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ON THE GROWTH OF ENTIRE FUNCTIONS WITH ZERO SETS HAVING INFINITE EXPONENT OF CONVERGENCE

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Abstract. Let Z be a sequence of complex numbers tending to infinity having infinite exponent of convergence. For $r > 0$, let $n(r)$ denote the number of members of Z of modulus at most r. If

$$\mu = \varlimsup_{r\to\infty} \frac{\log\log n(r)}{\log\log r},$$

it is shown that

$$\inf_f \varliminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \le \min\left(\pi + \log 2, \log\frac{\mu+1}{\mu-1}\right),$$

where the infimum is over all entire f vanishing precisely on Z and $M(r,f)$ denotes the maximum of $|f(z)|$ on $|z| = r$. This bound strengthens earlier results of Bergweiler.

1. Introduction

Let $Z = \{z_j : j = 1, 2, 3, \ldots\}$ be a sequence of complex numbers, not necessarily distinct, tending to infinity and ordered so that $|z_1| \le |z_2| \le \cdots$. We consider entire functions with zero set precisely Z, i.e., entire f with a zero of multiplicity m at a complex number β provided that $z_j = \beta$ for exactly m different values of j. We compare the rate of growth of the maximum modulus function

$$M(r,f) = \max_{|z|=r} |f(z)|$$

of such f with $n(r)$, the number of distinct values of j such that $|z_j| \le r$.

The exponent of convergence σ of the sequence Z is defined to be the infimum of all $\alpha > 0$ for which

$$\sum_{z_j\ne 0} |z_j|^{-\alpha} < \infty,$$

or equivalently by

$$\sigma = \varlimsup_{r\to\infty} \frac{\log n(r)}{\log r}. \tag{1.1}$$

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It is an easy consequence of Jensen's theorem that for any entire f vanishing precisely on Z, the order ρ of f, defined by

$$\rho = \varlimsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

satisfies $\rho \geq \sigma$. We shall be concerned exclusively with sequences Z with infinite exponent of convergence, and thus all entire f we consider will be of infinite order.

One measure of the growth of $n(r)$ for sequences with infinite exponent of convergence is

$$\mu = \varlimsup_{r\to\infty} \frac{\log\log n(r)}{\log\log r}. \tag{1.2}$$

It is elementary to verify that $\mu \geq 1$ if $\sigma > 0$ and that $\sigma = \infty$ if $\mu > 1$.

Bergweiler has studied the growth of entire functions vanishing precisely on a sequence Z with infinite exponent of convergence. Let

$$L(\mu) := \sup_Z \inf_f \varliminf_{r\to\infty} \frac{\log M(r,f)}{n(r)},$$

where the supremum is over all sequences Z satisfying (1.2) and the infimum is over all entire f vanishing precisely on Z. (In the case $\mu = 1$, we additionally require of Z that $\sigma = \infty$.) Bergweiler [2] proved the following two theorems.

Theorem A. *For every Z satisfying* (1.2), *there exists an entire f vanishing precisely on Z such that*

$$\varliminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \leq \alpha(\mu) := \begin{cases} \dfrac{3\mu}{4(\mu-1)}, & 1<\mu<\infty, \\ \dfrac{3}{4}, & \mu = \infty. \end{cases}$$

Theorem B. *For $1<\mu<\infty$ there exists Z satisfying* (1.2) *such that for every entire f vanishing precisely on Z,*

$$\varliminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \geq \beta(\mu) := (\mu-1)\log\frac{\mu-1}{\mu} + (\mu+1)\log\frac{\mu+1}{\mu}.$$

We note that Theorem A implies that $L(\mu) \leq \alpha(\mu)$ and Theorem B implies $L(\mu) \geq \beta(\mu)$. We also note that $\alpha(\mu)$ tends to infinity as $\mu \to 1^+$ and that Theorem A gives no information about $L(1)$. It is elementary that $\beta(\mu)$ is a decreasing function of μ on $(1,\infty)$ with $\beta(\mu) \to 2\log 2$ as $\mu \to 1^+$ and $\beta(\mu) \to 0$ as $\mu \to \infty$. Bergweiler [1] has asked (i) whether $L(\mu)$ is a bounded function for $1 \leq \mu \leq \infty$ and in particular if $L(1)$ is finite and (ii) whether $L(\mu) \to 0$ as $\mu \to \infty$.

We answer these questions by proving the following two theorems. We remark that our results give information about the minimum modulus of f as well as the maximum modulus.

Theorem 1. *If Z satisfies (1.2) for $1 < \mu \le \infty$, then*

$$\inf_f \varliminf_{r\to\infty} \frac{\left\|\log|f(re^{i\theta})|\right\|_\infty}{n(r)} \le \begin{cases} \log\dfrac{\mu+1}{\mu-1}, & 1<\mu<\infty, \\ 0, & \mu=\infty, \end{cases} \tag{1.3}$$

where the infimum is over all entire f vanishing precisely on Z.

Theorem 2. *If Z is any sequence with infinite exponent of convergence, then*

$$\inf_f \varliminf_{r\to\infty} \frac{\left\|\log|f(re^{i\theta})|\right\|_\infty}{n(r)} \le \pi + \log 2,$$

where the infimum is over all entire f vanishing precisely on Z.

We note that the combination of Theorems 1 and 2 yields

$$L(\mu) \le \begin{cases} \min\left(\pi+\log 2, \log\dfrac{\mu+1}{\mu-1}\right), & 1\le\mu<\infty, \\ 0, & \mu=\infty. \end{cases}$$

We further note that Theorem 1, in conjunction with Theorem B, shows that

$$\frac{1-o(1)}{\mu} \le L(\mu) \le \frac{2+o(1)}{\mu}, \qquad \mu\to\infty. \tag{1.4}$$

By a refinement of his proof of Theorem B, Bergweiler was able to show [2, p. 103] that $\liminf_{\mu\to1} L(\mu) > 1.6$. This result together with Theorem 2 yields

$$1.6 < \sup_Z \inf_f \varliminf_{r\to\infty} \frac{\log M(r,f)}{n(r)} \le \pi + \log 2, \tag{1.5}$$

where the supremum is over all Z with infinite exponent of convergence and the infimum is over all entire f vanishing precisely on Z. We have been unable to narrow the gap between the upper and lower bounds in either (1.4) or (1.5).

Our results involve upper bounds for $\log M(r,f)$ in terms of $n(r)$ on a sequence tending to infinity. It is observed in [2] that in general it is not possible to obtain such bounds on a large set of r-values, for example on a set of positive logarithmic density. Suppose, for example, that Z has infinite exponent of convergence, all members of Z are positive real numbers, and

$$\varlimsup_{\substack{r\to\infty\\ r\in E}} \frac{\log n(r)}{\log r} < \infty$$

for some set $E \subset [1,\infty)$ of upper logarithmic density 1. It is well known [5] that any entire f vanishing precisely on Z has infinite lower order. Thus

$$\lim_{\substack{r\to\infty\\ r\in E}} \frac{\log M(r,f)}{n(r)} = \infty$$

for any such f.

