 0\big)} \, d\theta,$$

where $k(w, a)$ denotes the cordial distance on the Riemann sphere given by

$$k(w, a) = \frac{|w - a|}{\sqrt{(1 + |a|^2)(1 + |w|^2)}}\,.$$

Thus

$$m_0(0, 0) = \log \frac{\sqrt{1 + |f(0)|^2}}{|f(0)|} = c_0 > 0.$$

It is also clear that $m_0(r, 0) > 0$. It then follows that

$$\begin{aligned} \sum_{|z_n|<r} \log \frac{r}{|z_n|} &= T_0(r, f) - m_0(r, 0) + c_0 \leq T_0(r, f) + c_0 \\ &\leq \frac{\|f\|^2}{2} \log\left(\frac{1}{1 - r}\right) + c_0. \end{aligned}$$

Now reasoning in a similar way as in [Hor, p. 625] shows that the inequality

$$\sum_{n=1}^{N} \log \frac{r}{|z_n|} \leq \frac{\|f\|^2}{2} \log\left(\frac{1}{1 - r}\right) + c_0$$

actually holds for $0 < r < 1$ and for all positive integers N. This in turn implies

$$\prod_{n=1}^{N} \frac{r}{|z_n|} \leq e^{c_0} \left(\frac{1}{1 - r}\right)^{\|f\|^2/2}.$$

Putting $r = 1 - 1/N$, we get for $N \geq 2$,

$$\prod_{n=1}^{N} \frac{1}{|z_n|} \leq 4e^{c_0} N^{\|f\|^2/2}.$$

Remark. We do not know if (1.4) is sharp for normal functions. In [Hor] the author has constructed analytic functions whose ordered zeros $0 < |z_1| \leq |z_2| \cdots$ satisfy

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = O(N^\alpha), \qquad \alpha > 0.$$

He has also proved that these functions are in A^p with some $p > 0$. Here we show that, at least in some cases, they are not normal. To this end consider

$$f(z) = \prod_{k=1}^{\infty} (1 - \mu z^{2^k}), \qquad \mu > 2. \tag{3.1}$$

It follows from [Hor, p. 698] that the ordered zeros of f satisfy

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = O(N^\alpha) \qquad \text{with } \alpha = \log \mu / \log 2.$$

To see that f given by (3.1) is not normal take $0 < x_n < 1$ such

$$x_n^{2^n} = \frac{1}{\mu}, \qquad n \geq 1. \tag{3.2}$$

Since

$$f'(z) = \sum_n (-\mu) 2^n z^{2^n - 1} \prod_{k \neq n} (1 - \mu z^{2^k}),$$

we see that

$$\begin{aligned} |f'(x_n)| &= 2^n \mu^{1/2^n} \prod_{k<n} |(1 - \mu\mu^{-2^{k-n}})| \prod_{k>n} (1 - \mu\mu^{-2^{k-n}}) \\ &= 2^n \mu^{1/2^n} \prod_{k=1}^{n-1} (\mu\mu^{-2^{-k}} - 1) f\left(\frac{1}{\mu}\right). \end{aligned}$$

By (3.2), $2^n = -\log \mu / \log x_n$, and

$$(1 - x_n) f^\sharp(x_n) = (1 - x_n)|f'(x_n)| > (1 - x_n) \frac{-\log \mu}{\log x_n} f\left(\frac{1}{\mu}\right) \prod_{k=1}^{n-1} (\mu\mu^{-2^{-k}} - 1).$$

Since

$$\lim_{x \to 1} \frac{(1 - x)}{\log x} = -1 \quad \text{and} \quad \prod_{k=1}^{\infty} (\mu\mu^{-2^{-k}} - 1)$$

diverges to $+\infty$, and since $f(x_n) = 0$,

$$\lim_{n \to \infty} (1 - x_n) f^\sharp(x_n) = +\infty.$$

4. Zeros of functions in N_0

The following theorem yields a necessary condition for zeros of the little normal functions.