It is perhaps worth remarking that no results such as Theorems 1 and 2 are possible for the ratio $\log M(r,f)/N(r)$, where

$$N(r) = \int_0^r \frac{n(t)}{t}\,dt$$

is the integrated counting function of value distribution theory. For in Bergweiler's examples in Theorem B, $n(r)/N(r) \to \infty$ as $r \to \infty$ and thus $\log M(r,f)/N(r) \to \infty$ for every entire f vanishing precisely on Z.

There is a vast literature concerning comparisons of the growth of $\log M(r,f)$ for an entire function f to the distribution of its zeros. An excellent collection of references appears in [2].

2. Preliminaries

The following lemma, used in the proof of Theorem 2, is due in its essential form to Newman [8]. (See also [3].) We are indebted to J. Fournier for bringing the lemma to our attention and for several helpful communications. For completeness we include Fournier's proof of the lemma in the precise form required for our purposes.

Lemma 1. *Suppose $0 < \varepsilon < 1$ and that*

$$P(\theta) = \sum_{k=-M}^{M} c_k e^{ik\theta}$$

is a real trigonometric polynomial with

$$|c_k| \le \frac{1}{2}\left(\frac{1}{M+1-\varepsilon-|k|}\right), \qquad 1 \le |k| \le M, \tag{2.1}$$

and

$$|c_0| \le \frac{1}{M+1-\varepsilon}. \tag{2.2}$$

Then there exists a real trigonometric polynomial

$$Q(\theta) = \sum_{k=-L}^{L} d_k e^{ik\theta}$$

where $L > M$ *such that* $d_k = c_k$ *for* $|k| \le M$ *and*

$$\|Q\|_\infty < \frac{\pi}{1-\varepsilon} + \varepsilon.$$

Proof. Let

$$R(\theta) = \sum_{k=-M}^{-1} c_k e^{ik\theta} + \frac{c_0}{2}$$

and let $b_j = c_{j-M-1}$ for $1 \le j \le M$ and $b_{M+1} = \frac{1}{2}c_0$. For $0 \le j \le M-1$, we have by (2.1)

$$|b_{j+1}| = |c_{j-M}| \le \frac{1}{2}\left(\frac{1}{M+1-\varepsilon+j-M}\right) < \frac{1}{2(1-\varepsilon)(j+1)}. \tag{2.3}$$

From (2.2) we conclude

$$|b_{M+1}| = \frac{|c_o|}{2} \le \frac{1}{2(M+1-\varepsilon)} < \frac{1}{2(1-\varepsilon)(M+1)}. \tag{2.4}$$

We define a linear functional T: $H^1 \to \mathscr{C}$ by

$$T(f) = T\left(\sum_{j=0}^{\infty} a_j z^j\right) = \sum_{j=0}^{M} a_j \overline{b_{j+1}}.$$

By a theorem of Hardy [4, p. 48],

$$|T(f)| \le \sum_{j=0}^{M} |a_j|\,|\overline{b_{j+1}}| \le \sum_{j=0}^{M} \frac{|a_j|}{2(1-\varepsilon)(j+1)} \le \frac{\pi\|f\|_1}{2(1-\varepsilon)},$$

where we have used (2.3) and (2.4). The Hahn–Banach theorem implies that T extends to a bounded linear functional on $L^1[-\pi,\pi]$ with norm at most $\pi/2(1-\varepsilon)$, and thus there exists a function h on $[-\pi,\pi]$ bounded by $\pi/2(1-\varepsilon)$ such that

$$T(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{h(\theta)}\,d\theta$$

for all $f \in H^1$, where $f(\theta) = \lim_{r\to 1} f(re^{i\theta})$.

Letting $f(\theta) = e^{i(n-1)\theta} \in H^1$ for $n \geq 1$, we get

$$T(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-1)\theta}\overline{h(\theta)}\,d\theta = \begin{cases} \overline{b_n}, & 1 \leq n \leq M+1, \\ 0, & n > M+1. \end{cases}$$

Taking conjugates we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta}H(\theta)\,d\theta = \begin{cases} b_n, & 1 \leq n \leq M+1, \\ 0, & n > M+1, \end{cases}$$

where

$$H(\theta) = e^{i\theta}h(\theta).$$

Setting $H^*(\theta) = e^{-i(M+1)\theta}H(\theta)$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n-M-1)\theta}H^*(\theta)\,d\theta = \begin{cases} b_n, & 1 \leq n \leq M+1, \\ 0, & n > M+1. \end{cases} \tag{2.5}$$

Using $c_{n-M-1} = b_n$ for $1 \leq n \leq M$ and $\frac{1}{2}c_o = b_{M+1}$, we rewrite (2.5) as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}H^*(\theta)\,d\theta = \begin{cases} c_k, & -M \leq k \leq -1, \\ \frac{1}{2}c_o, & k = 0, \\ 0, & k > 0. \end{cases} \tag{2.6}$$

Clearly $\|H^*\|_\infty = \|h\|_\infty \leq \pi/2(1-\varepsilon)$ and we note by (2.6) that H^* has the same Fourier coefficients as does R for $k \geq -M$. We set

$$G(\theta) = H^*(\theta) + \overline{H^*(\theta)}.$$

Thus $\|G\|_\infty \leq \pi/(1-\varepsilon)$ and G has the same Fourier coefficients as does P for $|k| \leq M$.

For all integers k, let

$$B_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}G(\theta)\,d\theta.$$

Consider the L^{th} Cesàro mean σ_L of G, i.e., let

$$\sigma_L(\theta, G) = \sum_{k=-L}^{L}\left(1 - \frac{|k|}{L+1}\right)B_k e^{ik\theta}.$$

Because Fejér's kernel is positive, we have

$$\|\sigma_L\|_\infty \leq \|G\|_\infty < \frac{\pi}{1-\varepsilon}.$$

Finally, set

$$Q(\theta) = \sigma_L(\theta, G) + \sum_{k=-M}^{M} \frac{|k|}{L+1} c_k e^{ik\theta}.$$

Since $|c_k| < 1/(1-\varepsilon)$ for $|k| \leq M$, we have

$$\|Q\|_\infty \leq \|\sigma_L\|_\infty + \frac{M(M+1)}{(L+1)(1-\varepsilon)} < \frac{\pi}{1-\varepsilon} + \varepsilon$$

provided that $L+1 > M(M+1)/\varepsilon(1-\varepsilon)$. Furthermore, for $|k| \leq M$, since $B_k = c_k$, we see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta} Q(\theta)\, d\theta = c_k,$$

proving Lemma 1.

The following elementary growth lemma is used in the proof of Theorem 1.

Lemma 2. *Suppose* $g\colon [0,\infty) \to [0,\infty)$ *is a nondecreasing function continuous from the right and suppose that* $0 < \alpha < \overline{\lim}_{x\to\infty} g(x)/x$. *Then there exists a nondecreasing sequence* $x_j \to \infty$ *such that*

$$g(x) - g(x_j) \leq \alpha(x - x_j), \qquad 0 \leq x \leq x_j.$$

Proof. Let $b_j \to \infty$ be an increasing sequence and let

$$x_j = \inf\{x \geq 0 : g(x) \geq \alpha x + b_j\}.$$

Clearly x_j is a well-defined sequence of real numbers such that $x_{j+1} \geq x_j \to \infty$. The continuity of g from the right ensures that $g(x_j) \geq \alpha x_j + b_j$. The definition of x_j implies that

$$g(x) < \alpha x + b_j, \qquad 0 \leq x < x_j,$$

and hence

$$g(x) - g(x_j) < \alpha(x - x_j), \qquad 0 \leq x < x_j,$$

completing the proof of Lemma 2.

The proofs of both Theorems 1 and 2 are based on an analysis of the Fourier series of $\log|H(re^{i\theta})|$ where H is entire. Suppose that H is entire with $|H(0)| = 1$. Suppose that H has zero sequence $\{z_\nu\}$ with due regard to multiplicity and that $|z_\nu| \neq r$ for $\nu = 1, 2, 3, \ldots$. Let $n(t)$ be the number of zeros of H counting multiplicity of modulus no more than t. We write

$$\log|H(re^{i\theta})| = \sum_{m=-\infty}^{\infty} c_m(r, H) e^{im\theta},$$

where the Fourier coefficients, first studied by F. Nevanlinna [7] (see also [6]), are given by the following formulas:

$$c_0(r, H) = N(r) = \int_0^r \frac{n(t)}{t}\, dt, \tag{2.7}$$

$$c_m(r, H) = \frac{\beta_m}{2} r^m + \frac{1}{2m} \sum_{|z_\nu|<r} \left(\frac{r}{z_\nu}\right)^m - \left(\frac{\overline{z_\nu}}{r}\right)^m, \qquad m \geq 1, \tag{2.8}$$

where

$$\log H(z) = \sum_{m=1}^{\infty} \beta_m z^m \tag{2.9}$$

near 0 for some branch of the logarithm, and

$$c_{-m}(r, H) = \overline{c_m(r, H)}, \qquad m \geq 1.$$

We collect certain estimates (see (2.14) and (2.15)) involving the Fourier series of $\log|H(re^{i\theta})|$ which are common to the proofs of Theorems 1 and 2. Suppose $z_\nu \neq 0$, $z_\nu \to \infty$, and H is the convergent product

$$H(z) = \prod_{\nu=1}^{\infty} E\left(\frac{z}{z_\nu}, \nu - 1\right), \tag{2.10}$$

where we recall that the logarithm vanishing at $z = 0$ of the usual Weierstrass factor $E(z, p)$ is given by

$$\log E(z, p) = -\frac{z^{p+1}}{p+1} - \frac{z^{p+2}}{p+2} - \cdots, \qquad |z| < 1. \tag{2.11}$$

Suppose that H has no zero of modulus r. From (2.8), (2.9), and (2.11) we conclude that if $|z_m| < r$, then

$$c_m(r, H) = \frac{1}{2m} \sum_{\substack{\nu > m \\ |z_\nu|<r}} \left(\frac{r}{z_\nu}\right)^m - \frac{1}{2m} \sum_{|z_\nu|<r} \left(\frac{\overline{z_\nu}}{r}\right)^m, \tag{2.12}$$

and if $|z_m| > r$ then

$$c_m(r,H) = -\frac{1}{2m}\sum_{\substack{\nu\le m\\ r<|z_\nu|}}\left(\frac{r}{z_\nu}\right)^m - \frac{1}{2m}\sum_{|z_\nu|<r}\left(\frac{\overline{z_\nu}}{r}\right)^m. \tag{2.13}$$

Since the modulus of each term on the right side of (2.13) is at most $1/2m$ and there are at most m such terms, we see that if $|z_m| > r$, then

$$|c_m(r,H)| \le \tfrac{1}{2}. \tag{2.14}$$

We now suppose that $|z_m| < r$. Taking into account the possibility of H having more than one zero of modulus $|z_m|$, we see from (2.12) that

$$\begin{aligned}|c_m(r,H)| &\le \frac{1}{2m}\int_{|z_m|}^{r}\left(\left(\frac{r}{t}\right)^m - \left(\frac{t}{r}\right)^m\right)dn(t)\\ &\quad + \frac{1}{2m}(n(|z_m|)-m)\left(\left(\frac{r}{|z_m|}\right)^m - \left(\frac{|z_m|}{r}\right)^m\right) + \frac{1}{2m}\sum_{\nu\le m}\left(\frac{|z_\nu|}{r}\right)^m\\ &\le \frac{1}{2}\int_{|z_m|}^{r}\left(\left(\frac{r}{t}\right)^m + \left(\frac{t}{r}\right)^m\right)\frac{n(t)}{t}\,dt + \frac{1}{2},\end{aligned} \tag{2.15}$$

where we have used integration by parts and the fact that the summation in the middle expression contains at most m terms, each of modulus no more than $1/2m$. A critical role in the proof of both of our theorems is played by (2.15).

3. Proof of Theorem 1

We first suppose $1 < \mu < \infty$. With no loss of generality, we may presume that $|z_1| > e$, for multiplication of f by a polynomial leaves (1.3) unaffected. Let ε in $(0,1)$ be such that $(1-\varepsilon)^2\mu > 1 + 2\varepsilon$. With the convention $\log^+ 0 = 0$, we apply Lemma 2 for $t \ge e$ with $x = \log\log t$, $g(x) = \log^+\log^+ n(e^{e^x})$, and $\alpha = (1-\varepsilon)\mu$ to conclude that there exists an increasing sequence r_j^* such that

$$\log^+\log^+ n(t) - \log\log n(r_j^*) \le (1-\varepsilon)\mu(\log\log t - \log\log r_j^*), \qquad e \le t \le r_j^*.$$

By continuity it is immediate that there exists $r_j > r_j^*$ such that $n(r_j) = n(r_j^*)$ and

$$\begin{aligned}&\log^+\log^+ n(t) - \log\log(1+\varepsilon)n(r_j)\\ &\qquad\le (1-\varepsilon)\mu(\log\log t - \log\log r_j), \qquad e \le t \le r_j.\end{aligned} \tag{3.1}$$