Theorem 3. *If $f \in N_0$, $f(0) \neq 0$, and $\{z_n\}$ are ordered zeros of f, then*

$$\sum_{n=1}^{N} \log \frac{1}{|z_n|} = o(\log N), \qquad \text{as } N \to \infty.$$

Proof. It follows from (1.2) that, given $\varepsilon > 0$, there is r_ε such that

$$|f^\sharp(z)|(1-|z|^2) < \varepsilon$$

for $r_\varepsilon < |z| < 1$. In [Y, p. 353] the following formula has been obtained

$$T_0(r,f) = \frac{1}{\pi} \int_{|z|<r} \left(f^\sharp(z)\right)^2 \log \frac{r}{|z|}\, dx\, dy.$$

Thus for $1 > r > r_\varepsilon$ we have

$$\begin{aligned} T_0(r,f) &= \frac{1}{\pi} \int_{|z|<r_\varepsilon} (f^\sharp(z))^2 \log \frac{r}{|z|}\, dx\, dy + \frac{1}{\pi} \int_{r_\varepsilon<|z|<r} \left(f^\sharp(z)\right)^2 \log \frac{r}{|z|}\, dx\, dy \\ &\leq C + 2\varepsilon^2 \int_{r_\varepsilon}^{r} \frac{t}{(1-t^2)^2} \log \frac{1}{t}\, dt. \end{aligned}$$

Using the inequality $\log(1/t) \leq (1-t)/t$, $0 < t < 1$, we get

$$T_0(r,f) \leq C + 2\varepsilon^2 \int_{r_\varepsilon}^{r} \frac{dt}{1-t} \leq C + 2\varepsilon^2 \log \frac{1}{1-r}.$$

A reasoning similar to that used in the proof of Theorem 2 shows that the inequality

$$\sum_{n=1}^{N} \log \frac{r}{|z_n|} \leq C + 2\varepsilon^2 \log \frac{1}{1-r}$$

holds for $1 > r > r_\varepsilon$ and for all positive integers N. Finally, putting $r = 1 - 1/N$ we see that for $N \geq 2$,

$$\sum_{n=1}^{N} \log \frac{1}{|z_n|} \leq C + \log 4 + 2\varepsilon^2 \log N,$$

which implies the desired result.

5. The class UBC

For $w \in \mathbf{D}$, we set

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}, \qquad z \in \mathbf{D}.$$

In [Y] the author defined the class of functions of uniformly bounded characteristic as follows: a meromorphic function f in $\mathbf{D}$ is said to be of bounded characteristic ($f \in$UBC) if and only if

$$\sup_{w \in \mathbf{D}} T_0(1, f_w) < \infty,$$

where $T_0(1, f) = \lim_{r \to 1} T_0(r, f)$ and $f_w = f \circ \varphi_w$. In [Y] the sharp inclusion

$$\mathrm{UBC} \subset N$$

was showed. Moreover, it is also clear that UBC is a subclass of the Nevalinna class (or the class of bounded characteristic, BC). Consequently, each non-zero $f \in$ UBC admits the decomposition

$$f = \frac{b_1 g}{b_2}, \tag{5.1}$$

where b_1 and b_2 are the Blaschke products whose zeros are precisely the zeros and poles of f, respectively, and $g \in$ BC has neither pole nor zero in $\mathbf{D}$. Yamishita [Y] also proved that if $f \in$UBC and (5.1) is satisfied, then g and $fb_2 = gb_1$ are also in UBC. The class UBC can be also considered as a meromorphic analogue of the space BMOA (i.e., the space of analytic functions of bounded mean oscillation, see e.g. [B]). The following characterization of zeros of functions of uniformly bounded characteristic has been motivated by the value distribution theorem for BMOA given in [B] and [Str].