We rearrange (3.1) to obtain

$$\frac{\log^+ n(t)}{\log(1+\varepsilon)n(r_j)} \le \left(\frac{\log t}{\log r_j}\right)^{(1-\varepsilon)\mu}, \qquad e \le t \le r_j. \tag{3.2}$$

Hence

$$\frac{\log^+ n(t)}{\log t} \le \left(\frac{\log t}{\log r_j}\right)^{(1-\varepsilon)\mu-1} \left(\frac{\log(1+\varepsilon)n(r_j)}{\log r_j}\right), \qquad e \le t \le r_j,$$

or

$$\frac{\log^+ n(t)}{\log t} \le \frac{\log(1+\varepsilon)n(r_j)}{\log r_j}, \qquad e \le t \le r_j. \tag{3.3}$$

To simplify notation, we define

$$u_j := \frac{\log(1+\varepsilon)n(r_j)}{\log r_j} \tag{3.4}$$

and rewrite (3.3) as

$$n(t) \le t^{u_j}, \qquad 0 \le t \le r_j. \tag{3.5}$$

Since Z has infinite exponent of convergence we observe from (1.1) and (3.3) that $u_j \to \infty$ as $j \to \infty$.

For a fixed $\alpha > 1$, consider the function $g(y) = y^\alpha - 1$, $0 \le y \le 1$. Clearly $g(0) = -1$, $g(1) = 0$, and g is increasing and convex on $(0,1)$. If y_0 in $(0,1)$ is determined by $y_0^{\alpha-1} = 1-\varepsilon$, then

$$\frac{g(y)-(-1)}{y-0} \le \frac{g(y_0)-(-1)}{y_0-0} = 1-\varepsilon, \qquad 0 < y < y_0,$$

and

$$\frac{g(y)-0}{y-1} \ge \frac{g(y_0)-0}{y_0-1} = \alpha\tilde{y}_0^{\alpha-1} > \alpha(1-\varepsilon), \qquad y_0 < y < 1,$$

for some $\tilde{y}_0$ in $(y_0, 1)$.

Recalling that $\alpha = (1-\varepsilon)\mu > 1$ and setting $y = (\log t)/\log r_j$, we conclude that there exists $\tilde{r}_j$ in $[1, r_j)$ with $\log \tilde{r}_j / \log r_j = y_0$ such that

$$\left(\frac{\log t}{\log r_j}\right)^{(1-\varepsilon)\mu} < (1-\varepsilon)\frac{\log t}{\log r_j}, \qquad 1 < t < \tilde{r}_j, \tag{3.6}$$

and

$$\left(\frac{\log t}{\log r_j}\right)^{(1-\varepsilon)\mu} - 1 < (1-\varepsilon)^2\mu\left(\frac{\log t}{\log r_j} - 1\right), \qquad \tilde{r}_j < t < r_j. \tag{3.7}$$

From (3.2) and (3.6) we have

$$\frac{\log^+ n(t)}{\log(1+\varepsilon)n(r_j)} < (1-\varepsilon)\frac{\log t}{\log r_j}, \qquad e \le t < \tilde{r}_j,$$

or

$$n(t) < t^{(1-\varepsilon)u_j}, \qquad 0 < t < \tilde{r}_j. \tag{3.8}$$

From (3.2) we have

$$\log^+ n(t) - \log(1+\varepsilon)n(r_j) < \big(\log(1+\varepsilon)n(r_j)\big)\left(\left(\frac{\log t}{\log r_j}\right)^{(1-\varepsilon)\mu} - 1\right), \tag{3.9}$$

for $e \le t \le r_j$. The combination of (3.7) and (3.9) yields

$$\log n(t) - \log(1+\varepsilon)n(r_j) < (1-\varepsilon)^2 \mu u_j(\log t - \log r_j), \qquad \tilde{r}_j < t < r_j,$$

or

$$n(t) < (1+\varepsilon)n(r_j)\left(\frac{t}{r_j}\right)^{(1-\varepsilon)^2\mu u_j}, \qquad \tilde{r}_j < t < r_j. \tag{3.10}$$

We define the entire function H by (2.10) and proceed to estimate $|c_m(r_j, H)|$ from above. First we note from (2.7), (3.4), and (3.5) that

$$\begin{aligned} |c_0(r_j, H)| &= \int_0^{r_j} \frac{n(t)}{t}\,dt \le \int_0^{r_j} t^{u_j-1}\,dt \\ &= \frac{r_j^{u_j}}{u_j} = \frac{(1+\varepsilon)n(r_j)}{u_j} = o\big(n(r_j)\big). \end{aligned} \tag{3.11}$$

For $|z_m| > r_j$, i.e., for $m > n(r_j)$, we have (2.14) with $r = r_j$. Thus we consider m such that $|z_m| < r_j$. First suppose $m > (1+\varepsilon)u_j$. From (2.15) and (3.5) we have

$$\begin{aligned} |c_m(r_j, H)| &\le \int_{|z_m|}^{r_j} \left(\frac{r_j}{t}\right)^m \frac{n(t)}{t}\,dt + \frac{1}{2} \\ &\le r_j^m \int_{|z_m|}^{r_j} t^{u_j-m-1}dt + \frac{1}{2} \le r_j^m\left\{\frac{|z_m|^{u_j-m}}{m-u_j} + \frac{1}{2r_j^m}\right\} \\ &< r_j^m\left\{\frac{1}{\varepsilon u_j |z_m|^{m(1-(1/(1+\varepsilon))}} + \frac{1}{2r_j^m}\right\}. \end{aligned} \tag{3.12}$$

Writing

$$c_m(r_j,H)=r_j^m\beta_m(j),\qquad (1+\varepsilon)u_j<m\le n(r_j), \tag{3.13}$$

we see from (3.12) that there exists $\delta_j\to 0$ such that

$$|\beta_m(j)|^{1/m}<\delta_j,\qquad (1+\varepsilon)u_j<m\le n(r_j). \tag{3.14}$$

We now suppose that $m\le[(1+\varepsilon)u_j]$. We have from (2.15) that

$$\begin{aligned}|c_m(r_j,H)|&\le\frac12\int_{|z_m|}^{\tilde r_j}\left(\left(\frac{r_j}{t}\right)^m+\left(\frac{t}{r_j}\right)^m\right)\frac{n(t)}{t}\,dt\\&\quad+\frac12\int_{\tilde r_j}^{r_j}\left(\left(\frac{r_j}{t}\right)^m+\left(\frac{t}{r_j}\right)^m\right)\frac{n(t)}{t}\,dt+\frac12\\&=c_m^a(r_j,H)+c_m^b(r_j,H)+\tfrac12,\end{aligned} \tag{3.15}$$

where of course the first term is omitted if $|z_m|\ge\tilde r_j$. From (3.10) we have

$$\begin{aligned}c_m^b(r_j,H)&\le\frac{(1+\varepsilon)n(r_j)}{2}\int_{\tilde r_j}^{r_j}\left(\left(\frac{r_j}{t}\right)^m+\left(\frac{t}{r_j}\right)^m\right)\left(\frac{t}{r_j}\right)^{(1-\varepsilon)^2\mu u_j}\frac{dt}{t}\\&\le\frac{(1+\varepsilon)n(r_j)}{2}\left(\frac{1}{(1-\varepsilon)^2\mu u_j-m}+\frac{1}{(1-\varepsilon)^2\mu u_j+m}\right).\end{aligned}$$