Theorem 4. *If $\{z_n\}$ is the sequence of zeros of $f \in$ UBC, then*

$$\sup\left\{\sum_{n=1}^{\infty} \log \frac{1}{|\varphi_w(z_n)|} : w \in \mathbf{D},\ |f(w)| \geq 1\right\} < \infty.$$

In the case when $|f(w)| < 1$ for all $w \in \mathbf{D}$ we assume that

$$\sum_{n=1}^{\infty} \log(1/|\varphi_w(z_n)|) = 0.$$

Proof. Let $f \in$UBC and let $\{z_n\}$ and $\{p_n\}$ be its zeros and poles, respectively. If $T(r,f)$ denotes the Nevanlinna characteristic, then

$$\begin{aligned} T(r,f) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta + \sum_{|p_n|<r} \log \frac{r}{|p_n|} \\ &\geq \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta + \sum_{|p_n|<r} \log \frac{r}{|p_n|} \\ &= \log|f(0)| + \sum_{|z_n|<r} \log \frac{r}{|z_n|}, \qquad 0<r<1, \end{aligned}$$

where the last equality follows from Jensen's formula. We also know that the Nevanlinna characteristic and Ahlfors–Schimizu characteristic differ by a bounded term, that is [Hay, p. 12],

$$\big|T(r,f) - T_0(r,f) - \log^+|f(0)|\big| \leq \tfrac{1}{2}\log 2.$$

Thus

$$T_0(r,f) \geq T(r,f) - \log^+|f(0)| - \tfrac{1}{2}\log 2.$$

It then follows that

$$T_0(1,f) \geq \log|f(0)| + \sum_{z_n} \log\frac{1}{|z_n|} - \log^+|f(0)| - \frac{1}{2}\log 2.$$

Replacing f by f_w, we see that if $|w|<1$ is such that $|f(w)|\geq 1$, then

$$\sum_{z_n} \log \frac{1}{|\varphi_w(z_n)|} \leq T_0(1, f_w),$$

which ends the proof.

Note that actually the following generalization of Theorem 4 is true.

Theorem 5. *If* $\{z_n\}$ *is a sequence of zeros of* $f \in$ UBC, *then for any positive* δ

$$\sup\left\{ \sum_{n=1}^{\infty} \log \frac{1}{|\varphi_w(z_n)|} : w \in \mathbf{D},\ |f(w)| \geq \delta \right\} < \infty.$$

References

[ACP] Anderson, J.M., J. Clunie, and Ch. Pommerenke: On Bloch functions and normal functions. - J. Reine Angew. Math. 240, 1974, 12–37.

[B] Baernstein, A.: Analytic functions of bounded mean oscillations. - In: Aspects of Contemporary Complex Analysis, edited by D.A. Brannan and J.G. Clunie. Academic Press, London, 1980, pp. 3–36.

[GNW] Girela, D., M. Nowak, and P. Waniurski: On the zeros of Bloch functions. - Math. Proc. Cambridge Philos. Soc. 139, 2000, 117–128.

[Hay] Hayman, W.K.: Meromorphic Functions. - Oxford, 1964.

[Hor] Horowitz, Ch.: Zeros of functions in the Bergman spaces. - Duke Math. J. 41, 1974, 693–710.

[L] Lappan, P.: Normal families and normal functions: results and techniques. - In: Function Spaces and Complex Analysis, Joensuu 1997, Univ. Joensuu, Department of Mathematics Rep. Ser. 2, 63–78.

[SS] Shapiro, H.S., and A.L. Shields: On the zeros of functions with finite Dirichlet integral and some related function spaces. - Math. Z. 80, 1962, 217–229.

[Str] Stroethoff, K.: Nevanlinna-type characterizations for the Bloch space and related spaces. - Proc. Edinburgh Math. Soc. 33, 1990, 123–141.

[U] Ulrich, D.: Kinchin's inequality and the zeros of Bloch functions. - Duke Math. J. 57, 1988, 519–535.

[T] Tse, K.F.: On sums and products of normal functions. - Comm. Math. Univ. St. Pauli 17, 1969, 63–72.

[Y] Yamashita, S.: Functions of uniformly bounded characteristic. - Ann. Acad. Sci. Fenn. Ser. A I Math. 7, 1982, 349–367.

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