Estimating the sums by the corresponding integrals we obtain for large j

$$\begin{aligned}\sum_{m=1}^{[(1+\varepsilon)u_j]}c_m^b(r_j,H)&\le\frac{(1+\varepsilon)n(r_j)}{2}\left(\frac{1}{(1-\varepsilon)^2\mu u_j+1}\right.\\&\qquad\left.+\log\frac{(1-\varepsilon)^2\mu u_j+(1+\varepsilon)u_j}{(1-\varepsilon)^2\mu u_j-(1+2\varepsilon)u_j}\right)\\&\le\frac{(1+\varepsilon)n(r_j)}{2}\left(\log\frac{(1-\varepsilon)^2\mu+1+\varepsilon}{(1-\varepsilon)^2\mu-(1+2\varepsilon)}\right)+o\big(n(r_j)\big).\end{aligned} \tag{3.16}$$

By (3.15) we may write

$$c_m(r_j,H)=c_m^\alpha(r_j,H)+c_m^\beta(r_j,H) \tag{3.17}$$

where $|c_m^\alpha(r_j,H)|\le c_m^a(r_j,H)$ and $|c_m^\beta(r_j,H)|\le c_m^b(r_j,H)+\frac12$. Thus by (3.4) and (3.16)

$$\sum_{m=1}^{[(1+\varepsilon)u_j]}|c_m^\beta(r_j,H)|\le\frac{(1+\varepsilon)n(r_j)}{2}\log\left(\frac{(1-\varepsilon)^2\mu+1+\varepsilon}{(1-\varepsilon)^2\mu-(1+2\varepsilon)}\right)+o\big(n(r_j)\big). \tag{3.18}$$

We next consider $c_m^a(r_j,H)$ for $(1-\frac{1}{2}\varepsilon)u_j < m \le [(1+\varepsilon)u_j]$. Writing

$$c_m^a(r_j,H) = r_j^m \beta_m(j), \qquad (1-\tfrac{1}{2}\varepsilon)u_j < m \le [(1+\varepsilon)u_j], \tag{3.19}$$

we have from (3.8) and (3.15) that

$$\begin{aligned} |\beta_m(j)| &\le \int_{|z_m|}^{\tilde{r}_j} \frac{n(t)}{t^{m+1}}\,dt \le \int_{|z_m|}^{\tilde{r}_j} t^{(1-\varepsilon)u_j-m-1}\,dt \\ &< \frac{|z_m|^{(1-\varepsilon)u_j-m}}{m-(1-\varepsilon)u_j} < \frac{2}{\varepsilon u_j |z_m|^{m(1-(1-\varepsilon)/(1-\varepsilon/2))}}. \end{aligned}$$

Thus there exists $\varepsilon_j \to 0$ such that

$$|\beta_m(j)|^{1/m} < \varepsilon_j, \qquad (1-\tfrac{1}{2}\varepsilon)u_j < m \le [(1+\varepsilon)u_j]. \tag{3.20}$$

Finally, suppose $m \le [(1-\frac{1}{2}\varepsilon)u_j]$. Let $m' = [(1-\frac{1}{4}\varepsilon)u_j]$. Using (3.4) and (3.8) we have for large j

$$\begin{aligned} c_m^a(r_j,H) &\le \int_{|z_m|}^{\tilde{r}_j} \left(\frac{r_j}{t}\right)^m \frac{n(t)}{t}\,dt \le \int_{|z_m|}^{\tilde{r}_j} \left(\frac{r_j}{t}\right)^{m'} \frac{n(t)}{t}\,dt \\ &= r_j^{m'} \int_{|z_m|}^{\tilde{r}_j} \frac{n(t)}{t^{m'+1}}\,dt \le r_j^{m'} \int_{|z_m|}^{\tilde{r}_j} t^{(1-\varepsilon)u_j-m'-1}\,dt \\ &< r_j^{m'} \int_1^{\infty} t^{(1-\varepsilon)u_j-m'-1}\,dt < r_j^{m'} \\ &< (1+\varepsilon)^{(1-\varepsilon/4)}\big(n(r_j)\big)^{1-\varepsilon/4}. \end{aligned}$$

Thus

$$\sum_{m=1}^{[(1-\varepsilon/2)u_j]} c_m^a(r_j,H) < (1+\varepsilon)^{(1-\varepsilon/4)}\big(n(r_j)\big)^{1-\varepsilon/4}(1-\tfrac{1}{2}\varepsilon)u_j = o\big(n(r_j)\big). \tag{3.21}$$

From elementary considerations, because H has no zeros on $|z| = r_j$, there exists $M_j > (1+\varepsilon)u_j$ such that

$$\sum_{|m|>M_j} |c_m(r_j,H)| < \varepsilon n(r_j). \tag{3.22}$$

We define $\kappa_m(j)$ for $(1-\frac{1}{2}\varepsilon)u_j < m \le M_j$ by

$$\kappa_m(j) = \begin{cases} \dfrac{-2c_m^\alpha(r_j,H)}{r_j^m}, & \big(1-\tfrac{1}{2}\varepsilon\big)u_j < m \le [(1+\varepsilon)u_j], \\ \dfrac{-2c_m(r_j,H)}{r_j^m}, & (1+\varepsilon)u_j < m \le M_j. \end{cases} \tag{3.23}$$

From (2.14), (3.13), (3.14), (3.19), and (3.20) we note that there exists $\gamma_j > 0$ such that whether $|z_m| < r_j$ or $|z_m| > r_j$,

$$|\kappa_m(j)|^{1/m} < \gamma_j \to 0, \qquad (1-\tfrac{1}{2}\varepsilon)u_j < m \leq M_j. \tag{3.24}$$

We consider a subsequence r_{j_k} such that

$$u_{j_{k+1}} > 2M_{j_k}, \qquad 1 \leq k < \infty, \tag{3.25}$$

and

$$\gamma_{j_{k+1}} < \frac{1}{2r_{j_k}}, \qquad 1 \leq k < \infty,$$

ensuring by (3.24) that for $k \geq 1$

$$|\kappa_m(j_p)| < \left(\frac{1}{2r_{j_k}}\right)^m, \qquad (1-\tfrac{1}{2}\varepsilon)u_{j_p} < m \leq M_{j_p}, \qquad p > k. \tag{3.26}$$

We define

$$T_j(z) = \sum_{m=1+[(1-\varepsilon/2)u_j]}^{M_j} \kappa_m(j) z^m, \qquad T(z) = \sum_{k=1}^{\infty} T_{j_k}(z), \tag{3.27}$$

and note by (3.25) that the powers of z appearing in the various T_{j_k} are distinct. We also note by (3.24) that T is entire. We set

$$f_k(z) = H(z)e^{T_{j_k}(z)}$$

and

$$f(z) = H(z)e^{T(z)}.$$

From (2.7), (2.8), (3.17), (3.23), and (3.27) we see that

$$c_m(r_{j_k}, f_k) = \begin{cases} c_o(r_{j_k}, H), & m = 0, \\ c_m^{\alpha}(r_{j_k}, H) + c_m^{\beta}(r_{j_k}, H), & 1 \leq m \leq [(1-\frac{1}{2}\varepsilon)u_{j_k}], \\ c_m^{\beta}(r_{j_k}, H), & (1-\frac{1}{2}\varepsilon)u_{j_k} < m \leq [(1+\varepsilon)u_j], \\ 0, & (1+\varepsilon)u_{j_k} < m \leq M_{j_k}, \\ c_m(r_{j_k}, H), & m > M_{j_k}. \end{cases} \tag{3.28}$$

From (3.11), (3.18), (3.21), (3.22), and (3.28) we see that

$$\sum_{m=-\infty}^{\infty} |c_m(r_{j_k}, f_k)| < (1+\varepsilon)n(r_{j_k}) \log\left(\frac{(1-\varepsilon)^2\mu + (1+\varepsilon)}{(1-\varepsilon)^2\mu - (1+2\varepsilon)}\right) + \big(\varepsilon + o(1)\big)n(r_{j_k}). \tag{3.29}$$

Since $|\kappa_m(j_k)| < 1$, it follows from (3.25) that

$$\Bigl\|\sum_{p<k} T_{j_p}(r_{j_k}e^{i\theta})\Bigr\|_\infty \le M_{j_{k-1}} r_{j_k}^{M_{j_{k-1}}} < (\tfrac{1}{2}u_{j_k})\bigl((1+\varepsilon)n(r_{j_k})\bigr)^{1/2} = o\bigl(n(r_{j_k})\bigr).$$

Also by (3.26)

$$\Bigl\|\sum_{p>k} T_{j_p}(r_{j_k}e^{i\theta})\Bigr\|_\infty \le \sum_{m>M_{j_k}} \frac{(r_{j_k})^m}{(2r_{j_k})^m} < 1,$$

implying that

$$\Bigl\|\log\Bigl|e^{\sum_{p\ne k} T_{j_p}(r_{j_k}e^{i\theta})}\Bigr|\Bigr\|_\infty = o\bigl(n(r_{j_k})\bigr). \tag{3.30}$$

The combination of (3.29) and (3.30) yields

$$\bigl\|\log|f(r_{j_k}e^{i\theta})|\bigr\|_\infty < (1+\varepsilon)n(r_{j_k})\log\biggl(\frac{(1-\varepsilon)^2\mu+1+\varepsilon}{(1-\varepsilon)^2\mu-(1+2\varepsilon)}\biggr) + \bigl(\varepsilon+o(1)\bigr)n(r_{j_k}).$$

Letting $\varepsilon \to 0^+$, we obtain (1.3) for $1 < \mu < \infty$.

If $\mu = \infty$, it is elementary from Lemma 2 that for every $\mu' < \infty$ there exists a sequence satisfying (3.1) with μ' in place of μ. Thus the conclusion of Theorem 1 holds with μ' in place of μ for every μ' in $(1,\infty)$, proving the theorem in the case $\mu = \infty$.

4. Proof of Theorem 2

In view of Theorem 1, we may restrict our attention to sequences Z for which $\mu < \frac{3}{2}$. With no loss of generality we may presume that $|z_1| > 1$. Suppose $0 < \varepsilon < \frac{1}{3}$.

Let m_j be an increasing sequence of positive integers. Let

$$r_j^* = \inf\biggl\{t \ge |z_1| : \frac{\log n(t)}{\log t} \ge m_j + 1 - \varepsilon\biggr\}.$$

Because Z has infinite exponent of convergence, it is elementary that r_j^* is a nondecreasing sequence of real numbers tending to infinity. It is immediate that

$$n(t) < t^{m_j+1-\varepsilon}, \qquad t < r_j^*. \tag{4.1}$$

Because $n(t)$ is continuous from the right, for each j there exists ν such that $|z_\nu| = r_j^*$ and

$$n(r_j^*) \ge (r_j^*)^{m_j+1-\varepsilon}. \tag{4.2}$$

We wish to focus on a sequence of circles containing no members of Z and thus choose $r_j > r_j^*$ such that $n(r_j) = n(r_j^*)$ and

$$\left(\frac{r_j}{r_j^*}\right)^{n(r_j)} < 1+\varepsilon. \tag{4.3}$$

For $j \geq 1$ we define

$$\tilde{r}_j = \inf\left\{t \geq |z_1| : \frac{\log n(t)}{\log t} \geq \frac{1}{2}(m_j+1-\varepsilon)\right\},$$

and note trivially that $\tilde{r}_j \leq r_j^*$ and

$$n(t) < t^{(m_j+1-\varepsilon)/2}, \qquad t < \tilde{r}_j. \tag{4.4}$$

Since $\log n(t)$ is continuous from the right, we see from the definition of $\tilde{r}_j$ that

$$\frac{\log n(\tilde{r}_j)}{\log \tilde{r}_j} \geq \frac{m_j+1-\varepsilon}{2}.$$

Since $\mu < \frac{3}{2}$, for large j we have $\log n(\tilde{r}_j) < (\log \tilde{r}_j)^{3/2}$ and thus

$$\frac{\log \tilde{r}_j}{m_j} > \frac{\log \tilde{r}_j}{m_j+1-\varepsilon} \geq \frac{(\log \tilde{r}_j)^2}{2\log n(\tilde{r}_j)} > \frac{(\log \tilde{r}_j)^{1/2}}{2} \to \infty \tag{4.5}$$

as $j \to \infty$.

We form the product (2.10) and now estimate $|c_m(r_j, H)|$. By (2.7), (4.1), and (4.3) we have

$$c_o(r_j, H) = \int_0^{r_j^*} \frac{n(t)}{t}\, dt + \int_{r_j^*}^{r_j} \frac{n(t)}{t}\, dt < \frac{(r_j^*)^{m_j+1-\varepsilon}}{m_j+1-\varepsilon} + \log(1+\varepsilon). \tag{4.6}$$

We next suppose $1 \leq m \leq m_j$. We note from (4.2) for large j that $n(r_j) > 2^{m_j} > m_j \geq m$ and thus $|z_m| < r_j$. From (2.15), (4.1), and (4.3) we conclude

$$\begin{aligned}
|c_m(r_j, H)| &\leq \frac{1}{2}\int_{|z_m|}^{r_j^*} \left(\left(\frac{r_j}{t}\right)^m + \left(\frac{t}{r_j}\right)^m\right)\frac{n(t)}{t}\, dt \\
&\quad + \frac{1}{2}\int_{r_j^*}^{r_j} \left(\left(\frac{r_j}{t}\right)^m + \left(\frac{t}{r_j}\right)^m\right)\frac{n(t)}{t}\, dt + \frac{1}{2} \\
&< \frac{(1+\varepsilon)(r_j^*)^{m_j+1-\varepsilon}}{2}\left(\frac{1}{m_j+1-\varepsilon-m} + \frac{1}{m_j+1-\varepsilon+m}\right) \\
&\quad + (1+\varepsilon)\log(1+\varepsilon) + \frac{1}{2}.
\end{aligned}$$

We conclude for $1 \le m \le m_j$ that there are expressions $c_m^\alpha(r_j)$ and $c_m^\beta(r_j)$ satisfying

$$c_m(r_j, H) = c_m^\alpha(r_j) + c_m^\beta(r_j), \tag{4.7}$$

$$|c_m^\alpha(r_j)| \le \frac{(1+\varepsilon)(r_j^*)^{m_j+1-\varepsilon}}{2(m_j+1-\varepsilon-m)}, \tag{4.8}$$

and

$$|c_m^\beta(r_j)| \le \frac{(1+\varepsilon)(r_j^*)^{m_j+1-\varepsilon}}{2(m_j+1-\varepsilon+m)} + 1. \tag{4.9}$$

We set $c_{-m}^\alpha(r_j) = \overline{c_m^\alpha(r_j)}$ for $1 \le m \le m_j$ and also set $c_0^\alpha(r_j) = c_0(r_j, H)$.

Now

$$\sum_{m=1}^{m_j} \frac{1}{m_j+1-\varepsilon+m} < \log 2,$$

and thus by (4.9) for large j

$$\begin{aligned}\sum_{m=1}^{m_j} |c_m^\beta(r_j)| &< \frac{(1+\varepsilon)(\log 2)(r_j^*)^{m_j+1-\varepsilon}}{2} + m_j \\ &< \frac{(1+2\varepsilon)(\log 2)(r_j^*)^{m_j+1-\varepsilon}}{2}.\end{aligned} \tag{4.10}$$

We now suppose $m_j < m \le n(r_j)$ and note that $|z_m| < r_j$. From (2.15), (4.1), (4.3), and (4.4) we have (with the omission of an obvious term if $|z_m| > \tilde{r}_j$)

$$\begin{aligned}|c_m(r_j, H)| &\le \int_{|z_m|}^{\tilde{r}_j} \left(\frac{r_j}{t}\right)^m \frac{n(t)}{t}\,dt + \int_{\tilde{r}_j}^{r_j^*} \left(\frac{r_j}{t}\right)^m \frac{n(t)}{t}\,dt \\ &\quad + \int_{r_j^*}^{r_j} \left(\frac{r_j}{t}\right)^m \frac{n(t)}{t}\,dt + \frac{1}{2} \\ &< \frac{r_j^m}{|z_m|^{m-(m_j+1-\varepsilon)/2}\big(m - \frac{1}{2}(m_j+1-\varepsilon)\big)} \\ &\quad + \frac{r_j^m}{\tilde{r}_j^{\,m-m_j-1+\varepsilon}(m-m_j-1+\varepsilon)} \\ &\quad + (1+\varepsilon)\log(1+\varepsilon) + \tfrac{1}{2} \\ &= r_j^m\big(\beta_m^a(j) + \beta_m^b(j) + \beta_m^c(j) + \beta_m^d(j)\big).\end{aligned} \tag{4.11}$$

Trivially

$$(4.12)\qquad \begin{aligned}(\beta_m^c(j)+\beta_m^d(j))^{1/m} &= \big((1+\varepsilon)\log(1+\varepsilon)+\tfrac{1}{2}\big)^{1/m}/r_j\\ &< \frac{1}{r_j}\to 0,\quad m_j<m\le n(r_j),\ j\to\infty.\end{aligned}$$

Likewise for $m_j < m \le n(r_j)$ we have for large j

$$(4.13)\qquad \big(\beta_m^a(j)\big)^{1/m} < \frac{1}{|z_m|^{1-(m_j+1-\varepsilon)/2m}} < \frac{1}{|z_{m_j}|^{1/4}} \to 0$$

as $j\to\infty$.

For $m_j < m \le n(r_j)$ we have

$$\beta_m^b(j) \le \frac{1}{\varepsilon \tilde r_j^{\,m-m_j-1+\varepsilon}}$$

and thus

$$(4.14)\qquad \log\big(\beta_m^b(j)\big)^{1/m} \le -\frac{\log\varepsilon}{m} - \frac{m-m_j-1+\varepsilon}{m}\log\tilde r_j.$$

We consider two cases. If $m_j < m \le 2m_j$, then

$$(4.15)\qquad \log\big(\beta_m^b(j)\big)^{1/m} < -\frac{\log\varepsilon}{m_j} - \frac{\varepsilon}{2m_j}\log\tilde r_j \to -\infty,\qquad j\to\infty,$$

by (4.5). For $m > 2m_j$ we have by (4.14)

$$(4.16)\qquad \begin{aligned}\log\big(\beta_m^b(j)\big)^{1/m} &\le -\frac{\log\varepsilon}{m} + \left(-1+\frac{m_j}{m}+\frac{1-\varepsilon}{m}\right)\log\tilde r_j\\ &< -\frac{\log\varepsilon}{2m_j} + \left(-\frac{1}{2}+\frac{1-\varepsilon}{2m_j}\right)\log\tilde r_j \to -\infty,\qquad j\to\infty.\end{aligned}$$

For $m > m_j$, including those for which $|z_m| > r_j$, the combination of (2.14), (4.11), (4.12), (4.13), (4.15), and (4.16) implies the existence of $\delta_j \to 0$ such that if

$$(4.17)\qquad c_m(r_j,H) = \beta_m(j) r_j^m,$$

then

$$(4.18)\qquad |\beta_m(j)|^{1/m} < \delta_j \to 0,\qquad j\to\infty.$$

Using (4.6) and (4.8) we apply Lemma 1 with $M = m_j$ to conclude for large j that there exists a trigonometric polynomial $Q_j(\theta)$ with

$$Q_j(\theta) = \sum_{m=-L_j}^{L_j} d_m(r_j) e^{im\theta},$$

where $L_j > m_j$, such that

$$d_m(r_j) = c_m^\alpha(r_j), \qquad |m| \le m_j, \tag{4.19}$$

and

$$\|Q_j\|_\infty < \left(\frac{\pi}{1-\varepsilon} + \varepsilon\right)(1+\varepsilon)(r_j^*)^{m_j+1-\varepsilon}. \tag{4.20}$$

We write

$$d_m(r_j) = \gamma_m(r_j) r_j^m, \qquad m_j < m \le L_j, \tag{4.21}$$

yielding by (4.20)

$$|\gamma_m(r_j)| \le \left(\frac{\pi}{1-\varepsilon} + \varepsilon\right)(1+\varepsilon) r_j^{m_j+1-\varepsilon-m}$$

and hence

$$\log|\gamma_m(r_j)|^{1/m} \le \frac{\log\bigl(\pi/(1-\varepsilon)+\varepsilon\bigr) + \log(1+\varepsilon)}{m} - \frac{m+\varepsilon-m_j-1}{m}\log r_j.$$

For $m_j < m \le \min(L_j, 2m_j)$, we have

$$\log|\gamma_m(r_j)|^{1/m} < \frac{\log\bigl(\pi/(1-\varepsilon)+\varepsilon\bigr) + \log(1+\varepsilon)}{m_j} - \frac{\varepsilon}{2m_j}\log r_j \to -\infty \tag{4.22}$$

as $j \to \infty$ by (4.5). If $L_j > 2m_j$, for $2m_j < m \le L_j$ we have

$$\log|\gamma_m(r_j)|^{1/m} < \frac{\log\bigl(\pi/(1-\varepsilon)+\varepsilon\bigr) + \log(1+\varepsilon)}{2m_j} + \left(-\frac{1}{2} + \frac{1-\varepsilon}{2m_j}\right)\log r_j \to -\infty \tag{4.23}$$

as $j \to \infty$. Combining (4.22) and (4.23), we conclude there exists $\kappa_j > 0$ such that

$$|\gamma_m(r_j)|^{1/m} < \kappa_j \to 0, \qquad m_j < m \le L_j, \tag{4.24}$$

as $j \to \infty$.

By elementary considerations, there exists $M_j > L_j$ such that

$$\sum_{|m|>M_j} |c_m(r_j, H)| < \varepsilon n(r_j). \tag{4.25}$$

For $m_j < m \le M_j$, we define

$$b_m(j) = \begin{cases} \dfrac{-2c_m(r_j, H) + 2d_m(r_j)}{r_j^m}, & m_j < m \le L_j, \\ \dfrac{-2c_m(r_j, H)}{r_j^m}, & L_j < m \le M_j. \end{cases} \tag{4.26}$$

From (4.17), (4.18), (4.21), and (4.24), we conclude that there exists $\eta_j > 0$ such that

$$|b_m(j)|^{1/m} < \eta_j \to 0, \qquad m_j < m \le M_j, \qquad j \to \infty. \tag{4.27}$$

Let

$$T_j(z) = \sum_{m_j < m \le M_j} b_m(j) z^m$$

and

$$f_j(z) = H(z) e^{T_j(z)}.$$

By (2.8), (4.7), and (4.26)

$$c_m(r_j, f_j) = \begin{cases} c_o(r_j, H) = c_o^\alpha(r_j), & m = 0, \\ c_m^\alpha(r_j) + c_m^\beta(r_j), & 1 \le m \le m_j, \\ d_m(r_j), & m_j < m \le L_j, \\ 0, & L_j < m \le M_j, \\ c_m(r_j, H), & m > M_j. \end{cases} \tag{4.28}$$

We also of course have $c_{-m}(r_j, f_j) = \overline{c_m(r_j, f_j)}$ for $m \ge 1$.

From (4.2), (4.10), (4.19), (4.20), (4.25), and (4.28), we have for all θ in $[-\pi, \pi]$ and large j that

$$\begin{aligned} \left|\log|f_j(r_j e^{i\theta})|\right| &\le \left| \sum_{|m| \le L_j} d_m(r_j) e^{im\theta} \right| \\ &\quad + \sum_{1 \le |m| \le m_j} |c_m^\beta(r_j)| + \sum_{|m| > M_j} |c_m(r_j, H)| \\ &< \left(\frac{\pi}{1-\varepsilon} + \varepsilon \right)(1+\varepsilon) n(r_j) + (\log 2 + 3\varepsilon) n(r_j). \end{aligned} \tag{4.29}$$

We consider a subsequence m_{j_k} satisfying

$$m_{j_{k+1}} > M_{j_k}, \qquad k \geq 1, \tag{4.30}$$

and

$$|b_m(j_p)| < \left(\frac{1}{2r_{j_k}}\right)^m, \qquad k \geq 1,\ p \geq k+1,\ m_{j_p} < m \leq M_{j_p}, \tag{4.31}$$

where we have used (4.27).

We set

$$T(z) = \sum_{k=1}^{\infty} T_{j_k}(z),$$

and note by (4.30) that the powers of z in the various T_{j_k} are distinct. From (4.27) we see that T is entire and from (4.31) it follows that the Maclaurin coefficients of T have modulus less than 1. Thus by (4.2), (4.3), and the fact that $\mu < \frac{3}{2}$, for $-\pi \leq \theta \leq \pi$ and large k we have

$$\sum_{p=1}^{k-1} |T_{j_p}(r_{j_k} e^{i\theta})| \leq M_{j_{k-1}} r_{j_k}^{M_{j_{k-1}}} < m_{j_k} r_{j_k}^{m_{j_k}} < \varepsilon n(r_{j_k}). \tag{4.32}$$

From (4.31) we have for $-\pi \leq \theta \leq \pi$ that

$$\begin{aligned} \left|\sum_{p>k} T_{j_p}(r_{j_k} e^{i\theta})\right| &\leq \sum_{p=k+1}^{\infty} \sum_{m_{j_p} < m \leq M_{j_p}} |b_m(j_p)| r_{j_k}^m \\ &\leq \sum_{p=k+1}^{\infty} \sum_{m_{j_p} < m \leq M_{j_p}} \left(\frac{1}{2}\right)^m < 1. \end{aligned} \tag{4.33}$$

We set

$$f(z) = H(z) e^{T(z)}$$

and note by (4.32) and (4.33) for large k and $-\pi \leq \theta \leq \pi$ that

$$\left|\log|f(r_{j_k} e^{i\theta})| - \log|f_{j_k}(r_{j_k} e^{i\theta})|\right| < \varepsilon n(r_{j_k}) + 1.$$

We conclude by (4.29) that for large k

$$\left\|\log|f(r_{j_k} e^{i\theta})|\right\|_\infty < \left(\left(\frac{\pi}{1-\varepsilon} + \varepsilon\right)(1+\varepsilon) + \log 2 + 5\varepsilon\right) n(r_{j_k}).$$

Since ε in $(0, \frac{1}{3})$ is arbitrary, we obtain the conclusion of Theorem 2.

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